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# VARIATIONAL METHODS FOR THE SOLUTION OF PROBLEMS OF EQUILIBRIUM AND VIBRATIONS

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As Henri Poincaré once remarked, "solution of a mathematical problem" is a phrase of indefinite meaning. Pure mathematicians sometimes are satisfied with showing that the non-existence of a solution implies a logical contradiction, while engineers might consider a numerical result as the only reasonable goal. Such one sided views seem to reflect human limitations rather than objective values. In itself mathematics is an indivisible organism uniting theoretical contemplation and active application.

This address will deal with a topic in which such a synthesis of theoretical and applied mathematics has become particularly convincing. Since Gauss and W. Thompson, the equivalence between boundary value problems of partial differential equations on the one hand and problems of the calculus of variations on the other hand has been a central point in analysis. At first, the theoretical interest in existence proofs dominated and only much later were practical applications envisaged by two physicists, Lord Rayleigh and Walther Ritz; they independently conceived the idea of utilizing this equivalence for numerical calculation of the solutions, by substituting for the variational problems simpler approximating extremum problems in which but a finite number of parameters need be determined. Rayleigh, in his classical work—*Theory of sound*—and in other publications, was the first to use such a procedure. But only the spectacular success of Walther Ritz and its tragic circumstances caught the general interest. In two publications of 1908 and 1909 [39], Ritz, conscious of his imminent death from consumption, gave a masterly account of the theory, and at the same time applied his method to the calculation of the nodal lines of vibrating plates, a problem of classical physics that previously had not been satisfactorily treated.

Thus methods emerged which could not fail to attract engineers and physicists; after all, the minimum principles of mechanics are more suggestive than the differential equations. Great successes in applications were soon followed by further progress in the understanding of the theoretical background, and such progress in turn must result in advantages for the applications.

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The following address will try to convey an idea of this interplay between practical and theoretical points of view. Completeness cannot be attempted; rather some selected topics with which the speaker has been personally concerned will be discussed.

Usually the solution of a difficult problem in analysis proceeds according to a general scheme: The given problem $P$ with the solution $S$ is replaced by a related problem $P_n$ so simple that its solution $S_n$ can be found with comparative ease. Then by improving the approximation $P_n$ to $P$ we may expect, or we may assume, or we may prove, that $S_n$ tends to the desired solution $S$ of $P$. The essential point in an individual case is to choose the sequence $P_n$ in a suitable manner.

Here we shall deal mainly with problems of equilibrium and vibrations (boundary value and eigenvalue problems,[1] respectively). They lead to linear self-adjoint differential equations for an unknown function $u(x, y)$:

$$L(u) = f, \tag{1}$$

or

$$L(u) + \lambda u = 0, \tag{2}$$

in a two-dimensional domain of the $x$, $y$-plane, or rather to equivalent variational problems for the kinetic and potential energies of the system.

## I. THE VARIATIONAL PROBLEMS

We assume our domain $B$ bounded by a piecewise smooth curve $C$, and we denote the arc length measured along $C$ by $s$, and differentiation in the direction of the inward normal by $\partial/\partial n$ or by a subscript $n$.

1. **Quadratic functionals.** Our variational problems refer to quadratic functionals

$$Q(v) = Q(v, v)$$

defined by symmetric bilinear expressions such as

$$D(v, w) = \iint_B (v_x w_x + v_y w_y) dx dy, \tag{3}$$

$$M(v, w) = \iint_B [\Delta v \Delta w + \alpha(v_{xx} w_{yy} + v_{yy} w_{xx} - 2 v_{xy} w_{xy})] dx dy, \tag{4}$$

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[1] Problems of critical loads (buckling) are likewise mathematically formulated as eigenvalue problems.

which occur in the study of membranes and plates, respectively. In (4) the symbol $\Delta$ denotes the Laplacian operator, and $\alpha$ a constant. Furthermore, we shall have to use other quadratic integrals defined by bilinear expressions such as

$$H(v, w) = \int\int_B vw\,dxdy, \tag{5}$$

$$K(v, w) = \int_C vw\,ds, \tag{6}$$

$$R(v, w) = \int_L v_{xx}w_{xx}\,dx, \tag{7}$$

where $L$ is a line $y = \text{const.}$ in $B$ and $C$ is the boundary of $B$. We then consider functionals such as

$$Q(v, w) = aD(v, w) + bM(v, w) + cK(v, w) + dR(v, w),$$

where $a$, $b$, $c$, $d$ are constants. Always the *"admissible" functions* $u$, $v$, $w$, $\cdots$ are restricted by the condition that all the occurring integrands be at least piecewise continuous.

The stable equilibrium of a plate or membrane under an external pressure $f$ is characterized by a variational problem of the type

$$Q(v) + 2\,H(v, f) = \text{minimum}, \tag{8}$$

for the deflection $v$, whereas vibrations of plates and membranes correspond to the problem of finding stationary values, $\nu^2 = \lambda$, of

$$Q(v)/H(v). \tag{9}$$

The values $\nu$ thus defined are the natural frequencies of the system. $Q(v)$ corresponds to the potential energy of the system in the case of equilibrium, while for vibrations $Q(v)$ and $H(v)$ are the "reduced" potential and kinetic energies, respectively.[3] Terms of the form $K$ and $R$ appearing in the expression $Q$ represent *additional energies concentrated along the boundary* $C$ of $B$ or along a line $L$ in $B$. For example, in the case of *reinforced plates* we would have $Q = M + dR$ if the reinforcement consists of a bar along the line $L$.

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[3] By "reduced" energies we mean the following: If we assume the free system vibrating with a frequency $\omega$ (not necessarily a natural frequency), then the deflection $u$ may be represented in the form $u = v \cos \omega t$, where $v$ is a function of position only. The potential energy $V$ may then be written in the form $Q(v) \cos^2 \omega t$ and the kinetic energy $T$ in the form $\omega^2 H(v) \sin^2 \omega t$; the quantities $Q$ and $H$ are termed the reduced energies.

2. **Rigid and natural boundary conditions.** (See [1, 2, 9].) The Euler differential equations (1) or (2) of our variational problems must be supplemented by appropriate boundary conditions. If we focus our attention on the differential equations as such, it is not at all obvious what boundary conditions belong to a particular problem of mechanics. However, from the point of view of the calculus of variations a complete clarification of this delicate question is almost automatically obtained if the mathematical reasoning follows our mechanical intuition. In the first place we observe that in a variational problem (not so in a problem of differential equations) we need not in advance impose boundary conditions in order to single out a specific solution. If the functions admissible in the competition are not subjected to restrictions at the boundary, we speak of a "*free problem.*" In these problems the first variation of the functional will contain terms referring to the boundary, and the vanishing of the first variation will imply not only Euler's differential equation for the domain $B$ but also conditions on the boundary $C$, which we call "*natural boundary conditions.*" Now the dominant fact is: *appropriate boundary conditions for differential equations are obtained as natural boundary conditions of corresponding variational problems.* In the latter they may, but they need not be prescribed in advance.

There is only the exceptional case, often termed the "simplest case of a variational problem" of *fixed boundary values* of $u$ or derivatives of $u$ or other expressions in $u$. Here the situation seems somewhat obscured. (The clamped membrane with the boundary condition $u=0$ and the clamped plate with the boundary conditions $u=u_x=u_y=0$ belong in this category.) Such fixed or *rigid* or *artificial boundary conditions* must be explicitly stipulated for the variational problem not only for the differential equation. However, we shall recognize them as limiting cases or degenerations of natural conditions.

Physically, rigid conditions correspond to rigid constraints of the system at the boundary $C$ while natural conditions express equilibrium of the system of $C$ if along $C$ partial or full freedom of motion is permitted.

To understand the significance of natural boundary conditions the following observation is essential: The Euler differential equations depend only on the domain integrals or the energies spread over $B$. But the *natural* boundary conditions are essentially affected by the boundary integrals representing those contributions to the energies which are concentrated along the boundary $C$. These terms lead to a great variety of possible natural boundary conditions for the same differential equation. In a somewhat different way we may formulate the fact: Natural boundary conditions, but not Euler's equations, are affected by divergence expressions in the domain integrals.

2a. **Examples. Plates, plane torsion for multiply-connected domains.** As an example we consider a membrane with a free boundary. A pressure $f$ of average value zero may act on the membrane. In this case we have a variational problem for $D(v)+2H(v,f)$; if $\zeta$ denotes an arbitrary variation of $v$, the variational condition is

$$D(v, \zeta) + H(\zeta, f) = 0.$$

Transforming $D(v, \zeta)$ by Green's formula into a domain integral plus a boundary integral, we find the natural boundary condition

$$\frac{\partial v}{\partial n} = 0,$$

expressing the fact that no force acts on the boundary $C$. For a plate, free at the boundary, $Q(v) = M(v)$, and the natural boundary conditions appear as the classical Kirchhoff conditions:

$$(1 + \alpha)\Delta v = \alpha(v_{xx}x_n^2 + 2v_{xy}x_n y_n + v_{yy}y_n^2),$$
$$\frac{\partial}{\partial n}\Delta v = \alpha(v_{xx}x_n x_s + v_{xy}[x_n y_s + x_s y_n] + v_{yy}y_n y_s),$$

where $x_n$, $y_n$ and $x_s$, $y_s$ are, respectively, direction cosines of the inner normal and tangent vector along $C$.

In $M(v)$ the term $\alpha(v_{xx}v_{yy}-v_{xy}^2)=\alpha\partial(v_x v_{yy})/\partial x-\alpha\partial(v_x v_{xy})/\partial y$ is a divergence expression, very essential for Kirchhoff's natural boundary conditions but irrelevant for Euler's differential equation $\Delta\Delta v=0$ and without consequence for the clamped plate.

Similar remarks apply to *natural "discontinuity conditions"* that arise if energy is concentrated along lines $L$ interior to $B$, such as in the case of *reinforced plates*. For example, for a rectangular plate clamped at the boundary but reinforced by a bar along the line $y=0$, the variational problem becomes

$$Q(v) + 2H(v, f) + kR(v) = \iint_B [(\Delta v)^2 + 2\alpha(v_{xx}v_{yy} - v_{xy}^2)]dxdy$$
$$+ k\int_L v_{xx}^2 dx + 2\iint_B vf dxdy = \text{min}.$$

with the condition that $v=v_x=v_y=0$ on $C$ and $v$, $v_x$, $v_y$ are continuous in $B$ while the second derivatives of $v$ are at least piecewise continuous in $B$. If $\zeta$ is an arbitrary variation of $v$ satisfying these conditions, we obtain as above not only the Euler equation $\Delta\Delta v+f=0$, but also the ***natural discontinuity condition***

$$[(\Delta v + \alpha v_{xx})]_{-}^{+} = \frac{\partial^4}{\partial x^4} v(x, 0),$$

where the symbol $[g]_{-}^{+}$ means the amount of discontinuity suffered by a function $g$ in crossing the line $y=0$ from positive to negative values of $y$. Another example of a somewhat different character is the the problem of ***torsion of long columns with multiply connected cross sections***. The contour $C$ of the cross section may include a domain $B$ in the $x$, $y$-plane from which are removed holes $B_1, B_2, B_3, \cdots$ with contours $C_1, C_2, C_3, \cdots$ and areas $A_1, A_2, A_3, \cdots$. The multiply-connected domain between $C$ and $C_1, C_2, C_3, \cdots$ may be called $B^*$.

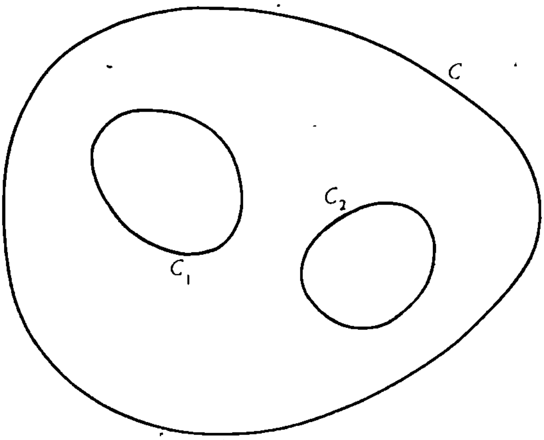

FIG. 1

Then the adequate variational formulation of the torsion problem in proper units is: To find a function $\phi=u$ continuous in $B+C$, having piecewise continuous first derivatives in $B$, having the boundary values zero on $C$ and constant, but not prescribed values $c_i$ in the holes $B_i$, such that for the whole domain $B$

$$D(\phi) = \iint [(\phi_x^2 + \phi_y^2) + 2\phi]dxdy$$

attains its least value $d$ for $\phi=u$. The function $u$ then will give the stresses in the cross section by differentiation.

Note that this problem requires not only to determine $u$ in $B^*$ as a solution of Euler's equation

$$\Delta u = 1$$

with the boundary conditions $u=0$ on $C$ and $u=\text{const.}=c_i$ on $C_i$, but also to find the constants $c_i$. The freedom for these parameters in the functions $\phi$ admissible in the variational problem must lead to natural conditions from which the $c_i$ are to be characterized.

Indeed, from the variational condition $D(u, \zeta)\times\iint\zeta dxdy=0$ valid for arbitrary $\zeta$ that vanishes on $C$ and has constant values in each $B$: we can only obtain Euler's equation, but likewise immediately the natural boundary conditions

$$\int_{C_i} \frac{\partial u}{\partial n} ds + c_i A_i = 0$$

which connect the unknown boundary constants with the known areas $A_i$.

Incidentally, for the special choice $\zeta=u$ we obtain (in line with more general results) that the solution, representing a state of equilibrium, satisfies the relation

$$S = D(u) = -\iint u dxdy.$$

The quantity $S$ represents the "*total stiffness*" of the column with respect to torsion.

In the appendix we shall see how this problem, which as an ordinary boundary value problem of a partial differential equation would be rather formidable, can be attacked numerically with success from the point of view developed here.

2b. **Rigid constraints as limiting cases.** If we have additional energy concentrated at the boundary and expressed, for example, by the term $K(v)=\int_C v^2 ds$, then not the Euler equation but the natural boundary conditions will be influenced by these terms. Thus, for the free membrane with

$$Q(v) = D(v) + \gamma K(v)$$

we obtain the natural boundary condition

$$\frac{\partial v}{\partial n} - \gamma v = 0. \tag{10}$$

We observe that as the parameter $\gamma$ increases indefinitely, that is, as the elastic restoring forces at the boundary approach a rigid constraint, the condition (10) tends to the boundary condition $v=0$ of the clamped membrane. This is in harmony with the fact that for large positive $\gamma$ the boundary values of $v^2$ though free, must in the average be small in order to keep the energy $Q(v)$ in bounds.

Quite generally rigid boundary conditions should be regarded as limiting cases of natural conditions in which a parameter tends to infinity. This corresponds to the physical fact that *rigid constraints are only idealised limiting cases of very large restoring forces.*[3] It may be mentioned that this interpretation of fixed boundary conditions is the key to a more penetrating analysis of the question what we may and what we may not prescribe at the boundary (see [1]).

3. **General conclusions.** The mere formulation of our problem in terms of maxima and minima leads to further important applications. As an example we mention a famous *principle* first formulated by Lord Rayleigh [38]: If a vibrating system whose energies are expressed by quadratic integrals is changed into another system by an increase of masses or by a decrease of elastic forces, then all the natural frequencies of the system can only change toward lower values. This fact was recognized by the speaker (see [9]) as a consequence of the following theorem, which is easily proved: The $n$th natural frequency of any of our vibration problems is the *highest value of the lowest frequency* of all systems obtained from the given system by imposing $n-1$ constraints; or the $n$th eigenvalue $\lambda_n$ is the largest value attained by the minimum $d(w_1, \cdots, w_{n-1})$ of the quotient $Q(v)/H(v)$ if $v$ is subjected to $n-1$ linear conditions of the form

$$H(v, w_i) = 0, \qquad i = 1, \cdots, n-1,$$

the $w_i$ being arbitrarily chosen functions. Here "minimum" refers to a fixed set of $n-1$ functions $w_i$.

This principle can render useful service for appraising the change in the natural frequencies resulting from changes in the given vibrating system. We recognize immediately that a stiffening of the system by the introduction of new elastic forces (which lead to an increase in potential energy) must produce higher natural frequencies throughout. Likewise, imposing new rigid constraints will have the same effect. Even quantitative results can be obtained by a simple application of this principle. The asymptotic behavior of the natural frequencies $\nu_n = \lambda_n^{1/2}$ for large $n$ was found by A. Sommerfeld, H. Weyl, and the speaker. For example, the $n$th eigenvalue of $\Delta v + \lambda v = 0$ for a free or clamped boundary and domain of area $A$ behaves asymptotically like $4\pi n/A$. (See [9].)

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[3] It might be mentioned that A. Weinstein's method (see §3) is a somewhat different way of presenting rigid boundaries as limiting cases of problems with fewer restrictions.

Recently Alexander Weinstein has made an interesting application of this maximum-minimum principle to clamped plates [4]. The eigenvalue problem for a "supported" plate (with $\alpha = 0$) refers to the quadratic energy expressions $Q(v) = \iint_B (\Delta v)^2 dxdy$, $H(v) = \iint_B v^2 dxdy$. It is then required to make the quotient $Q/H$ stationary only under the boundary condition $v = 0$. From this variational problem, we obtain the Euler equation $\Delta\Delta v - \lambda v = 0$ and one natural boundary condition, $\Delta v = 0$; the solutions for this problem are in this case identical with those of the clamped membrane. Now Weinstein, by imposing successively a denumerable number of boundary conditions of the form $\int_C (\partial v/\partial n)\phi_i ds = 0$, $i = 1, 2, 3, \cdots$, where the $\phi_i$ form a complete system of functions on the boundary $C$, obtains the problem of the clamped plate as a limiting case. The approximating problems stipulate conditions less restrictive than the limiting problem of a clamped plate, consequently they lead to smaller values of the minima and hence of the maxi-minima, and therefore provide lower bounds for the natural frequencies of the clamped plate. It is remarkable that the approximating problems could be solved explicitly in terms of solutions of the membrane problem [20, 21, 22].

## II. RAYLEIGH-RITZ METHOD

We now discuss the question of attacking a variational problem numerically. In principle, there are many ways in which such a variational problem may suggest approximations by simpler problems. The Rayleigh-Ritz method is only one of them.

1. **The principle and theoretical aspects.** Suppose we seek the minimum $d$ of an integral expression or any other variational expression $I(\phi)$ (for example, our quadratic functionals of the preceding section). We then start with a ***minimizing sequence***

$$\phi_1, \phi_2, \phi_3, \cdots, \phi_n, \cdots, \tag{11}$$

that is, a sequence of functions, admissible in our variational problem, for which

$$\lim_{n\to\infty} I(\phi_n) = d, \tag{12}$$

$d$ being the lower bound of the functional $I(\phi)$. The existence of the lower bound $d$ is obvious or may be easily proved in all relevant problems and the *existence* of the minimizing sequence (11) is then a logical consequence.

However, the problem in applications is one, not of the existence, but of the practical *construction* of such a minimizing sequence. Ritz's method is nothing but a recipe for such a construction. A minimizing sequence immediately furnishes an approximation to $d$ (sometimes this is all we wish to know, for example, if we are interested in the natural frequencies of a vibrating system). Moreover, it may be assumed and in many cases it can be proved that the minimizing sequence itself will furnish a good approximation for the function $u$ which actually solves the problem.

Ritz's construction proceeds as follows: We start with an arbitrarily chosen system of "*coordinate functions*"

$$\omega_1, \omega_2, \cdots, \omega_n, \cdots, \tag{13}$$

which should satisfy the two conditions:

(a) Any linear combination

$$\phi_n = c_1\omega_1 + c_2\omega_2 + \cdots + c_n\omega_n \tag{14}$$

of them is admissible in the variational problem.

(b) They should form a complete system of functions in the sense that *any* admissible function $\phi$ and its relevant derivatives may be approximated with any degree of accuracy by a linear combination of coordinate functions and of their corresponding derivatives, respectively.

If we begin with such a system of coordinate functions, it is clear that for $n$ sufficiently large and for a suitable choice of the coefficients $c_1, c_2, \cdots, c_n$ of (14) we can find admissible functions $\phi_n$ for which $I(\phi_n)$ differs arbitrarily little from $d$. In other words, it is possible to find a minimizing sequence $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ as a sequence of linear combinations of the coordinate functions. In order to obtain such a minimizing sequence we choose the $c_i$ in the following manner. We consider any function $\phi_n$ defined by (14) and substitute it in our variational problem. $I(\phi_n)$ then becomes a function $F(c_1, \cdots, c_n)$ of the $n$ parameters $c_i$ which we may now determine from the ordinary minimum problem of the calculus

$$I(\phi_n) = F(c_1, \cdots, c_n) = \text{min.} \tag{15}$$

In the problems considered here, $I(\phi)$ is a quadratic or bilinear functional, and (15) leads to a system of $n$ linear equations for the parameters,[4] a system which may be solved by established methods. Thus the minimizing sequence, $\phi_n$ $(n=1, 2, \cdots)$ is found. Recalling our general scheme we may identify the construction of the $\phi_n$ with the problem $P_n$.

At this point one important remark may be made [1]. While the convergence of $I(\phi_n)$ to $d$ is assured, it is by no means generally true that $\phi_n$ tends to $u$, the solution of the original minimum problem, or, even less so, that the derivatives of $\phi_n$ tend to the corresponding derivatives of $u$. However, a comparative investigation of different types of variational problems reveals that, generally speaking, the *convergence* of a minimizing sequence $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ and the sequences of the derivatives of the $\phi_n$ is *improved if the order of the occurring derivatives becomes higher*. On the other hand, there is a tendency toward *worse convergence as the number of independent variables increases*. For example, in the one-dimensional problem of an elastic string $(Q(v)=\int_0^1 v'^2 dx)$, the convergence of the $\phi_n$ to $u$ is assured, the derivatives $\phi_n'$, however, need not converge to $u'$. But for the corresponding problem of the bar $(Q(v)=\int_0^1 v''^2 dx)$, not only does $\phi_n$ converge to $u$ but also $\phi_n'$ to $u'$. On the other hand, in the membrane problem even the $\phi_n$ need not converge to $u$, while in the case of the plate the convergence of $\phi_n$ to $u$ is assured. The first success attained by Ritz depended largely on his good fortune in attacking the seemingly more difficult problem of the plate rather than that of the membrane.

These facts which are intimately related to more profound questions in the general theory of the variational calculus have suggested the following method of obtaining better convergence in the Rayleigh-Ritz method. Instead of considering the simple variational problem for the corresponding boundary value problem, we modify the former problem without changing the solution of the latter. This is accomplished by adding to the original variational expression terms of higher order which vanish for the actual solution $u$. For example, we may formulate the equilibrium problem for a membrane under the external pressure $f$ as follows:

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[4] This suggests the following interpretation and generalization of the procedure whereby reference to a minimum problem need no longer be made: we replace our differential equation $L(u)=0$ by the condition that $L(u)$ should be orthogonal to $n$ functions of a previously selected complete system of functions. If, then, $n$ tends to infinity, the totality of all these relations will be substituted for the differential equation, and for any fixed $n$, $u$ may be chosen, for example, as a linear combination (14): $u=\phi_n$. In this general pattern that goes back to Galerkin, there is more freedom left for the choice of the approximations to the solution $u$. However, in the Rayleigh-Ritz method proper, generally speaking the question of convergence is more easily investigated. (The generalized view interpretation is implicitly mentioned in Ritz' papers and was later developed by several authors. See for example [18, 19].)

$$I(v) = \iint_B (v_x^2 + v_y^2 + vf)\,dxdy + \iint_B k(\Delta v - f)^2\,dxdy = \text{min.},$$

where $k$ is an arbitrary positive constant or function. Such additional terms make $I(v)$ more *sensitive* to variations of $v$ without changing the solution. In other words, minimizing sequences attached to such a "sensitized" functional will by force behave better as regards convergence [7].

The practical value of the method of sensitizing the integral by the addition of terms of higher order has not yet been sufficiently explored. Certainly the sensitizing terms will lead to a more complicated system of equations for the $c_i$. This means that a compromise must be made for a suitable choice of the arbitrary positive function $k$ so that good convergence is assured while the necessary labor is kept within bounds.

2. **Practical viewpoints.** Theoretically the Rayleigh-Ritz method consists merely in the construction of the minimizing sequence. However, the difficulty that challenges the inventive skill of the applied mathematician is to find suitable coordinate functions and to estimate the accuracy of the result. From a practical point of view almost any success depends on the selection of coordinate functions. If these functions are chosen without proper regard for the individuality of the problem the task of computation will become hopeless. A choice should be made so that the system of linear equations for the $c_i$ obtained from (15) will have a preponderance of terms along the diagonal of their matrix, and that the number of terms to be taken into account be kept small. Since only a few of the coordinate functions will enter into the calculation, the theoretical completeness of the $\phi_n$ is irrelevant. It is important that the initial function be already a fair approximation, and furthermore that the functions $\phi_1$, $\phi_2, \cdots$ should be sufficiently different[5] so that increasing the number of terms will lead to an actual improvement of the approximation. Importance should also be attached to the need for making the numerical evaluation of the coefficients of our linear equations practicable.

In many cases the use of polynomials for coordinate functions is most advantageous. Si Luan Wei has shown in his thesis that results for plates can be obtained by polynomials more readily and more accurately than by the coordinate functions originally used by Ritz [40].

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[5] Such a "difference" can be measured. See Courant-Hilbert, ***Methoden der mathematischen Physik***, vol. I, p. 52.

Recently physicists have chosen coordinate functions according to the following pattern. One starts with a choice of the function $\omega_1$, which is expected to be a fair approximation to the actual solution. Then $\omega_2$ is defined by $\omega_2 = L(\omega_1)$. Finally one chooses $\omega_3 = L(\omega_2) = LL(\omega_1)$, $\omega_4 = L(\omega_3) = LLL(\omega_1)$, and so on. This choice makes the calculation of the matrix elements of our linear equations for the $c_\nu$ comparatively simple. Since the operator $L$ introduces higher derivatives which may lead to cumbersome complications at boundaries, it is understandable that such a choice of the $\omega_n$ seems feasible primarily for infinite domains where no boundary is given and where the natural boundary conditions are equivalent to the finiteness of the variational integrals.

3. **Boundary conditions.** For rigid boundary conditions the approximation by the Rayleigh-Ritz method is comparatively good. Few admissible coordinate functions would in most cases suffice to yield a result near the desired solution. This pleasant feature, however, is offset by the restriction imposed by the rigid boundary conditions which in general precludes the choice of simple coordinate functions.

For free boundaries and natural boundary conditions the choice of coordinate functions is eased considerably since no boundary conditions need be stipulated in advance. As a general rule we might well use, for the $\phi_n$, polynomials with undetermined coefficients. For this great advantage, however, we must pay a price, namely, the necessity for using many more terms to secure reasonable accuracy. Therefore it is sometimes preferable if we can satisfy in advance, at least approximately, the natural boundary conditions, by a proper choice of functions. Still, the advantage gained by making the $\phi_n$ polynomials might be decisive.

In this connection it seems to be of importance that *rigid boundaries can be considered as a limiting case of free boundaries*, as indicated previously in §I, 2. As an example we may consider the following problem for a membrane:

$$D(v) + 2H(v, f) + \gamma K(v) = \iint_B (v_x^2 + v_y^2 + 2vf)\,dxdy$$

$$+ \gamma \int_C v^2 ds = \min.$$

If $\gamma$ is very large, then the free boundary problem relating to this expression is almost identical to the corresponding problem for a clamped membrane. By letting $\gamma$ tend to infinity we *"freeze"* the boundary. This suggests an attempt to solve the equilibrium problem of the clamped plate by choosing a large numerical value for $\gamma$ and then treating the problem as one for a free membrane. However, to obtain reasonable accuracy for the free problem, the larger the quantity $\gamma$ the more terms will be necessary. Hence the practical application of the method again requires a compromise—$\gamma$ must be chosen large enough to approximate rigidity but small enough to keep the necessary labor within reasonable bounds. From a theoretical as well as from a practical point of view it would seem worth while to study the preferable choices of these artificial parameters.

4. **Estimates.** A weak point in the Rayleigh-Ritz procedure is that it does not contain a principle for estimating the accuracy of the approximation. This is not the place to give an account of the numerous efforts made to fill this gap at least theoretically. For problems of equilibrium, estimates for the minimum $d$ can be obtained by a method suggested by Castigliano's principle in the theory of elasticity. The mathematical idea is to represent the minimum value $d$ of the given problem as a maximum value of another variational problem (see [**28**]). The margins obtained are quite narrow. But the practical value of this accuracy does not appear great since the value of $d$ in problems of equilibrium is of little interest,[8] whereas we are concerned with the deflection $u$ and its derivatives. In the case of vibrations, where the minimum values, as squares of the frequencies, are of considerable importance, methods for the estimation of accuracy are in general less satisfactory [**29**].

5. **Objections to the Rayleigh-Ritz method.** The vagueness as to the accuracy of the approximation obtained is only one of the objections to the Rayleigh-Ritz method that may be raised. More annoying is that a suitable selection of the coordinate functions is often very difficult and that laborious computations are sometimes necessary. For these reasons, alternative methods must be studied.

## III. METHOD OF FINITE DIFFERENCES. GENERAL RANDOM STATISTICAL METHODS

As far as practical experience goes the most important of these methods is that of finite differences. In this well known procedure we replace differential quotients by difference quotients and integral expressions by finite sums defined over a set of net-points in the plane, for example, a quadratic net in the domain $B$ formed by the intersection of the lines $x=\gamma h$, $y=\mu h$ $(\gamma=1, 2, \cdots, N; \mu=1, 2, \cdots, M)$. The simplified problem $P_n$ of difference equations can very often be solved with relative ease. Then if we permit the mesh, $h=1/n$, of the net to tend to zero with increasing $n$, not only does $P_n$ tend to $P$, but the solutions $S_n$ of the difference equations approach the solution $S$ of the original problem exceedingly well. Furthermore, we have the remarkable fact that all relevant difference quotients of first and higher order converge to the corresponding derivatives pertaining to the original problem.

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[8] See, however, E. Trefftz, Math. Ann. vol. 108 (1933), p. 595 where the deflection at a point $x_0$, $y_0$ is represented as such a minimum $d$ of a modified functional.

On these grounds the method of finite difference as a general procedure is often preferable to the Rayleigh-Ritz method. The latter might lend itself more readily to the solution of specific problems where suitable analytic expressions are available for coordinate functions. However, in other cases experience points to the superiority of the method of finite differences. One of the underlying reasons is that finite differences are attached directly to the values of the function itself without an interceding medium such as the more or less arbitrary coordinate functions.

If the variational problems contain derivatives not higher than the first order the method of finite difference can be subordinated to the Rayleigh-Ritz method by considering in the competition only functions $\phi$ which are linear in the meshes of a sub-division of our net into triangles formed by diagonals of the squares of the net. For such polyhedral functions the integrals become sums expressed by the finite number of values of $\phi$ in the net-points and the minimum conditions become our difference equations. Such an interpretation suggests a *wide generalization* which provides great flexibility and seems to have considerable practical value. Instead of starting with a quadratic or rectangular net we may consider from the outset any polyhedral surfaces with edges over an *arbitrarily chosen* (preferably triangular) *net*. Our integrals again become finite sums, and the minimum condition will be equations for the values of $\phi$ in the net-points. While these equations seem less simple than the original difference equations, we gain the enormous advantage of better adaptability to the data of the problem and thus we can often obtain good results with very little numerical calculation. (See appendix.)

This procedure of finite differences was analyzed from a mathematical point of view—and in particular the convergence for the quadratic mesh width $h \to 0$ was proved—first by Philipps in unpublished papers, then by Philipps and Wiener [14], and at the same time by Lusternik and by Courant [5], and later by Courant, Friedrichs, and Lewy [8].

As was already observed by Philipps and Wiener, equations of finite differences can be interpreted by processes of probability—the so-called random walk. This connection has led Courant, R. Lueneburg [16], and Petrovsky [17] to a more general attack upon our problems, as follows: Instead of simply replacing our integral expressions by finite sums defined over a net, we may replace them in various ways by other simple functionals and thus obtain a greater variety of possible approximate problems $P_n$. Consider, for example, the non-negative function $K(x, y; \xi, \eta)$ which is assumed to be symmetric, to be defined and piecewise continuous in the entire plane and to satisfy $\iint_{-\infty}^{+\infty} K(x, y; \xi, \eta)\, d\xi d\eta = 1$. We then focus our attention on integrals of the following form

$$(16)\quad T(\omega) = \iiiint_{-\infty}^{+\infty} K(x, y; \xi, \eta)\left[\omega(x, y) - \omega(\xi, \eta)\right]^2 dx dy d\xi d\eta$$

taken over the entire plane, and investigate the problem $P_n$,

$$(16\text{a})\qquad T(\omega) = \text{min.},$$

for which the admissible functions are prescribed in the domain $\bar{B}$, complementary to $B$, that is, in the portion of the plane outside of $B$.

For example, if $K = f(r)$, where $r^2 = (x-\xi)^2 + (y-\eta)^2$ and $f(r) \geqq 0$, while $f(r) = 0$ for $r > h$, it is easily verified that, as the parameter $h$ tends to zero, we have $T(\omega) \to D(\omega)$ so that $P_n$ tends to the boundary value problem for $\Delta u = 0$. It can be shown generally that the nucleus $K$ may be chosen dependent on a parameter $h$ so that $T(\omega)$ tends to a given quadratic functional which yields as Euler equation any prescribed homogeneous elliptic differential equation.

Now the variational condition for the problem (16a) is no longer a differential equation but a Fredholm integral equation, which is more easily treated:

$$(17)\qquad u(x, y) = \iint_B K(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta + g(x, y),$$

where

$$g(x, y) = \iint_{\bar{B}} K(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta$$

is known because $u$ is prescribed in $\bar{B}$. The solution $u_n$ of (17) for $K=K_n$ converges to the solution $u$ of the problem $P$. Moreover, and this is the important point for practical applications, it can be proved that the integral equation (17) is solvable by the Neumann method of iteration under very wide conditions for the nucleus $K$. Thus we may develop a new method for attacking boundary value problems which, however, so far has not been tested in practical applications.

It may be remarked that, generally speaking, the iteration process will converge more slowly if the approximation of $P_n$ to $P$ becomes better. Hence again a compromise for practical purposes is indicated.

In passing, we mention the intimate *connection* of this method *with statistics*. Let us suppose a substance to be distributed with a density $u(x, y)$ over the plane. We now imagine that the substance is redistributed in distinct steps in such a manner that a unit mass concentrated at the point $P(x, y)$ will be spread over the plane with a density $K(x, y; \xi, \eta)$ at the point $Q(\xi, \eta)$. The integral equation (17) then characterizes a state of statistical equilibrium, if the density $u(x, y)$ is prescribed in the complementary domain $\bar{B}$ of $B$. Of course, if we interpret the integrals as Stieltjes' integrals, we may include in our formulation even problems of finite differences and random walk problems of the classical type. For us here, the main objective is to point out a method that enables us in principle to find approximately solutions of boundary value problems.

## IV. METHOD OF GRADIENTS

Still another alternative to the Rayleigh-Ritz procedure should be mentioned. This is the method of gradients, which goes back to a paper published by Hadamard in 1907 [12]. Highly suggestive as is Hadamard's attempt, difficulties of convergence were encountered. However, recent developments in the theory of conformal mapping and in Plateau's problem throw new light on Hadamard's idea, so that it seems justified now to expect from it not only purely mathematical existence proofs but also a basis for numerical treatment in suitable cases [13, 10, 11].

The principle of the method may be understood from the elementary geometric concept of a vector gradient. Let $u=f(x_1, \cdots, x_n)$ be a non-negative function of the $n$ variables $x_i$, or as we might say of the position vector $X=(x_1, \cdots, x_n)$, and let us seek to determine a vector $X=X_0$ for which $u$ is at least stationary. We then proceed as follows: on the surface $u=f(x)$ we move a point $(x_1, \cdots, x_n, u)$ so that $x_i(t)$ and $u(t)$ become functions of a time-parameter $t$. Then the velocity of ascent or descent along the line $X = X(t)$, $u = u(t)$ on the surface is

$$\frac{du}{dt} = \dot{u} = \sum_{i=1}^{n} \dot{x}_i f_{x_i} = \dot{X} \cdot \operatorname{grad} f.$$

We now choose the line along the line along which the motion proceeds so that the descent is as steep as possible (*lines of steepest descent*). This means to make $u$ negative and as large "as possible" in absolute value, for example, by choosing

$$(18) \qquad \dot{X} = - \operatorname{grad} f,$$

so that

$$\dot{u} = - (\operatorname{grad} f)^2.$$

Hence the position vector $X$ moves according to the system of ordinary differential equations (18) along the lines of steepest descent with respect to the function $f$. Under very general assumptions, it is clear that $X$, starting from an *arbitrary* initial position, will, for $t \to \infty$, approach a position for which $\operatorname{grad} f = 0$, and therefore for which $f$ is stationary and possibly a minimum. However, instead of using the continuous procedure given by the differential equation (18), we may proceed stepwise, correcting a set of approximations $x$ to the solutions of the equations $\operatorname{grad} f = 0$ by corrections proportional to the respective components of $-\operatorname{grad} f$.

This elementary idea can be generalized to variational problems. If we wish to determine a function $u(x, y)$ defined in $B$ and having prescribed boundary values such that $u$ is the solution of a variational problem

$$(19) \qquad I(v) = \iint_B F(x, y, v, v_x, v_y)\,dxdy = \text{min.},$$

then we interpret the desired function $u$ as the limit for $t \to \infty$ of a function $v(x, y, t)$, whose values may be chosen arbitrarily for $t = 0$ and for all $t$ thereafter are determined in such a way that the expression $I(v)$, considered as a function $I(t)$ of $t$, decreases as rapidly as possible toward its minimal value. Of course the boundary values of $v(x, y, t)$ are the same as those for $u(x, y)$, so that $v_t$ must vanish at the boundary. If we choose $v = v(x, y, t)$, we find

$$(20) \qquad \dot{I}(t) = - \iint_B v_t L(v)\,dxdy,$$

where $L(v)$ is the Euler expression corresponding to (20). To consider a concrete example, we suppose that

$$I(v) = \int\int_B (v_x^2 + v_y^2)dxdy, \tag{21}$$

so that our minimum problem amounts to determining the equilibrium of a membrane with given boundary deflections $g(\mathfrak{s})$. Then $L(v) = -2\Delta v$. Incidentally (20) displays an analogy between the Euler expression and the gradient of a function $f(x_1, \cdots, x_n)$ of $n$ independent variables. The variation or "velocity" of $I(v)$ is expressed as an "inner product" of the velocity of the "independent function" $v$ with the Euler expression $L(v)$, the *gradient of a functional in function space.*

We now assure ourselves of a steady descent or decrease of $I(t)$ by choosing $v_t$ in accordance with the differential equation

$$v_t = -kL(v), \tag{22}$$

where $k$ is a positive arbitrary function of $x, y$. (21) then becomes

$$I(t) = -\int\int_B k[L(v)]^2 dxdy,$$

and again we can infer that, for $t \to \infty$, $v(x, y, t)$ will tend to the solution $u(x, y)$ of the corresponding boundary value problem $L(u) = 0$.

For the case of the membrane the differential equation (22) becomes

$$v_t = \Delta v, \tag{23}$$

the equation of *heat transfer*. In our interpretation this equation describes a rapid approach to a stationary state along the "lines of steepest descent." While for the equations (23) or (22) the convergence of $v$ for $t \to \infty$ can be proved, serious difficulty arises if we want to replace our continuous process by a process of stepwise corrections as would be required for numerical applications. Each step means a correction proportional to $\Delta v$, thus introducing higher and higher derivatives of the initial function $v$. Another great difficulty is presented by rigid boundary values.[7]

Yet there do exist classes of problems where such difficulties can be overcome if the method is extended properly. First of all we may observe that it is not necessary to select the *steepest* descent along the gradient; it suffices to secure a *safe* descent at a suitably fast rate. Furthermore, if we consider problems for which the boundary value problem of the differential equation presents no difficulties for the domain $B$, but for which a degree of freedom in the boundary values is left, then the problem reduces to one for finding these boundary values, and now all our difficulties disappear. A typical example is the problem of the conformal mapping of a circle $B$ onto a simply-connected domain $G$, and it may be that the method of gradients opens a path for the attack of the problem of conformal mapping for multiply-connected domains and other problems as well.

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[7] Incidentally, if we apply this procedure to a problem for a finite net, it converges very well and is, as a matter of fact, nothing but a natural method of solving a system of linear equations by a method of iteration.

This address has emphasized more theoretical aspects. However, some of the general principles described may be helpful in enlarging our equipment for practical purposes.

## APPENDIX[8]

### NUMERICAL TREATMENT OF THE PLANE TORSION PROBLEM FOR MULTIPLY-CONNECTED DOMAINS

The computation of the stiffness $S$ defined in §I, 2a furnishes an example of independent interest which permits to compare the practical merits of some of the methods described in this address. Numerical calculations were carried out for the cross sections of the following diagrams, a square from which a smaller square is cut out; and a square, from which four squares are cut out. In the first case our quadratic frame was supposed to be bounded by the lines $x = \pm 1$, $y = \pm 1$ and $x = \pm 3/4$, $y = \pm 3/4$. To apply the Rayleigh-Ritz method for the domain as a whole would already be cumbersome because of the boundary conditions for admissible functions $\phi$. However, this difficulty disappears if we exploit the symmetry of the domain and the resulting symmetry of the solution; thus we may confine ourselves to considering only one-eighth of the domain $B^*$, namely the quadrangle $ABCD$. For this polygon any function of the type

$$\phi = a(1 - x)[1 + (x - 3/4)P]$$

where $P(x, y)$ is a polynomial, is admissible, and its substitution in the integral leads to simple linear equations for the cofficients. Thus for the simplest attempt

$$\phi = a(1 - x)$$

which leaves only one constant $a$ to be determined, we find with a negligible amount of numerical labor $S=.339$ and $c=-.11$. A refined attempt with the function

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[8] Addition not contained in the original address.

$$\phi = a(1-x)[1+\alpha(x-3/4)y]$$

yielded $S=.340$ and $c=-.109$ with little more labor.

These results were checked with those obtained by our generalized method of finite differences where arbitrary triangular nets are permitted. The diagrams are self-explanatory. Unknown are the net-point-values $u_i$, ($c=u_0$). In the net-triangles our functions were chosen as linear, so that the variational problem results in linear equations for the $u_i$. The results, easily obtainable, were: case (a) with two unknowns: $S=.344$, $u_0=-.11$; case (b) with three unknowns: $S=.352$, $u_0=-.11$; case (c) with five unknowns $S=.353$, $u_0=-.11$; case (d) with nine unknowns, corresponding to the ordinary difference method $S=.353$, $u_0=-.11$.

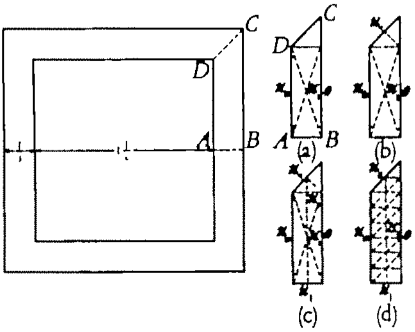

FIG. 2

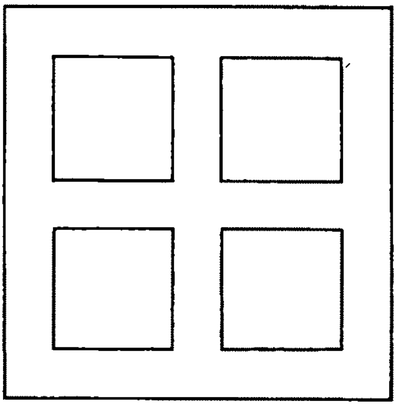

FIG. 3

p 35689

These results show in themselves and by comparison that the generalized method of triangular nets seems to have advantages. It was applied with similar success to the case of a square with four holes, and it is obviously adaptable to any type of domain, much more so than the Rayleigh-Ritz procedure in which the construction of admissible functions would usually offer decisive obstacles.

In a separate publication it will be shown how the method can be extended also to problems of plates and to other problems involving higher derivatives.

Of course, one must not expect good local results from a method using so few elements. However, it might be expected that a smooth interpolation of the net functions obtained will yield functions which themselves with their derivatives are fairly good approximations to the actual quantities.

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NEW YORK UNIVERSITY

# THE OCTOBER MEETING IN NEW YORK

The three hundred ninetieth meeting of the American Mathematical Society was held at the Hotel Pennsylvania on Friday and at Columbia University on Saturday, October 30–31, 1942. The attendance included the following one hundred fifty-eight members of the Society:

A. A. Albert, T. W. Anderson, R. G. Archibald, L. A. Aroian, S. F. Barber, J. L. Barnes, A. E. Basch, M. F. Becker, Stefan Bergman, P. G. Bergmann, E. E. Betz, Felix Bernstein, Garrett Birkhoff, G. D. Birkhoff, Gertrude Blanch, Salomon Bochner, H. F. Bohnenblust, S. G. Bourne, Joseph Bowden, H. W. Brinkmann, G. W. Brown, Hobart Bushey, J. H. Bushey, S. S. Cairns, M. E. Carlen, M. D. Clement, I. S. Cohen, Harvey Cohn, T. F. Cope, Richard Courant, J. E. Crawford, M. T. Curran, J. H. Curtiss, Jesse Douglas, Arnold Dresden, Nelson Dunford, J. E. Eaton, M. L. Elveback, Paul Erdös, A. D. Fialkow, M. M. Flood, Tomlinson Fort, M. C. Foster, R. M. Foster, K. O. Friedrichs, B. P. Gill, L. M. Graves, Bernard Greenspan, C. C. Grove, E. J. Gumbel, Margaret Gurney, Jacques Hadamard, F. C. Hall, L. A. Hazeltine, E. R. Hedrick, Edward Helly, Aaron Herschfeld, Max Herzberger, E. H. Hildebrandt, Einar Hille, Abraham Hillman, Banesh Hoffmann, T. R. Hollcroft, M. W. Hopkins, E. M. Hull, R. P. Isaacs, S. A. Joffe, Fritz John, Edward Kasner, J. R. Kline, B. O. Koopman, Mark Kormes, Arthur Korn, H. N. Laden, M. E. Ladue, J. A. Larrivee, Solomon Lefschetz, B. A. Lengyel, Marie Litzinger, E. R. Lorch, A. N. Lowan, R. K. Luneberg, A. W. McMillan, Brockway McMillan, L. A. MacColl, C. C. MacDuffee, Saunders MacLane, H. F. MacNeish, H. B. Mann, A. J. Maria, M. H. Maria, Imanuel Marx, Walther Mayer, A. E. Meder, H. L. Mintzer, E. C. Molina, Parry Moon, Marston Morse, G. W. Mullins, C. A. Nelson, C. O. Oakley, L. F. Ollmann, Oystein Ore, J. C. Oxtoby, E. L. Post, R. G. Putnam, Hans Rademacher, J. F. Randolph, L. L. Rauch, Mina Rees, Moses Richardson, J. F. Ritt, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Arthur Sard, R. D. Schafer, Henry Scheffé, M. G. Scherberg, I. E. Segal, Abraham Seidenberg, Stephan Serghiesco, Bernard Sherman, Seymour Sherman, Max Shiffman, James Singer, R. R. Singleton, P. A. Smith, Ernst Snapper, A. H. Sprague, E. R. Stabler, J. J. Stoker, W. C. Strodt, S. R. R. Struik, M. M. Sullivan, J. L. Synge, J. D. Tamarkin, W. R. Transue, C. A. Truesdell, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, D. F. Votaw, L. I. Wade, W. R. Wasow, Louis Weisner, M. E. Wells, H. S. White, A. L. Whiteman, P. M. Whitman, J. E. Wilkins, S. S. Wilks, John Williamson, H. A. Wood, Alexander Wundheiler, Leo Zippin, O. J. Zobel, Antoni Zygmund

Friday afternoon the Society joined with the Optical Society of America in a symposium on *Mathematics in the field of optics*. Mr. R. M. Foster presided at the session which included the following addresses: *The theory of the Schmidt telescope* by Professor J. L. Synge of the University of Toronto, *The mathematics of turbid media* by Dr. S. Q. Duntley of the Massachusetts Institute of Technology, *A matrix treatment of optical systems, with polarizing, birefringent, and optically active elements* by Dr. R. C. Jones of the Bell Telephone

Laboratories, and *New methods of calculating illumination* by Professor Parry Moon of the Massachusetts Institute of Technology.

The Society met in two sections on Saturday morning, one for presentation of papers in analysis and geometry with Professor Antoni Zygmund presiding, and another for algebra, statistics and applied mathematics with Professor H. W. Brinkmann presiding.

President Marston Morse presided at the general session Saturday afternoon at which Professor Salomon Bochner of Princeton University gave an address on *Continuation of analytic functions in several variables.*

Members of the Society joined with members of the Optical Society of America and the Society of Rheology at lunch at the Hotel Pennsylvania on Friday. Dr. F. B. Jewett, Vice President of the American Telephone and Telegraph Company, was the speaker at the luncheon. Members of the Society were invited to attend the annual dinner of the Optical Society of America which was held at the Hotel Pennsylvania Friday evening. One of the speakers was President Marston Morse who discussed the interest of mathematicians in applied mathematics.

The Council met on Saturday, October 31, at 12:00 M. in the Men's Faculty Club at Columbia University.

The Secretary announced the election of the following three persons to membership in the Society:

Mr. Robert Steiner Fouch, Army Air Forces Pre-Flight School, Maxwell Field, Ala.;
Mr. Paul W. Healy, Case School of Applied Science;
Mr. Alvin Spiro, Ann Arbor, Mich.

The following appointments by President Marston Morse were reported: as a Committee on Printing Contracts, Dean Tomlinson Fort (chairman), Dean M. H. Ingraham, Professor J. D. Tamarkin; as representative of the Society at the celebration of the Fiftieth Anniversary of the Founding of The Woman's College of the University of North Carolina on October 4–5, 1942, Professor J. M. Thomas; as representative of the Society at the inauguration of John Nelson Russell Score as President of Southwestern University (Georgetown, Texas) on October 6, 1942, Professor E. L. Dodd; as representative of the Society at the inauguration of Charles Albert Anderson as President of Coe College (Cedar Rapids, Iowa) on November 12, 1942, Professor E. W. Chittenden.

The Secretary reported that Professor Harry Bateman had accepted the invitation to deliver the Gibbs Lecture at the Annual Meeting of 1943; the title of the Lecture will be *The control of elastic* *fluids*. Professor Einar Hille has accepted the invitation to deliver the Colloquium Lectures at the Summer Meeting of 1944 on the following subject: *The analytical theory of semi-groups*.

It was announced that Professor Nathan Jacobson would give an invited address at the February, 1943, meeting in New York City, entitled *Some topics in the theory of semi-linear transformations*.

The Secretary reported that Professor F. D. Murnaghan had been named chairman of the temporary editorial committee for Mathematical Surveys and that Professor T. H. Hildebrandt had been appointed a member of the committee, to replace Professor E. J. McShane.

Professor E. J. Moulton and Dean W. G. Simon were appointed representatives on the Council of the American Association for the Advancement of Science for 1943.

Professors T. H. Hildebrandt, Hassler Whitney, and G. T. Whyburn were appointed representatives on the Editorial Board of the Annals of Mathematics for a period of three years beginning with 1943.

The Council appointed Professor A. D. Michal of California Institute of Technology as Associate Secretary in the far west, to fill the unexpired term of Dean T. M. Putnam.

At the Summer Meeting of 1942 a committee consisting of Professors O. E. Neugebauer and G. B. Price was appointed to investigate the possibilities of scientific communication with other countries. A report of this committee on the Russian situation was published in Volume 48, Number 11, of this Bulletin. The committee is continuing its investigations, with particular emphasis on communication with the Indian and Australian mathematicians.

Titles and cross references to the abstracts of papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1–2 were read in the section for analysis and geometry, those numbered 3–8 in the section for algebra, statistics and applied mathematics; those numbered 9–27 were read by title. Paper 2 was read by Professor Kasner, and paper 7 by Dr. Bers. Dr. Kogbetliantz was introduced by Professor J. A. Shohat, and Dr. Wong by Professor T. R. Hollcroft.

1. J. E. Wilkins: *The first canonical pencil*. (Abstract 49-1-82.)

2. Edward Kasner and John DeCicco: *A generalized theory of dynamical trajectories*. (Abstract 48-11-329.)

3. Ernst Snapper: *The resultant of a linear set*. (Abstract 48-11-309.)

4. R. P. Isaacs: *Application of polygenic functions to two-dimensional elasticity.* (Abstract 48-11-328.)

5. Garrett Birkhoff: *A reversibility paradox in hydrodynamics.* (Abstract 48-11-325.)

6. H. B. Mann: *The construction of orthogonal Latin squares.* (Abstract 48-11-336.)

7. Lipman Bers and Abe Gelbart: *On solutions of the differential equations of gas dynamics.* I. (Abstract 48-11-324.)

8. Max Herzberger: *Direct methods in geometrical optics.* (Abstract 49-1-64.)

9. R. H. Cameron and W. T. Martin: *Infinite linear difference equations with arbitrary real spans and first degree coefficients.* (Abstract 48-9-257-*t.*)

10. Nathaniel Coburn: *The linear yield condition in the plane plasticity problem.* (Abstract 48-11-326-*t.*)

11. John DeCicco: *New proofs of the theorems of Beltrami and Kasner on linear families* (Abstract 48-9-274-*t.*)

12. C. J. Everett: *Sequence completion of lattice moduls.* (Abstract 48-11-302-*t.*)

13. Abraham Hillman and H. E. Salzer: *Complex roots of* $\sin z = z$. (Abstract 48-11-314-*t.*)

14. Irving Kaplansky: *Systems of congruences in a valuation ring.* (Abstract 48-11-303-*t.*)

15. Irving Kaplansky: *The direct product of rings.* (Abstract 48-11-304-*t.*)

16. Edward Kasner: *Differential equations of the type:* $y''' = Gy'' + Hy''^2$. (Abstract 48-9-277-*t.*)

17. Edward Kasner: *Differential equations of the type:* $y^{\mathrm{iv}} = Ay'''^2 + By''' + C$. (Abstract 48-9-278-*t.*)

18. Edward Kasner: *The close packing of spheres.* (Abstract 49-1-72-*t.*)

19. Ervand Kogbetliantz: *Inequality for definite integrals.* Preliminary report. (Abstract 48-11-315-*t.*)

20. A. N. Lowan and H. E. Salzer: *Coefficients in formulae for numerical integration.* (Abstract 48-7-238-*t.*)

21. Szolem Mandelbrojt: *Quasi-analyticity and properties of flatness of entire functions.* (Abstract 48-11-318-*t.*)

22. J. F. Ritt: *Bézout's theorem and algebraic differential equations.* (Abstract 48-11-308-*t.*)

23. R. M. Robinson: *Bounded analytic functions.* (Abstract 48-11-320-*t.*)

24. J. A. Shohat: *An inequality for definite integrals.* Preliminary report. (Abstract 48-11-323-*t*.)

25. A. D. Wallace: *A characterization of unicoherence.* (Abstract 48-11-345-*t*.)

26. Y. C. Wong: *Some Einstein spaces with conformally separable fundamental tensors.* (Abstract 49-1-83-*t*.)

27. Jakob Levitski: *On the radical of a general ring.* (Abstract 49-1-7-*t*.)

T. R. HOLLCROFT,
*Associate Secretary*

## BOOK REVIEWS

*Sur les ensembles de distances des ensembles de points d'un espace Euclidien.* (Mémoires de l'Université de Neuchatel, vol. 13.) By Sophie Piccard. L'Université de Neuchatel, 1939. 212 pp.

Let $A$ denote a set of points on the real number-line $R$. The set of all number $|a-a'|$, for any $a$ and $a'$ in $A$, is the set of distances of the set $A$, and is denoted by $D(A)$.

A connection between $A$ and $D(A)$ arises naturally in the following way. Let $r$ be a real number, and let $x \rightarrow y = x + r$ be a translation of $R$ carrying $x$ into $y$. Let this translation carry $A$ into $A(r)$. A number $z$ will be in both $A$ and $A(r)$ if and only if $z$ is in $A$ and also at a directed distance $r$ from some point of $A$. Thus the intersection $A \cdot A(r)$ contains exactly one point for each distinct unordered pair $(a, a')$ of elements of $A$ such that $|a-a'| = |r|$. It is not surprising, then, to find many theorems based on the cardinal number of those $r$ for which the cardinal number of $A \cdot A(r)$ is restricted in some way.

This particular connection is of largely auxiliary interest, however, for the main purpose of the present memoir is to examine for their own sake the relations between properties of $A$ and $D(A)$. The results are so numerous and so detailed that we are obliged merely to cite a few typical items from each of the four chapters. Definitions and notation are as follows: $A$ is congruent to $B$ ($A \cong B$) when and only when a real number $k$ can be found such that one of the transformations $x \rightarrow y = \pm x + k$ of $R$ carries $A$ into $B$. The complement of $A$ in $R$ is denoted by $CA$. The complement of $D(A)$ in the set $0 \leqq x < \infty$ is denoted by $D'(A)$ (reviewer's notation); since 0 is in $D(A)$ naturally, any member of $D'(A)$ is positive. If $A$ and $B$ are two sets, $D(A, B)$ is the set of all numbers $|a-b|$ for any $a$ in $A$ and $b$ in $B$.

The most important material in the first chapter is a summary of general theorems, due mainly to Sierpinski, whose acknowledged influence appears clearly throughout the book. We learn, for instance, that, if $A$ is open, an $F_\sigma$, a denumerable $G_\delta$, Borel-measurable and with $A \cdot A(r)$ at most denumerable for each $r$, analytic, or of the second Baire category, then $D(A)$ is, respectively, a $G_\delta$, an $F_\sigma$, a $G_{\delta\sigma}$, Borel-measurable, analytic, or of the second Baire category. A short proof is given of Steinhaus' important theorem that, if $A$ has positive measure, then $D(A)$ contains an interval $0 \leqq x < k$ for some $k$. Finally, a partial answer is given to the perfectly natural question: Evidently $A \cong B$ implies $D(A) = D(B)$; when does $D(A) = D(B)$ imply $A \cong B$? If $A$ and $B$ are finite, it is necessary and sufficient to assume

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that, for each $r$, $A \cdot A(r)$ and $B \cdot B(r)$ each contains at most one element; if $A$ and $B$ are infinite, this assumption is insufficient.

In the second chapter, the set $\Delta(A) = D'(A) \cdot D'(CA)$ is of great importance. If $\Delta(A)$ is empty, one of its factors is empty. $\Delta(A)$ is not empty when and only when $A \cong CA$. The second chapter investigates $D(A)$ mainly on this hypothesis, which entails $D(A) = D(CA)$ and $\Delta(A) = D'(A) = D'(CA)$. Under these circumstances, $A(\pm d) = CA$ (and $(CA)(\mp d) = A$) if and only if $d$ is in $D'(A)$. Further, if $p > 0$ and $k_1, \cdots, k_p$ are any integers, $m_1, \cdots, m_p$ are any members of $D'(A)$, and $r = k_1m_1 + \cdots + k_pm_p$, then $|r|$ is in $D'(A)$ if and only if the integer $k_1 + \cdots + k_p$ is odd. $D'(A)$ is a proper subset of that one of $A$ and $CA$ which does not contain zero. If $D(A)$ contains an interval, $D'(A)$ contains no decreasing sequence (and hence is denumerable). If $\Delta(A)$ contains two incommensurable numbers, then $\Delta(A)$ contains a decreasing sequence, and is everywhere dense on the interval $0 \leqq x < \infty$, while $A \cong CA$ is everywhere dense on $R$. Many other results deal with measurability and the Baire categories.

The next chapter is devoted to perfect sets. If $A$ is closed and bounded, or dense-in-itself, or perfect and bounded, then $D(A)$ is, respectively, closed, dense-in-itself, or perfect. A perfect set $P$ is said to be of the first kind (Mirimanoff) if it arises from an interval $a \leqq x \leqq b$ by a process generalized in a specific way from the construction of the Cantor ternary set. If $P$ is of the first kind, $D(P)$ is the interval $0 \leqq x \leqq b - a$. If $P$ and $Q$ are of the first kind, $D(P, Q)$ has positive Lebesgue-measure mes $D(P, Q)$. There exist bounded perfect sets $P$ and $Q$ such that mes $D(P) = 0$ and mes $D(Q) = 0$, while mes $D(P, Q) > 0$; there exist bounded perfect sets $P$ and $Q$ such that mes $D(P) > 0$ and mes $D(Q) > 0$, while mes $D(P, Q) = 0$. If mes $D(A) = 0$, then mes $D(A, A(r)) = 0$ for any real $r$. The remainder of this chapter contains a plethora of specific information on $D(A)$ and $D(A, B)$, where $A$ and $B$ consist of those numbers expressible in special ways in systems of notation with assigned bases. Presumably as an example, the base 4 receives extraordinarily searching attention.

The final chapter studies conditions on a subset $D$ of the interval $0 \leqq x < \infty$ in order that there may exist an $A$ for which $D = D(A)$. If $D$ is finite, it is necessary and sufficient that some subset of $D$, taken in increasing order, shall form a sequence $S$ such that $D$ coincides with the sums of the gapless subsequences of $S$. (It may be necessary to repeat elements of $D$ when forming $S$. Example: $A = \{0, a, 2a\} = D(A)$ with $a > 0$, $S = 0, a, a$.) Results which increase rapidly in complication are obtained directly when $D$ has not more than six elements, or when $A\cdot A(r)$ or $D\cdot D(r)$ are suitably restricted. With conditions on $A$, the criterion for finite $D$ generalizes to infinite $D$. If $D(A)$ and (hence) $A$ are denumerably (nondenumerably) infinite, then $D(A)\cdot(D(A))(r)$ is necessarily denumerably (nondenumerably) infinite for denumerably (nondenumerably) many $r$'s. If $D'$ is of measure zero or of the first Baire category, then the continuum-hypothesis implies the existence of an $A$ for which $D=D(A)$. Finally, many properties of $D$ are enumerated which exclude the existence of the required $A$.

Aside from occasional use of transfinite induction and the continuum-hypothesis, the proofs may fairly be called elementary. On the other hand, they are often quite intricate, simple results requiring the examination of a myriad cases. Pushed with patient energy, this study has yielded a remarkable amount of detailed information, from which one may form a definite idea of the difficulties and rewards to be met in the directions here pursued. One also finds general results of great interest, which do not claim to be complete. Many problems are explicitly proposed, and many others at once suggest themselves. For this solid progress we are very substantially indebted to the author, who doubtless shares the hope that a still deeper analysis—possibly along somewhat different lines—may presently yield a more satisfying theory.

Remarks: 1. Considerable data are given on sets $A$ in higher dimensional spaces, but such results have been ignored here for the sake of brevity.

2. Though the argument is apparently not vitiated by its omission in the text (for reasons which will not escape a reader), the reviewer feels compelled to record here the fact that the property $P$ (p. 54) is not invariant under translation, is accordingly peculiar to the position of a set on the line, and may not be shared by a set congruent to a set which has it.

3. One should take care, on p. 39, line 19, to read "$D(A)+D(CA)$" instead of "$D(A)$" (see Property 2, p. 46).

4. This book regrettably upholds the secretive tradition under which the index is omitted.

F. A. FICKEN

*Tables of probability functions*, Vol. II. New York, Work Projects Administration, 1942. 21+344 pp. $2.00.

The first volume of *Probability functions* appeared in 1941; it was reviewed in Bull. Amer. Math. Soc. vol. 48 (1942) p. 201. The present volume tabulates the functions

$$I = \frac{1}{(2\pi)^{1/2}} e^{-x^2/2}, \qquad H = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

from 0.0000 to 1.0000 at intervals of 0.0001 to fifteen places of decimals, and from 1.000 to 8.285 at intervals of 0.001 to fifteen places. Tables II include the functions $I$ and $1-H$ from 6.00 to 10.00 at intervals of 0.01 to six places of decimals. The bibliographical data provided give valuable information concerning uses of the tables. The extensive and systematic tests for accuracy are as in the earlier volumes of the series.

VIRGIL SNYDER

*Tables of sine and cosine integrals for arguments from* 10 *to* 100. New York, Work Projects Administration, 1942. 185 pp. $2.00.

The present volume is a continuation of *Tables of sine, cosine and exponential integrals*, vols. I and II which appeared in 1940 and were reviewed in Bull. Amer. Math. Soc. vol. 47 (1941) pp. 677–678, except that exponential integrals are not included over the new interval. New features in the present volume are graphs of $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ and a bibliography of applications as well as of tables and of reference texts.

The Tables themselves are to ten places, at intervals of 0.01 from 10 to 100, arranged as in the earlier volumes. Then follow $n\pi/2$ to fifteen places, $n$ ranging by integers from 1 to 100, and $p(1-p)$ and $p(1-p^2)$; each to six places.

The Tables are reproduced by the photo-offset process as were the previous volumes. Such use has been made in the checks and controls described in the Introduction as to secure a very high degree of accuracy in the results.

VIRGIL SNYDER

# NOTES

The editors of the Bulletin wish to make grateful public acknowledgement of the services rendered by the following persons who have referred papers: R. P. Agnew, Leonidas Alaoglu, Emil Artin, Reinhold Baer, Walter Bartky, Harry Bateman, E. F. Beckenbach, E. T. Bell, A. A. Bennett, D. G. Bourgin, A. T. Brauer, Richard Brauer, H. E. Buchanan, J. A. Clarkson, Nathaniel Coburn, Max Coral, D. R. Curtiss, J. H. Curtiss, M. M. Day, John DeCicco, Samuel Eilenberg, G. C. Evans, L. R. Ford, W. B. Ford, T. C. Fry, M. R. Hestenes, Ralph Hull, D. H. Hyers, R. D. James, R. L. Jeffery, B. W. Jones, F. B. Jones, Solomon Lefschetz, D. H. Lehmer, Walter Leighton, Howard Levi, Harry Levy, E. R. Lorch, E. J. McShane, Saunders MacLane, W. T. Martin, G. M. Merriman, A. P. Morse, F. J. Murray, C. V. Newsom, Oystein Ore, G. H. Peebles, Sam Perlis, George Polya, Hillel Poritsky, G. B. Price, Hans Rademacher, Tibor Rado, G. Y. Rainich, Moses Richardson, J. F. Ritt, M. S. Robertson, R. M. Robinson, J. B. Rosser, I. J. Schoenberg, I. M. Sheffer, L. L. Smail, F. C. Smith, I. S. Sokolnikoff, N. E. Steenrod, M. H. Stone, Otto Szász, Gabor Szegö, Alfred Tarski, W. J. Trjitzinsky, J. V. Uspensky, R. J. Walker, H. S. Wall, Henry Wallman, J. L. Walsh, Louis Weisner, Marie J. Weiss, G. T. Whyburn, W. M. Whyburn, S. S. Wilks, Aurel Wintner, J. W. T. Youngs and Antoni Zygmund.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers offered for publication in volumes 51 and 52: Emil Artin, E. F. Beckenbach, Garrett Birkhoff, G. D. Birkhoff, L. M. Blumenthal, H. E. Buchanan, J. A. Clarkson, A. B. Coble, C. J. Everett, K. O. Friedrichs, L. M. Graves, Marshall Hall, G. A. Hedlund, M. S. Knebelman, N. H. McCoy, H. M. McNeille, C. B. Morrey, A. P. Morse, F. J. Murray, Gordon Pall, George Polya, G. B. Price, Tibor Rado, G. Y. Rainich, H. W. Raudenbush, P. V. Reichelderfer, A. C. Schaeffer, Otto Szász, J. D. Tamarkin, Morgan Ward, D. V. Widder, and L. R. Wilcox.

The Kosciuszko Foundation announces that for May 23, 1943, it is planning a scientific tribute in commemoration of the four hundredth anniversary of the death of Nicolas Copernicus, the great Polish astronomer. This is also the four hundredth anniversary of the publication of his work, *De revolutionibus orbium coelestium.*

M. Krylov will head a special committee of the Soviet Academy of Sciences to arrange for celebration of the tercentenary of Newton's birth.

Applications for the Benjamin Peirce instructorships at Harvard University for the academic year 1942–1943 should be sent to the chairman of the department of mathematics. Candidates should have received the Ph.D. degree or have had equivalent training.

Haverford College has conferred the doctorate of science upon Professor H. V. Gummere, who retired from its staff in June, 1942.

Assistant Professor E. F. Beckenbach of the University of Michigan has been appointed to an associate professorship at the University of Texas.

Professor R. E. Bruce of Boston University has been given the title professor emeritus.

Assistant Professor P. N. Carpenter of Grove City College has been promoted to an associate professorship.

Professor Helen E. Clarkson of Jacksonville College, Jacksonville, Texas, has been appointed to an assistant professorship of physics at Hardin-Simmons University.

Professor L. W. Cohen of the University of Kentucky has been appointed visiting lecturer at the University of Wisconsin for this academic year.

Mr. H. W. Eves of the mathematics staff of the Tennessee Valley Authority has been appointed to an assistant professorship of applied mathematics at Syracuse University.

Dr. Mary C. Graustein of Radcliffe College has been appointed to an assistant professorship at Oberlin College.

Professor Emeritus J. G. Hardy of Williams College will be visiting professor of mathematics at Reed College during the current academic year.

Dr. H. A. Jordan of the Catholic University of America has been appointed to an assistant professorship at the College of William and Mary.

Assistant Professor F. H. Miller of the Cooper Union School of Engineering has been promoted to an associate professorship.

Dr. J. W. Odle has been appointed to an assistant professorship at Pennsylvania State College.

Dr. E. W. Paxson of Wayne University has been promoted to an assistant professorship.

Dr. Grace S. Quinn has been appointed associate in mathematics at George Washington University.

Assistant Professor W. J. Robinson of Washington College, Chestertown, Maryland, has been appointed to an assistant professorship at Marshall College, Huntington, West Virginia.

Professor A. J. Smith of Susquehanna College has been appointed to an assistant professorship at the College of William and Mary.

Assistant Professor F. H. Steen of Georgia School of Technology has been appointed to an assistant professorship at Allegheny College.

Assistant Professor N. E. Steenrod of the University of Chicago has been appointed to an assistant professorship at the University of Michigan.

Professor H. P. Thielman of the College of St. Thomas, St. Paul, Minnesota, has been appointed to an assistant professorship at Iowa State College.

Dr. G. R. Thurman of the University of Missouri has been appointed to an assistant professorship at Tusculum College, Greenville, Tennessee.

Dr. W. R. Van Voorhis of Fenn College has been promoted to an assistant professorship.

Dr. P. A. White has been appointed to an assistant professorship at Louisiana State University.

Associate Professor J. H. Zant of Oklahoma Agricultural and Mechanical College has been promoted to a professorship.

The following appointments to instructorships have been announced: Cornell University: Mr. R. L. Beinert, Mr. C. D. Firestone, Mr. R. L. Hull, Mr. A. R. Turquette; Grays Harbor Junior College, Aberdeen, Washington: Miss Ruth E. Porter; Illinois Institute of Technology: Dr. W. B. Caton; Louisiana State University: Mr. J. C. Stewart; University of Maryland: Mr. H. F. Gingerich, Dr. Margaret S. Matchett; Massachusetts Institute of Technology: Dr. W. F. Whitmore; Michigan State College: Dr. L. V. Toralballa, Miss Elaine

Van Aken, Mr. K. C. Walters; New York University: Dr. Wolfgang Wasow; Queens College: Dr. Deborah M. Maria; University of Wyoming: Dr. Hans Samelson.

A recent message received through the American Red Cross announces the death in Germany on July 5, 1942, of Professor Oskar Bolza at the age of 85 years. He had been a member of the Society since 1891.

Nature reports the death of the English mathematician D. B. Mair.

Professor A. D. Campbell of Syracuse University died September 23, 1942. He had been a member of the Society since 1920.

Professor C. N. Haskins of Dartmouth College died November 14, 1942, at the age of 68 years. He had been a member of the Society since 1913.

Professor Emeritus D. A. Lehman of Goshen College died September 8, 1942, at the age of 82 years.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

## ALGEBRA AND THEORY OF NUMBERS

### 1. I. S. Cohen: *Some theorems on local rings.*

Let $\mathfrak{R}$ be a $p$-series ring (as defined by Krull, J. Reine Angew. Math. vol. 179 (1938) p. 205); let $n$ be its dimension. It is shown that if $\mathfrak{a}$ is an ideal in $\mathfrak{R}$ having a basis of $r$ elements and of dimension at most $n-r$, than $\mathfrak{a}$ is unmixed, of dimension $n-r$. This is a generalization of a well known theorem of Macaulay. If $\mathfrak{R}$ is complete and if $\mathfrak{S}$ is a $p$-series ring integrally dependent on $\mathfrak{R}$, then the rank of $\mathfrak{S}$ over $\mathfrak{R}$ is equal to the ramification order of $\mathfrak{S}$ with respect to $\mathfrak{R}$ (defined as the length of the primary ideal obtained by extending to $\mathfrak{S}$ the maximal ideal of $\mathfrak{R}$) multiplied by the degree of the residue field of $\mathfrak{S}$ over that of $\mathfrak{R}$. These results depend on the following theorem concerning the structure of a complete $p$-series ring: If the characteristics of $\mathfrak{R}$ and its residue field are the same, then $\mathfrak{R}$ is a power series ring over this field; if the characteristics are different, then under a simple additional hypothesis, $\mathfrak{R}$ is a power series ring over a complete discrete valuation ring. It is also shown that every complete local ring is a homomorphic map of a ring of one of these two types. (Received November 24, 1942.)

### 2. Franklin Haimo: *Periodic functions on algebraic systems.*

A single-valued function $F$ over a group $G$ is said to have a period $p$ if $F(xpy) = F(xy)$ for every $x$ and $y$ in $G$. The periods of $F$ form a normal subgroup of $G$. All single-valued functions over $G$ with range in a class $C$ are at least trivially periodic and are partitioned into classes each containing one and only one distinct homomorphism of $G$. Single-valued functions over a quasi-field $H$ may have periods which are both additive and multiplicative. Such periods form a normal subgroup of the multiplicative group of $H$. Single-valued functions over lattices may have join-periods and meet-periods. Meet-periods form a join-ideal and dually; while non-constant single-valued functions over Boolean algebras have no members which are both join- and meet-periods. (Received November 23, 1942.)

### 3. P. R. Halmos: *On automorphisms of compact groups.* I.

If $T$ is a continuous automorphism of a compact abelian group $G$ then, because of the uniqueness of Haar measure, $T$ is a measure preserving transformation. $T$ is ergodic (in fact strongly mixing) if and only if the adjoint automorphism $T^*$ of the character group $G^*$ has no finite orbits. If an automorphism (such as $T^*$) of a discrete abelian group has no finite orbits then it has an infinite number of orbits. It follows easily that the spectral type of the unitary operator induced by an ergodic $T$ on $L_2(G)$ depends only on the cardinal number of $G^*$. In case $G$ is a finite-dimensional toral group (so that $T$ is defined by a unimodular matrix) a simple condition on the coefficients of $T$ is equivalent to ergodicity. This last remark puts in evidence many new and simple examples of analytic mixing transformations. A purely measure theoretic invariant of measure preserving transformations is proposed (along the lines of the characteristic equation theory of matrices); it is suggested that this invariant will serve to distinguish between transformations lumped together by the cruder spectral invariants. (Received November 17, 1942.)

### 4. Gerald Harrison: *The structure of algebraic moduls.*

A 1-1 correspondence between the submoduls of an algebraic modul of degree $n$ and the set of all $n\times n$ matrices whose elements are rational integers may be set up if equivalence of matrices is defined in the usual manner. Nonsingular matrices whose elements are rational integers have multiplicative decompositions as the product of prime matrices, that is, matrices whose norms are primes. The number of such decompositions which a matrix has may be described in terms of the modular lattice which the set of all nonsingular matrices forms. This lattice has a direct product decomposition which is at the same time multiplicative, that is, if $a=(a_1, a_2, \cdots)$ and $b=(b_1, b_2, \cdots)$ then $a\cup b=(a_1\cup b_1, a_2\cup b_2, \cdots)$, $a\cap b=(a_1\cap b_1, a_2\cap b_2, \cdots)$, and $ab=(a_1b_1, a_2b_2, \cdots)$, where the multiplication referred to is that of algebraic moduls. (Received November 23, 1942.)

### 5. Olaf Helmer: *An extension of the elementary divisor theorem.*

A Prüfer ring is a domain of integrity in which all ideals possessing a finite basis are principal ideals. The elementary divisor theorem is known to hold in principal ideal rings as well as in rings with Euclidean algorithm; whether it will ever be possible to extend the theorem to Prufer rings is questionable. In this paper it will be proved for a subclass of all Prufer rings, to be called adequate rings, which more than comprise the class of rings hitherto known to be covered by the theorem. An adequate ring is defined as a Prüfer ring in which every nonzero element $a$ can, for any element $b$, be split into two factors: $a=a_1\cdot a_2$, such that $a_1$ is a largest factor of $a$ prime to $b$: $(a_1, b)=1$, largest in the sense that no factor $a_3$ of $a_2$ is prime to $b$: $(a_3, b)\neq 1$. The proof proceeds along familiar lines, after the following lemma has been proved: Let $M=(a_{ij})$, $(i=1, \cdots, r; j=1, \cdots, n)$ be a matrix with coefficients in an adequate ring $R$, let $1<r\leqq n$, rank $M=r$, and $(a_{11}, a_{12}, \cdots, a_{rn})=d$; then there exist $t_1, t_2, \cdots, t_{r-1}$ in $R$ such that $(A_1, A_2, \cdots, A_n)=d$, where $A_j=a_{1j}t_1+\cdots+a_{r-1,j}t_{r-1}+a_{rj}$. (Received November 23, 1942.)

### 6. Jakob Levitzki: *On semi-nilpotent ideals.*

A ring $T$ is called semi-nilpotent (in short: s.n.) if each subring of $T$ which is generated by a finite subset of $T$ is nilpotent; otherwise $T$ is called semi-regular. Thus the sum $\overline{N}$ of all nilpotent right (left) ideals of any ring $S$ is s.n. In a previous note the author has proved that each ring $S$ possesses a uniquely determined maximal two sided s.n. ideal $N$ which contains also all one-sided s.n. ideals of the ring. In the present note it is shown that if $S$ is a ring with minimal condition for two-sided s.n. ideals, then $N$ is nilpotent, and hence $N=\overline{N}$. Applied to semi-primary rings, this theorem yields certain generalizations of recent results due to C. Hopkins (Duke

Math. J. vol. 4 (1938) pp. 664–667), J. Levitzki (Compositio Math. vol. 7 (1939) pp. 214–222) and K. Asano (Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 288–291). (Received November 23, 1942.)

### 7. Jakob Levitski: *On the radical of a general ring.*

In the present note a definition of the radical is suggested which retains its significance also in the general case. This definition is based on the notion of s.n. (semi-nilpotent) ideal (its counterpart is the semi-regular ideal) which is defined as follows: $R$ is a s.n. ideal if each finite set of elements in $R$ generates a nilpotent ring. Each nilpotent ideal is s.n., and each s.n. ideal is a nil-ideal. The sum $N$ of all right s.n. ideals is a s.n. two-sided ideal, which is called the radical. The radical $N$ of any ring $S$ contains all one-sided (and two-sided) s.n. ideals, and the radical of $S/N$ is zero. These theorems lose their validity for general rings if the radical is defined (as has been hitherto the case) by replacing s.n. ideals either by nilpotent ideals (specialized radical) or nil-ideals (generalized radical). The radical contains the specialized radical, and is a subset of the generalized radical (in case the latter exists). The results described above are applied to primary rings. (Received October 5, 1942.)

### 8. Saunders MacLane and B. A. Lengyel: *Integral invariants of a tensor under the symmetry operations of the* 32 *crystal classes.* Preliminary report.

The theory of elastic and plastic deformations of an isotropic body makes use of the invariants of the stress tensor under the group of all orthogonal transformations. To extend this theory to crystalline media it is necessary to know the invariants of the finite groups of transformations characteristic of the crystal classes. The effect of the various symmetry operations on the components of a tensor was studied and an integrity basis was set up for the invariant polynomials for each of the 32 classes. The method of finding the integrity bases is elementary and consists of reducing the problem to that of the vector invariants of the symmetric and alternating groups, respectively. (Received November 24, 1942.)

### 9. C. J. Nesbitt and W. M. Scott: *Some remarks on algebras over an algebraically closed field.*

In this paper are examined some aspects of an algebra $A$ which may have a radical and whose coefficient field is algebraically closed. A main concept is that of basic algebra. The basic algebra of $A$ is a semi-primary subalgebra which for the algebra $A$ plays a role in some respects analogous to that of division algebras in the theory of simple algebras. Related to the basic algebra are the Cartan basis systems and systems of elementary modules (W. M. Scott, Ann. of Math. (2) vol. 43 (1942) pp. 147–160). An algebra $B$ is said to be similar to $A$ if the basic algebra of $B$ is isomorphic to that of $A$. Similar algebras form a class, and have corresponding representations. The commutator-algebras of matrix representations of $A$ are analyzed. The linear symmetric functions of $A$ (abstract 44-11-452) are determined. The regular representations of $A$ are written in terms of the elementary modules. (Received October 26, 1942.)

### 10. Ivan Niven: *An unsolved case of the Waring problem.*

The Dickson-Pillai-Vinogradow solution of Waring's problem (L. E. Dickson, Amer. J. Math. vol. 58 (1936) p. 535) leaves unsolved the case in which $r$, the remainder upon division of $3^n$ by $2^n$, equals $2^n-q-2$, $q$ being the quotient. The present paper shows that in this case $g(n)$, the minimum number of positive or zero $n$th powers necessary to express every positive integer, has the "ideal" value $2^n+q-2$. The method used is Dickson's, sharpened somewhat. (Received October 14, 1942.)

### 11. Ivan Niven: *The Pell equation in quadratic fields.*

In a previous report (*Quadratic Diophantine equations in the rational and quadratic fields*, Trans. Amer. Math. Soc. vol. 52 (1942) p. 2, Theorem 4) the author gave necessary and sufficient conditions that the equation $\xi^2-\gamma\eta^2=1$ have an infinite number of integral solutions $\xi$, $\eta$ in any quadratic field, $\gamma$ being a given integer of the field. It is shown here that if the equation has an infinitude of solutions, they can be obtained from one least solution (as in the case of the Pell equation in the rational field) if and only if $\gamma$ is not a totally positive non-square integer of a real quadratic field. In case $\gamma$ has this property it is shown that the equation has infinitely many solutions with $\xi$ and $\eta$ bounded. (Received October 16, 1942.)

### 12. Rufus Oldenburger: *The characteristic of a sum of quadratic forms.*

It is proved that the Loewy characteristic of a sum of quadratic forms is determined by properties of adjoint forms, and nonzero components in characteristic splittings of quadratic forms. (Received November 21, 1942.)

### 13. Edward Rosenthall: *Diophantine equations in arbitrary algebraic number fields.*

The complete solution in integers $XY=ZW$ in any algebraic integral domain is given by $eX=US$, $eY=VT$, $eZ=UT$, $eW=VS$ where $e$ takes only the finite set of rational integral values, each equal to the norm of a representative ideal from each class, and $U$, $S$, $V$, $T$, are arbitrary integers of the field. Complete integer solutions of other multiplicative equations are deduced and in particular from one of these equations all sets of rational integers satisfying $ax^2+by^2=z^3$ are obtained in terms of arbitrary integral parameters subject to a congruential condition modulo $e$. (Received November 20, 1942.)

### 14. Edward Rosenthall: *Diophantine equations in biquadratic fields.*

Explicit complete rational integer solutions are obtained for some diophantine equations reducible in certain so-called special Dirichlet biquadratic fields. The equation $N(X)=N(Y)$ is solved completely in rational integers, where $N(A)$ denotes the normal of the biquadratic integer $A$; in particular this equation in $Ra[2^{1/2}+i]$ yields the complete rational integer solution of $x^4+y^4=N(s+it+2^{1/2}u+i2^{1/2}v)$ with at most one linear relationship connecting the coordinates $s$, $t$, $u$, $v$. A multiplicative equation in $Ra[3^{1/2}+i]$ gives the complete rational integer solution of $x^3+y^3=u^2+v^2$. The equation $x^3+y^3=u^3+v^3$ is also solved completely in Gaussian integers. (Received November 20, 1942.)

### 15. T. L. Wade: *Euclidean concomitants of the triangle.*

The results of the writer (Bull. Amer. Math. Soc. vol. 47 (1941) pp. 475–478 and vol. 48 (1942) pp. 589–601) which relate tensor algebra and invariant theory are used in the consideration of the elementary problem of the three-line configuration. (Received November 23, 1942.)

### 16. T. L. Wade: *On conjugate tensors.*

How a contravariant (skew-symmetric) tensor $V$ of order $n-p$ may be associated with a covariant skew-symmetric tensor $U$ of order $p$, in an $n$-dimensional coordinate system, is well known (see Veblen and von Neumann, *Geometry of complex domains*). This standard association holds only when $U$ is skew-symmetric. The purpose of this note is to show how a contravariant tensor $V$, of defined order and symmetry, can be associated with the covariant tensor $U$, where $U$ is of any type $[\alpha]$ of symmetry. (Received November 23, 1942.)

### 17. T. L. Wade: *On the factorization of rank tensors.*

Let $C^{(\omega)}_{(\omega)} = D^{(\omega)}_{(\omega)} + E^{(\omega)}_{(\omega)}$, where $D^{(\omega)}_{(\omega)}$ and $E^{(\omega)}_{(\omega)}$ are mutually orthogonal idempotent numerical tensors. Expressions for the contravariant and covariant factors of the rank tensor (see Amer. J. Math. vol. 64 (1942) pp. 725–752) of $C^{(\omega)}_{(\omega)}$ in terms of like factors of the rank tensors of $D^{(\omega)}_{(\omega)}$ and $E^{(\omega)}_{(\omega)}$ are established in this paper. (Received November 23, 1942.)

### 18. T. L. Wade and R. H. Bruck: *Types of symmetries.*

This paper considers some aspects of symmetries with tensorial significance which are believed not to have appeared in the literature. (Received November 23, 1942.)

### 19. André Weil: *Differentiation in algebraic number-fields.*

Analogies with function-fields have long ago led E. Noether and others to the conjecture that the theory of the different in number-fields can be built upon some arithmetical analogue of differentiation. This is now done, by defining a derivation modulo an ideal $\mathfrak{a}$ in a number-field as an operator $D$ with the following properties: (a) $D$ maps the ring $\mathfrak{o}$ of integers in the field into the ring $\mathfrak{o}/\mathfrak{a}$; (b) $D(\alpha+\beta) = D\alpha + D\beta$; (c) if $\bar{\alpha}$, $\bar{\beta}$ are the classes of $\alpha$, $\beta$ mod $\mathfrak{a}$, then $D(\alpha\beta) = \bar{\alpha} \cdot D\beta + \bar{\beta} \cdot D\alpha$; $D$ is essential if there is $\alpha$ in $\mathfrak{o}$, such that $D\alpha$ is not a zero-divisor in $\mathfrak{o}/\mathfrak{a}$. The different is then the least common multiple of all ideals modulo which there exists an essential derivation. This is easily extended to the relative different, to $p$-adic fields, and so on. (Received November 9, 1942.)

### 20. Alexander Wundheiler: *An algebraic definition of affine space.*

A simple set of axioms for affine geometry based on one operation $C = kAB$, where $k$ is a real number, $A$, $B$ points and $C$ the point collinear with $A$ and $B$, and such that $CA/CB = k$, is given. There are essentially five axioms, only one of them involving more than two (namely, three) points. An "affine calculus," which permits the writing of every affine theorem as an implication between formulas, arises from the mentioned operation. (Received November 20, 1942.)

## ANALYSIS

### 21. R. P. Agnew: *Euler transformations.*

Let $E(r)$ denote the Euler transformation $\sigma_n = \sum_{k=0}^{n} C_{nk} r^k (1-r)^{n-k} s_k$ by means of which a sequence $s_n$ is summable to $\sigma$ if $\sigma_n \to \sigma$ as $n \to \infty$. The fundamental properties of the transformations $E(r)$ are developed for the general case in which $r$ is complex. The family of transformations $E(r)$ for which $r \neq 0$ is a consistent family even though $E(r)$ is regular only when $r$ is real and $0 < r \leqq 1$. Relations between the methods $E(r)$ and other methods of summability are established. The members of a family of series-to-series transformations are shown to be equivalent to the transformations $E(r)$ when the parameters are suitably restricted. The open sets in which power series are summable are characterized. If $r_1, r_2, \cdots$ is a sequence of real numbers for which $r_n \to 0$ and $nr_n \to \infty$, then the transformation $\sigma_n = \sum_{k=0}^{n} C_{n,k} r_n^k (1-r_n)^{n-k} s_k$ includes $E(r)$ for each $r > 0$. (Received October 13, 1942.)

### 22. R. P. Agnew: *On sequences with vanishing even or odd differences.*

Let $x_0, x_1, x_2, \cdots$ be a sequence of real or complex numbers, and let $d_n = \sum_{k=0}^{n} (-1)^k C_{n,k} x_k$, $n = 0, 1, 2, \cdots$, denote its sequence of differences. By use of a theorem on Euler methods of summability, a short proof of the two following theorems is obtained. If $x_n$ if bounded and $d_n = 0$ when $n$ is even, then $x_n = 0$ for each $n$. If $x_n$ is bounded and $d_n = 0$ when $n$ is odd, then $x_n = x_0$ for each $n$. (Received October 13, 1942.)

### 23. I. A. Barnett: *A note on skew-symmetric kernels.*

It is proved in this note that a necessary and sufficient condition that $\Gamma(x, y; \lambda)$ be the resolvent kernel of a skew-symmetric kernel is that $\Gamma(x, y; \lambda) + \Gamma(y, x; \lambda) + 2\lambda \int_a^b \Gamma(x, t; \lambda)\, \Gamma(y, t; \lambda)\, dt$ be identically zero in $x$, $y$ and $\lambda$. This is the analogue of the fact that the adjoint $C(\lambda)$ of $A - \lambda I$ satisfies the equation $\det(A - \lambda I)[C(\lambda) + C'(\lambda)] + 2\lambda C(\lambda) C'(\lambda) = 0$, where $A$ is a skew-symmetric matrix, and $C'(\lambda)$ denotes the transpose of $C(\lambda)$ (cf. Amer. Math. Monthly vol. 49 (1942) pp. 169–170). As a consequence of the first relation, it follows that a solution of the quadratic integral equation $\phi(x, x) = \lambda \int_a^b \phi^2(x, t)\, dt$ is $\phi(x, y) = -\Gamma(x, y; \lambda)$ where $\Gamma$ is the resolvent of an arbitrary skew-symmetric kernel. (Received November 20, 1942.)

### 24. E. F. Beckenbach: *The stronger form of Cauchy's integral theorem.*

According to the stronger form of Cauchy's integral theorem, if $f(z)$ is holomorphic on the interior $D$ of a simply closed rectifiable curve $C$, and continuous on $D + C$, then $\int_C f(z)\, dz = 0$. A brief and simple though not elementary proof is given, based on considerations of conformal mapping, the length of level curves of the Green's function, and a simple property of a Stieltjes integral. (Received November 23, 1942.)

### 25. Henry Blumberg: *On arbitrary point transformations.*

The transformations $\tau$ considered are correspondences $\tau: y = f(x)$ which associate with each point $x: (x_1, x_2, \cdots, x_m)$ of euclidean $m$-space $S_m$ a point $y: (y_1, y_2, \cdots, y_n)$ of euclidean $n$-space $S_n$. No other conditions are imposed on $\tau$ except "one-valuedness," which, too, is not an essential one. A series of theorems are proved, of descriptive and matric character, for the structure of $\tau$. The principal results are of the latter sort, and relate to the notion of "salient point," defined as a point of non-infinitesimal approach—in a certain readily suggested sense—of the "graph" of $\tau$ in

$S_{m+n}$, a euclidean space of $m+n$ dimensions. Theorems are obtained, among others on the distribution, closure properties, and measurability—in a certain sense—of the set of salient points. Finally, necessary and sufficient conditions are established characterizing such a set. (Received November 19, 1942.)

### 26. R. F. Clippinger: *General remarks about the set of products of positive powers of n-by-n matrices and the associated manifold.* Preliminary report.

Let $P$ be the set of products of arbitrary positive powers of the exponentials of $m$ given $n$-by-$n$ matrices, $A_i$, of real numbers. Let $M$ be the corresponding manifold of points of $n^2$ Euclidean space. Let $\bar{P}$ and $\bar{M}$ be the closure of $P$ and the corresponding manifold. $P$ is a groupoid. $M$ is connected and $r$-dimensional if $r$ is the number of linearly independent matrices $B_j$ among $A_i$ and their alternants. If $C_l$ is an arbitrary linear combination of the $B_j$, there exists a number $l$ such that any matrix of $P$ is a product of at most $l$ matrices exp $C_k$. $\bar{P}$ contains all matrices of the form exp $A$ if $A$ is any linear combination with positive coefficients of $A_i$. $\bar{P}$ is identical with the closure of the set of particular values of solutions of the differential equation $dY/dt=YB$, $Y(a)=1$ where $B$ is a linear combination of $A_i$ with coefficients which are arbitrary non-negative functions of class $C^\infty$. (Received November 21, 1942.)

### 27. R. F. Clippinger: *Matrix products of matrix powers.* Preliminary report.

If $A$ and $B$ are 2-by-2 matrices with constant elements $a, b, c, d, e, f, g, h$ and distinct characteristic roots such that $A$, $B$, $(AB)$ and $(A(AB))$ are linearly independent and $[(bg+cf-ah-de)(e+h)+2(eh-fg)(a+d)][(bg+fc-ah-de)(a+d)+2(ad-bc)(e+h)]$ is negative, and if $\alpha, \beta, \gamma, \delta, \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots$ are arbitrary non-negative numbers, then the closure of the set of matrices $\exp \alpha_1 A \exp \beta_1 B \exp \alpha_2 A \cdot \exp \beta_2 B \cdots$ is identical with the closure of the set of matrices $\exp \alpha A \exp \beta B \exp \gamma A \cdot \exp \delta B$. (Received November 21, 1942.)

### 28. R. F. Clippinger: *Mean value theorems for a certain linear matrix differential equation.* Preliminary report.

Let $A$ and $B$ be two, $n$-by-$n$ matrices, with distinct characteristic roots, which verify the condition $AB-BA=\lambda A+\mu B$, $\mu\neq 0$. If $\rho(t)$ and $\sigma(t)$ are arbitrary, non-negative, bounded measurable functions of $t$ on the interval $a\leqq t\leqq b$, and if $Y(b)$ is the particular value for $t=b$ of the solution of the matrix differential equation $dY(t)/dt=Y(t)[\rho(t)A+\sigma(t)B]$, $Y(a)=1$, then there exist non-negative numbers $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $\zeta$ such that, without changing $Y(b)$, $\rho(t)$ and $\mu(t)$ may be replaced by: (a) $\alpha$ and $\beta$ on $a\leqq t\leqq b$; or, (unless $\mu>0$ and $\lambda>0$), (b) $\gamma$ and 0 on $a\leqq t\leqq (a+b)/2$; 0 and $\delta$ on $(a+b)/2\leqq t\leqq b$; or, (unless $\mu<0$ and $\lambda<0$), (c) 0 and $\delta$ on $a\leqq t\leqq (a+b)/2$; $\gamma$ and 0 on $(a+b)/2\leqq t\leqq b$. (Received November 21, 1942.)

### 29. Max Coral: *Solution of quasi-linear partial differential equations through a characteristic initial curve.*

A new method is presented of passing a solution of a quasi-linear first order partial differential equation in two independent variables through an initial curve $C$ which is characteristic for the equation. It is shown that $C$ yields a solution $S$ of the equations of variation of the characteristic differential equations. If $T$ is any solution of these equations of variation which is independent of $S$, then a one-parameter family of characteristic curves imbedding $C$ can be found, whose variation along $C$ is $T$. These characteristic curves envelop the desired solution. The present method has the advantage that the solution thus found contains the entire extent of the initial curve. (Received October 30, 1942.)

### 30. G. M. Ewing: *Minimising an integral on a class of continuous curves.*

Hahn published an example in 1925 of a positive semidefinite positive quasi-regular variation problem which admits no rectifiable minimizing curve. Menger has obtained several existence theorems for problems in very general spaces which apply to the example of Hahn. In the present paper attention is confined to variation problems in euclidean spaces. The integral $J$ is defined for a continuous curve $C$ as g.l.b. lim inf $J(C_n)$, the greatest lower bound being on the class of sequences $C_n$ of rectifiable continuous curves converging to $C$. Using $\delta$-length based on any distance $\delta(p, q)$ satisfying the axioms of a metric space, a set of admissible curves of bounded $\delta$-length is compact. The method involves finding a suitable metric $\delta$ and determining hypotheses under which the problem admits a minimizing sequence of rectifiable curves of bounded $\delta$-length. Existence theorems based on several choices of $\delta$ and related lower-semicontinuity theorems are discussed. (Received October 14, 1942.)

### 31. N. A. Hall: *Confluence of the basic hypergeometric series.*

The general properties of the basic hypergeometric series are reviewed with special attention being given to the introduction of a satisfactory comprehensive notation and to the establishment of criteria for identifying analogues of theorems concerning generalized hypergeometric series. In particular two analogues of the confluence of generalized hypergeometric series are discussed and applied to several identities appearing in mathematical literature. Examples are given relating these confluent basic series to identities of modular functions. It is shown that the whole field of theta function, mock-theta function, modular function, and Raman Ramanujan identities may probably be expressed very elegantly and completely in terms of the basic hypergeometric series and their established relations. (Received November 19, 1942.)

### 32. Einar Hille and Max Zorn: *Open additive semi-groups of complex numbers.*

The main object of this paper is to determine the open, connected and additive semi-groups of the complex number plane. These sets are important as parameter manifolds of semi-groups of linear transformations which have been studied in detail by Hille. It is shown that the open semi-groups of the plane depend upon the upper-semi-continuous solutions of the inequality $f(x+y) \leqq f(x)+f(y)$, the function $f$ being defined for all real or all positive (negative) numbers $x$. These sets are automatically connected and simply connected. They are also shown to be the maximal domain of existence for suitable 1-parameter semi-groups of linear transformations in appropriate Banach spaces. Many of the results about the parameter manifolds permit considerable generalizations. (Received November 28, 1942.)

### 33. H. K. Hughes and Cleota G. Fry: *Asymptotic developments of certain integral functions.*

The authors obtain asymptotic developments valid for value of $s$ of large modulus, of the function $f_\alpha(s)=\sum_{n=0}^{\infty}g(n)s^n$, where $g(n)$ is of the form $h(n)\{\Gamma(\alpha n+p)\}^{-1}$; $\alpha>0$; $p$ is any constant, real or complex. It is assumed that the function $g(w)$, where $w=x+iy$, is single-valued and analytic in the finite $w$-plane, and that $h(w)$ is developable in a factorial series of the form $\sum_{n=0}^{\infty}a_n\Gamma(\alpha w+p)/\Gamma(\alpha w+p+n)$. The paper may be regarded as a generalization of results already obtained by W. B. Ford, and appearing in chap. 6 of his book entitled *Asymptotic developments of functions defined by Maclaurin series* (Michigan Science Series, vol. 11, Ann Arbor, 1936). The methods employed in the paper are essentially those employed by Ford. (Received November 20, 1942.)

### 34. Glenn James: *On equivalence of methods of summing divergent series.*

This paper investigates the uniqueness of sums given by the general, regular, definition $\lim_{n\to\infty}\sum_1^n s_i\alpha_i=S$, where the $\alpha_i$'s are positive or zero and converge uniformly to zero and $\lim_{n\to\infty}\sum_1^n\alpha_i=1$. It is shown that $S$ is generally arbitrary. A fourth restriction is then placed upon the $\alpha_i$'s which, at least for most ordinary summable series, makes $S$ unique, and enables one actually to determine $S$. This restriction is that the limit of the quotient of the number of changes in monotony in the sequence by $n$, as $n$ increases, shall be zero. (Received October 31, 1942.)

### 35. H. F. S. Jonah and A. H. Smith: *Zero order summability of the series conjugate to the derived Fourier series.*

This paper proves that a theorem of Takahashi (Jap. J. Math. vol. 10 (1933) pp. 127–132) concerning the Cesàro summability of the series conjugate to the derived Fourier series remains true if the Cesàro method of summation is replaced by the zero order method of Bosanquet-Linfoot (J. London Math. Soc. vol. 6 (1931) pp. 117–126), which is weaker than the Cesàro method of any order $\alpha>0$. (Received October 29, 1942.)

### 36. Knox Millsaps: *Characterization of the abstract exponential function.*

After defining the abstract exponential function over normed rings, it is possible to characterize this function as the solution of a first order general differential system. The validity of the theorem is established by the use of two theorems of Kerner on the existence of a solution for such equations and the symmetry of second order Fréchet differentials. (Received October 31, 1942.)

### 37. C. N. Moore: *On the relationship between Nörlund means of a certain type.*

The purpose of this paper is to obtain relationships of inclusion between Nörlund means defined in terms of the coefficients of the power series developments of functions analytic within the unit circle and having a singularity of algebraic-logarithmic type at the point $z=1$. The inclusion relationship is found to depend on the relative order of infinity of the functions involved in the neighborhood of the singular point. (Received November 19, 1942.)

### 38. G. B. Price: *Cauchy-Stieltjes and Riemann-Stieltjes integrals.*

This note treats the equivalence of the Riemann-Stieltjes and Cauchy-Stieltjes integrals (abbreviated RS and CS integrals) and conditions for the existence and equality of the latter. The RS and the left and right CS integrals are defined as limits of the sums $\sum_1^n f(\xi_i)[g(x_i)-g(x_{i-1})]$, $x_{i-1}\leqq\xi_i\leqq x_i$, $\sum_1^n f(x_{i-1})[g(x_i)-g(x_{i-1})]$, and $\sum_1^n f(x_i)[g(x_i)-g(x_{i-1})]$. When $g(x)=x$, these integrals are the Riemann and the two Cauchy integrals; Gillespie (Ann. of Math. vol. 17 (1915) pp. 61–63) proved that they are equivalent. Examples show (i) that the CS integrals may exist when the RS does not; (ii) that the two CS integrals may exist and have different values; (iii) and that one CS integral may exist when the other does not. One theorem is the following: if $g$ is non-decreasing, if $f$ and $g$ have no common discontinuities on the same side, and if the left (right) CS integral exists, then the RS integral exists and has the same value, the integrals being limits in the sense of increasing refinement of subdivisions. (Received November 23, 1942.)

### 39. Maxwell Reade: *On a theorem of Fédoroff and Binney.*

A theorem on harmonic functions, due to Fédoroff (Rec. Math. (Mat. Sbornik) N.S. vol. 40 (1933) pp. 168–179) and Binney (Trans. Amer. Math. Soc. vol. 37 (1935) pp. 254–265), has an analogue for subharmonic functions. The result is the following theorem. If $u(x, y)$ and $v(x, y)$ are continuous in the unit circle, and if $\int_R u dy - v dx \geqq 0$ and $\int_R u dx + v dy = 0$ for all oriented rectangles, $R$ lying in the unit circle, then there exists a function $f(x, y)$ that is subharmonic in the unit circle for which $\partial f/\partial x = u(x, y)$, $\partial f/\partial y = v(x, y)$. The proof follows Levin (*Contributions to the calculus of variations*, Chicago, 1942, pp. 363–410). (Received October 17, 1942.)

### 40. Maxwell Reade: *Remarks on a paper of Beckenbach.*

Beckenbach's Theorem 1 (Duke Math. J. vol. 8 (1941) pp. 393–400) is similar to the classic Montel-Rado theorem characterizing functions having subharmonic logarithms; therefore it admits of Saks-Kierst types of generalizations (S. Saks, Acta Univ. Szeged vol. 5 (1930–1932) pp. 187–193). A typical result of this paper is the following theorem. Let $f(t)$ be of class $c^2$, with $f'(t) > 0$ for $-\infty < t < \infty$. If $v(x, y)$ is of class $c^2$ in a domain $D$, and if $f(\log[(x-\alpha)^2+(y-\beta)^2]+v(x, y))$ is subharmonic for each choice of the real constants $\alpha$ and $\beta$, then $v(x, y)$ is subharmonic in $D$. In addition Beckenbach's Theorem 2 is given a slightly more general form. (Received October 16, 1942.)

### 41. Raphael Salem: *On some singular monotonic functions which are strictly increasing.*

The paper gives simple direct constructions of some singular monotonic functions which are strictly increasing. It contains also some new indications on Minkowski's singular function, particularly concerning its modulus of continuity. (Received October 9, 1942.)

### 42. H. M. Schwartz: *On sequences of Stieltjes integrals.* II.

Given a sequence $g_n(x)$ $(n = 1, 2, \cdots)$ of functions of bounded variation, the problem of determining sets $C(F)$ of conditions necessary and sufficient for the convergence of the sequence $\int_a^b f dg_n$ for all functions $f(x)$ of a given family $F$, is an extension of the problem treated by Lebesgue in his paper *Sur les intégrales singulières* (Ann. Fac. Sci. Univ. Toulouse vol. 23 (1909) pp. 25–117) and which corresponds to the case of absolutely continuous $g_n(x)$. Considering first the case of Riemann-Stieltjes integration over a finite interval, conditions $C(F)$ were obtained earlier (Bull. Amer. Math. Soc. abstract 48-1-55) for certain families $F$, where $F$ are proper subsets of the set $F^*$ of all functions which are defined in $(a, b)$ and are simultaneously integrable with respect to all $g_n$. The principal result of the present paper is the determination of a set $C(F^*)$. In the special case of absolutely continuous $g_n$, this result provides a solution for the Lebesgue problem in the case, not treated by Lebesgue, when $F$ consists of the class of Riemann integrable functions. The paper is concerned also with the extension of the results to the case of an infinite interval $(a, b)$ and to more general Stieltjes integrals of Riemann type. (Received November 20, 1942.)

### 43. I. M. Sheffer: *Note on a linear transformation of "analytic" type.*

A linear problem concerning analytic functions is often expressed in terms of the power series coefficients of the functions, thus leading the problem to a system of linear equations in the coefficients: (1) $A_n[X]=\sum_{k=0}^{\infty} a_{nk}x_k=c_n$, $n=0, 1, \cdots$, where $X$ represents the sequence $\{x_n\}$. Let lim sup $|x_n|^{1/n}$ be called the type of sequence $\{x_n\}$. As a first step in the analysis of transformation (1) examine the following question (together with some of its consequences): Under what conditions will sequences $\{x_n\}$ of given type transform into sequences of prescribed type under (1)? A typical result is the following: In order that $\{A_n[X]\}$ be of type at most $k$ for every $\{x_n\}$ of type at most $r$ it is necessary and sufficient that: (i) If $r_n$ is the radius of convergence of $A_n(t)=\sum_{k=0}^{\infty} a_{nk}t^k$, and $r^*=$g.l.b. $\{r_n\}$, then $r^*>r$. (ii) Let $A_n^*(t)=\sum_{k=0}^{\infty} |a_{nk}| t^k$. For every $\epsilon>0$ there corresponds an $r'$ in $r<r'<r^*$ such that $\{A_n^*(r')\}$ is of type at most $k+\epsilon$. (Received November 24, 1942.)

### 44. W. S. Snyder: *Non-parametric surfaces and inscribed polyhedra.* Preliminary report.

Let $z=f(x, y)$ define a continuous surface over an oriented rectangle of the $xy$-plane. Kempisty (*Sur la methode triangulaire* $\cdots$, Bull. Soc. Math. France vol. 64 (1936)) and the author have studied certain functions of rectangles associated with the surface and have derived sufficient conditions that the Burkill integrals of the functions exist and equal the Lebesgue area of the surface. This paper undertakes the study of the simple Burkill integrals (Burkill, *Functions of intervals*, Proc. London Math. Soc. (2) vol. 22 (1924)) of these functions. Necessary conditions that the simple Burkill integrals of these functions exist and equal the Lebesgue area of the surface are derived, and somewhat stronger conditions are shown to be sufficient for this result to hold. (Received November 23, 1942.)

### 45. W. C. Strodt: *Analytic solutions of nonlinear difference equations.* Preliminary report.

A new approach to difference equations is outlined, and applied to a large class of nonlinear difference equations. Special solutions are obtained as uniform limits of analytic solutions of $q$-difference equations, equations which yield more readily to power series treatment. The given equation $P[x, y(x), y(x+\omega_1), \cdots, y(x+\omega_n)]=0$ (where the $\omega_k$ are arbitrary positive numbers, and $P$ is a polynomial in the $y(x+\omega_k)$, analytic in $x$) is replaced by the approximating functional equation $P[x, y(x), y(q_1x+\omega_1), \cdots, y(q_nx+\omega_n)]=0$, where $q_i=1-\omega_i\eta$, $\eta$ being any small positive number. Each transformation $x\rightarrow q_ix+\omega_i$ has $1/\eta$ for fixed point. A translation of origin to this point transforms this functional equation into the $q$-difference equation $P[u+1/\eta, z(u), \cdots, z(q_nu)]=0$, where $u=x-1/\eta$, $z(u)=y(u+1/\eta)$. This equation has for solution a unique power series $z(u;\eta)$ in $u$ whose convergence is shown by the calculus of limits. The uniform limit of $z(x-1/\eta;\eta)$, as $\eta$ assumes a suitable sequence of values tending to 0, is an analytic solution of the given difference equation. There is evidence that the solution generalizes to nonlinear equations Norlund's "principal solution." The procedure applies formally to essentially every difference equation; a preliminary transformation $y(x)=x^sY(x)$ is sometimes needed. (Received October 16, 1942.)

### 46. Otto Szász: *On Abel and Lebesgue summability.*

The main results of this paper are: If the real numbers $a_n$ satisfy the condition (*) $\sum_n^{2n}(|a_\nu|-a_\nu)=O(1)$ as $n\rightarrow\infty$, then Abel summability of the series $\sum_1^\infty a_n$ implies its Lebesgue summability; more explicitly the series $\sum a_n(\sin nt/n)=F(t)$ converges uniformly and $\lim_{t\downarrow 0}t^{-1}F(t)=s$ exists. At the same time the series $\sum a_n$ may diverge; in fact a series $\sum a_n$ is constructed which is Abel summable and satisfies (*), while $\limsup |a_n|=1$, so that $\sum a_n$ diverges. On the other hand it is proved that if (*) holds and if $\sum a_n$ converges, then the series $\sum a_n(\sin nt/nt)=t^{-1}F(t)$ converges uniformly in the interval $0<t<\pi$, that is, $\sum a_n$ is Lebesgue summable. It is to be noted that convergence alone does not imply Lebesgue summability. The method is similar to the one used in a previous paper: Amer. J. Math. vol. 64 (1942) pp. 575–591. (Received October 20, 1942.)

### 47. W. C. Taylor: *Asymptotic behavior of the Abel sums of the Laguerre expansion.*

For the Laguerre expansion $f(x)\sim\sum c_\nu L_\nu^{(\alpha)}(x)$ the behavior of the Abel sum $A(x,w)=\sum c_\nu L_\nu^{(\alpha)}(x)w^\nu$ is studied. Conditions are obtained which insure that $A(x,w)-f(x)$ is $o(1)$ or $o(f(x))$ under various limiting processes involving $x\rightarrow\infty$ and/or $w\rightarrow 1$. (Received November 21, 1942.)

### 48. W. J. Trjitzinsky: *Singular nonlinear integral equations.*

The author studies problems (1) $\phi(x)+\int K(x,t)f(t,\phi(t))dt=0$, (2) $\phi(x)+\int\Gamma(x,y,\phi(y))dy=0$ ($\phi$ is the unknown; the integrals are definite), where $\Gamma(x,y,z)=\int K(x,t)k(t,y,z)dt$. The functions $f(t,u)$, $k(t,y,u)$ are subject to Lipschitz conditions in $u$. Past investigations were under the supposition that the kernel $K(x,t)$ is regular in the sense that the characteristic values of $K(x,t)$ form a discrete set, Fredholm theory being applicable. The author proceeds with more general kernels, not regular in the above sense, belonging to $L_2$ in each of the variables separately; such kernels are singular. Among other results a number of existence theorems for (1), (2) are established. These problems are of importance in themselves as well as in many phases of mathematical physics; also, many nonlinear differential problems with linear boundary conditions are reducible to the problem (1). This theory naturally depends in an essential manner on the theory of singular kernels as previously developed by Carleman and later by the present author. (Received November 20, 1942.)

### 49. S. M. Ulam: *On equivalence of functions.*

Two functions, $f(x)$ and $g(x)$ defined on $0 \leqq x \leqq 1$ are said to be equivalent, if there exists a one-to-one transformation of the interval $h(x)$ such that $f(x) = hgh^{-1}(x)$. This notion of equivalence is investigated under restriction of $f(x)$ and $g(x)$ to various classes of functions. Among others, it is proved that if $f(x)$ is a one-to-one analytic function, it is equivalent to a Baire function of finite class. Necessary and sufficient conditions are given for equivalence of two continuous functions—and an enumeration of possible equivalence types of such functions is obtained. (Received November 23, 1942.)

### 50. Wolfgang Wasow: *On a boundary layer problem for a certain linear partial differential equation.*

Let $u(x, y, \lambda)$ be the function satisfying the differential equation $(1/\lambda)\Delta u + u_x = f(x, y)$ in the interior of a convex domain $B$ and assuming prescribed boundary values, not depending on $\lambda$, on the boundary $C$ of $B$. As $\lambda \rightarrow \infty$ $(\lambda > 0)$ the differential equation reduces to the limiting differential equation $u_x = f(x, y)$. In the paper the author proves that $v(x, y) = \lim_{\lambda \rightarrow \infty} u(x, y, \lambda)$ exists and satisfies the limiting differential equation. But in general $v(x, y)$ will satisfy the prescribed boundary condition only on a certain arc $C_1$ of $C$. Along the remaining arc $C_2$ of $C$ the convergence of $u(x, y, \lambda)$ is non-uniform and gives rise to a boundary layer phenomenon. The proof is based on certain estimates for Green's function and uses an idea suggested in a paper by E. Rothe. (See E. Rothe, *Asymptotic solution of a boundary value problem*, Iowa State College Journal for Science vol. 13 (1939).) (Received November 23, 1942.)

### 51. Alexander Weinstein: *On a general variational method for the determination of eigenvalues.*

Let $U(v)$ and $H(v)$ be two positive definite quadratic functionals and let $P$ be the eigenvalue problem relative to the variational problem $U(v)/H(v) = \text{Min.}$ under certain boundary conditions $BC$ which express the vanishing of $v$, $dv/dn$, and so on, on certain parts of the boundary. Let $P_0$ be the eigenvalue problem obtained from $P$ by cancelling a certain number of the boundary conditions $BC$. The eigenvalues in $P_0$ are obviously not greater than those in $P$. It can be shown that the eigenvalues in $P$ can be expressed in terms of the solutions of $P_0$. This result is obtained by linking $P_0$ with $P$ by a chain of intermediate problems $P_1, P_2, \cdots$ defined in a similar way as those used in particular cases by the author (Mémorial des Sciences Mathématiques, no. 88). (Received November 16, 1942.)

### 52. J. E. Wilkins: *A class of functions in the calculus of variations for multiple integrals in parametric form.*

In order that a multiple integral with integrand $f(y^1, \cdots, y^n, p_1^1, \cdots, p_m^n)$, where $p_\alpha^i = \partial y^i/\partial t_\alpha$, be independent of the parametric representation $y^i = y^i(t)$ of the variety for which it is computed, it is necessary and sufficient that the equation $f(y, pA) = |A| f(y, p)$ hold for every $m$-rowed square matrix $A$ with positive determinant. It was proved in the author's doctoral dissertation that there exist constants $N^{\beta;\epsilon}_{\alpha;\gamma}$ such that the equation, $p^{\iota}_{\alpha} f^{\beta\epsilon}_{\iota\kappa} = M^{\beta;\epsilon}_{\alpha;\gamma} f^{\gamma}_{\kappa}$ holds for every function $f$ satisfying the above relation. In this equation $\iota = (i_1, \cdots, i_r)$, $\alpha = (a_1, \cdots, a_r)$, $\beta = (b_1, \cdots, b_s)$, $\epsilon = (e_1, \cdots, e_h)$, $\kappa = (k_1, \cdots, k_h)$, $\gamma = (c_1, \cdots, c_s)$, $p^{\iota}_{\alpha}$ stands for the product of the corresponding $p$'s, and $f^{\gamma}_{\kappa}$ stands for the derivative of $f$ with respect to the corresponding $p$'s. It is the purpose of this paper to give an explicit definition of the constants $M^{\beta;\epsilon}_{\alpha;\gamma}$ in terms of a certain set of permutations. On the basis of this definition certain properties of the $M$'s are readily deduced. (Received October 27, 1942.)

### 53. J. E. Wilkins: *A note on the Weierstrass condition for multiple integrals in the calculus of variations.*

For the purpose of obtaining a sufficiency theorem for multiple integral problems in the calculus of variations, Carathéodory has introduced an $E$-function $E_*(x, y, p, P)$ different from the usual $E$-function $E$. It is proved here that $E_* = E$ in case $P - p$ has rank one. Consequently, a necessary condition for a minimum is that $E_* \geqq 0$ for all $P$ such that $P - p$ has rank one. But an example is given to show that it is not true that $E_* \geqq 0$ for all $P$ is necessary for a minimum. (Received October 30, 1942.)

### 54. J. E. Wilkins: *Definitely self-conjugate adjoint integral equations.*

This paper contains a new definition for definitely self-conjugate adjoint integral equations which is substantially weaker than that originally given by Reid. For such systems it is found that all of the expansion theorems obtained by Reid remain valid, but that there need not now be a denumerable infinity of characteristic values. Moreover it is shown that the index of a characteristic value is equal to its multiplicity as a zero of the Fredholm determinant. (Received November 25, 1946 )

### 55. Fumio Yagi: *On a certain Stieltjes integral equation.*

Let $g(s)$ and $p(\lambda)$ be two given functions which satisfy certain norm conditions. It is known (Cameron-Martin, *Infinite linear difference equations with arbitrary real spans and first degree coefficients*, to be published) that the integral equation $\int_{-\infty}^{\infty} f(s-\lambda)dp(\lambda) = g(s)$ has a unique analytic solution in $a < \mathrm{Im}\, s < b$ of the same norm as $g(s)$, provided that $P(w) = \int_{-\infty}^{\infty} e^{w\lambda} dp(\lambda)$ is non-vanishing in a strip $c < \mathrm{Re}\, w < d$. In this paper the author allows the function $P(w)$ to have a finite number of zeros in $c < \mathrm{Re}\, w < d$ and shows that there exists an analytic solution $f(s)$ in $a < \mathrm{Im}\, s < b$ of the same norm as $g(s)$. Moreover it is proved that every other solution of the required type is the form $f(s) + \sum_{n=1}^{N} \sum_{m=1}^{a_n} A_{mn} e^{-w_n s} s^{m-1}$ where $A_{mn}$ are constants, $w_n$ $(n = 1, 2, \cdots, N)$ are the zeros of $P(w)$, and $a_n$ is the order of zero at $w_n$. (Received November 20, 1942.)

### 56. J. W. T. Youngs: *On the additivity of the Lebesgue area.*

Suppose a surface $S$ is given in parametric form by the continuous triple $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ with $(u, v)$ in the unit square $Q$. Each division of $Q$ into two rectangles $R_1$ and $R_2$ by a line yields a decomposition of $S$ into two surfaces $S_1$ and $S_2$. This paper provides a direct proof for the theorem that, if the continuous triple carries the dividing line into a single point, then the Lebesgue area is additive:

$L(S)=L(S_1)+L(S_2)$ (Rado and Reichelderfer, *On a stretching process for surfaces*, Amer. J. Math. vol. 61 (1939)). (Received November 28, 1942.)

### 57. Antoni Zygmund: *A property of the zeros of Legendre polynomials.*

Suppose that $n<m$ are positive integers and that a polynomial $\phi(x)$ of degree $n$ does not exceed $M$ in absolute value at the zeros of the Legendre polynomial $P_m(x)$. Then $|\phi(x)| \leqq A(\delta)M$ for $-1\leqq x\leqq +1$, where $A(\delta)$ depends only on the number $\delta$ defined by the equations $m/n=1+\delta$. Similar results hold for the integrals $\int_{-1}^{+1}|\phi(x)|^r dx$ with $r\geqq 1$. (Received November 23, 1942.)

## Applied Mathematics

### 58. Stefan Bergman: *A formula for the stream function of compressible fluid flow.*

Let $q=ve^{i\theta}$ denote the velocity vector. A flow, $\mathfrak{F}$ is said to be of the type $D_n$, if the boundary of the domain, $B$, in which $\mathfrak{F}$ is defined consists of $2n$ segments $S_K$ such that along each $S_{2K}$, $K=1, 2, \cdots, n$, $\theta=\theta_K$ is constant and along each $S_{2K-1}$, $v$ is constant. ($S_{2K}$ are segments of straight lines, $S_{2K-1}$ are so-called "free boundaries.") The image of $B$ in the logarithmic plane (see *Notes on hodograph method in the theory of compressible fluid*, publication of Brown University, p. 6) is a polygonal domain. In the case of an incompressible fluid the stream function of $\mathfrak{F}$ can be represented as a closed expression with $n$ parameters. The author considers subsonic flows, $\mathfrak{C}$, of compressible fluid. Using certain linear operators (see above mentioned *Notes*, §§6 and 10, and Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155) he derives a similar explicit formula with $n$ parameters for the flows $\mathfrak{C}$ of "nearly type $D_n$," that is to say, for flows whose boundaries consist of $2n$ segments along which $\theta$ or $v$ assume nearly constant values. The angles $\theta_K$ may be prescribed. (Received November 21, 1942.)

### 59. R. M. Foster: *On the average resistance of an electrical network.*

In an electrical network composed of two-terminal resistance elements, let $J_i$ designate the total resistance measured across the terminals of an element, the internal resistance of this element being $r_i$, and let $S_i$ be the driving-point resistance measured in the branch containing this element $r_i$. It is shown in this paper that, for any network configuration whatsoever, $\sum J_i/r_i=R$ and $\sum r_iS_i=N$ (the summation being extended over all the elements of the network), with $R=V-P$ and $N=E-V+P$, where $E$ is the number of elements, $V$ the number of vertices, and $P$ the number of separate, unconnected parts of the configuration. The average values of the ratios $J_i/r_i$ and $r_i/S_i$ are thus $R/E$ and $N/E$, respectively. If all the elements of the network have the same internal resistance $r$, and if there is complete symmetry among the elements so that the resistance measured across any one element is necessarily equal to that across any other element, then $J_i=rR/E$ and $S_i=rE/N$. These results are extended to generalized impedances, and to infinite networks. (Received November 23, 1942.)

### 60. A. H. Fox: *Integral representation of the flow of a compressible fluid around a cylinder.*

The steady irrotational two-dimensional flow of a compressible fluid may be approximated by the flow of a hypothetical incompressible fluid in which the pressure is represented by a polynomial in the reciprocal of the density. The stream function of the flow may be expanded in a series in terms of a parameter $q$ that appears in the polynomial. For $q=0$ this reduces to the von Kármán-Tsien method of attacking similar problems. The differential equation of the flow in the hodograph plane may be reduced to the form $\psi^*_{s\bar{s}}+F\psi^*=0$, for which the integral representation $\psi^*$ is the real part of $\int_{-1}^{1}E^*(s, \bar{s}, t)(1-t^2)^{-1/2}f[1/2s(1-t^2)]dt$ has been established by Bergman. The function $f$ is a function of one complex variable determined by the pattern of the flow in the hypothetical fluid. In the present paper, the method is applied to the flow of compressible fluid around a circular obstacle. (Received November 24, 1942.)

### 61. K. O. Friedrichs: *On nonlinear vibrations of third order.*

Problems of self-excited oscillations frequently lead to differential equations of higher than second order. While for the order two, Bendixson's theorem furnishes a general criterion for the existence of periodic solutions, no similar general criterion is available in cases of higher order. The present paper analyzes such a problem of order three (occurring in the theory of electrical oscillations in circuits involving vacuum tubes). First, the existence of periodic solutions is obtained from the fixed-point theorem by a special construction. In addition, periodic solutions are investigated in the neighborhood of degenerate cases of order two by a perturbation method, which yields asymptotic expansions. (Received November 23, 1942.)

### 62. Hilda Geiringer: *New convergence cases for iteration methods applied to linear equations.*

To obtain the solution $x^{\rightarrow}$ of the system $\sum_{k=1}^{n}a_{ik}x_k+r_i=0$ $(i=1, \cdots, n; |A|\neq 0)$, by iteration through "successive displacements" new values $x_i^{(\nu+1)}$ are successively computed from the $i$th equation using $x_1^{(\nu+1)}, \cdots, x_{i-1}^{(\nu+1)}, x_{i+1}^{(\nu)}, \cdots, x_n^{(\nu)}$, $(\nu=0, 1, 2, \cdots; x^{(0)\rightarrow}$ arbitrary). (See Bull. Amer. Math. Soc. abstract 48-5-202.) The only sufficient condition for the convergence of $x^{(\nu)\rightarrow}\rightarrow x^{\rightarrow}$ known so far is that the matrix $(a_{ik})$ is positive definite. In this case the procedure reduces to Southwell's relaxation-method. Now the necessary and sufficient condition for convergence is that the roots $\rho_i$ of a certain equation $F_{n-1}(\rho)=0$ of degree $n-1$ are all less than one in absolute value, a condition assuming no symmetry. If, however, the matrix is symmetric and $a_{ii}>0$ then positive-definiteness is necessary for convergence from an arbitrary starting point $x^{(0)\rightarrow}$. In addition it is proved that all sufficient conditions known for the "ordinary" iteration, that is, for the iterated linear homogeneous transformation of the "error-vector" $\epsilon_i^{(\nu)}=x_i^{(\nu)}-x_i$, assure convergence of the successive displacements, but by no means vice versa; and a new sufficient condition for both methods is established. In all these cases any order is admissible for the successive displacements; also the convergence may be improved by "group-iteration." (Received November 23, 1942.)

### 63. A. E. Heins: *Some remarks on the solution of dual integral equations.* I.

The solutions of dual integral equations whose kernels can be expressed in a series of Bessel functions are considered. Explicit solutions for these equations are given and their properties are discussed. (Received November 23, 1942.)

### 64. Max Herzberger: *Direct methods in geometrical optics.*

The author presents a new approach to the problem of geometrical optics deviating from the methods of W. I. Hamilton. Hamilton has shown the existence of a characteristic function, the knowledge of which gives all the information necessary about an optical system. However, the construction of Hamilton's function for any given optical system seems to lead to insurmountable mathematical difficulties. The author proposes instead to calculate the coordinates of the image rays directly as functions of the coordinates of the object rays. Instead of getting a single function like Hamilton, he finds four functions connected by three differential equations and one finite equation. He has succeeded in obtaining the explicit form of these functions for any given optical system and developing an image error theory which is more adapted to the practical problems of lens design. (Received October 3, 1942.)

### 65. Fritz John: *Instability of certain nonlinear vibrations.*

The differential equation $(E)$: $x''(t)+f(x)=-F\sin\lambda t$, where $f(0)=0$, g.l.b. $f'(x)>0$, $\lambda$ and $F$ positive constants, represents the motion of a spring with nonlinear restoring force and without damping under a periodic disturbing force. It is shown here: For every $M>0$ there exists and is uniquely determined a value $\lambda>0$ and a solution $x(t)$ of $(E)$ of period $2\pi/\lambda$, for which $x=M$, $x'=0$ for $t=\pi/2\lambda$ and $x'>0$ for $|t|<\pi/2\lambda$ ($x(t)$ then is a periodic solution of phase difference $\pi/\lambda$). The value $\lambda$ determined by $M$ may define the "resonance function" $\lambda=\lambda(M)$ (this is only one branch of the complete resonance curve). For $d\lambda/dM>0$ the corresponding periodic solutions $x(t)$ are shown to be unstable in the following sense: there is an $\epsilon>0$, such that for every $\delta>0$ there exists a solution $x_1(t)$ of $(E)$, for which $|x_1-x|<\delta$, $|x_1'-x'|<\delta$ for some $t$, and $|x_1-x|>\epsilon$ for other $t$. One always has $d\lambda/dM\leqq 0$ for "soft" springs, that is, if $f'(x)\leqq f(x)/x$. (Received November 23, 1942.)

### 66. Edward Kasner: *Trajectories in a resisting medium.*

Consider the motion of a particle moving in a general positional field and influenced by a resisting medium $R$. The five fundamental properties of a pure positional field (*Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications vol. 3, 1913, chap. 1) are of course in general changed. However if $R=Av^2+B$, property I remains valid. If properties I and II are valid then the resistance is found to vary as the square of the speed. If I, II, III hold then $R$ vanishes. The discussion is carried out for both two- and three-dimensional fields. Fields of simple and complicated character are studied where either all, or the maximum infinitude, of trajectories are circles (a particular example is the Maxwell central force). (Received November 18, 1942.)

### 67. Arthur Korn: *On vibrational vortices.*

The vortex equations in fluids are nonlinear differential equations, but in some special cases they may be replaced by linear equations. A special case is the problem of vibrational velocities (vectors): $u=u_0+u_1\cos\nu t+u_2\sin\nu t$, where $\nu$ is an exceedingly great frequency and the first derivatives of $u_0$, $u_1$, $u_2$ with respect to the time $t$ and the first derivatives with respect to the coordinates $x$, $y$, $z$ of $u_1$, $u_2$ can be neglected in comparison with $\nu u_0$, $\nu u_1$, $\nu u_2$. Then one remarks that on the left-hand side of the vortex equations the main terms are $(-\operatorname{curl} u_1 \sin\nu t+\operatorname{curl} u_2\cos\nu t)\nu/\rho$, $\rho$ being the density of the fluid. In order to obtain on the right-hand side such great terms, the first derivatives of $u_0$ with respect to $x$, $y$, $z$ must be exceedingly great. Excluding exceedingly great values of $\rho u_0^2/2$ such cases cannot be realized in incompressible fluids, but they are perfectly possible in compressible fluids, even if the conditions div $u_1=0$ and div $u_2=0$ are retained. $u_2$ must have wave character with an exceedingly small wave length $\lambda$. Such cases can be handled by means of linear differential equations. (Received November 21, 1942.)

### 68. Brockway McMillan: *Networks of mechanisms*. Preliminary report.

A mechanism $M$ maps a class $I$ of input histories $i(t)$ upon a class $O$ of output histories $o(t)$, $-\infty<t<\infty$. It is single-valued and has the property that whenever $i_1(t)\equiv i_2(t)$ for $t\leqq t_0$, then $o_1(t)\equiv o_2(t)$ for $t\leqq t_0$. If $o_1(t)\equiv o_2(t)$ for $t\leqq t_0+\lambda$, uniformly in $i_1$, $i_2$, and $t_0$, then $M$ has the latency $\lambda$. Suppose that $I$ is closed under an operation of addition, that a null function $i_0(t)\equiv\phi$ is in $I$, that every $i(t)$ is identically $\phi$ near $t=-\infty$, and that $O\subseteq I$. Let $(M_k)$ be a collection of mechanisms from $I$ to $O$ such that (a) each has latency at least $\lambda>0$, and (b) each maps the null function on itself. The inputs and outputs of the $M_k$ are interconnected to form a network $N$. Arbitrary inputs $i(t)\in I$ at each junction make the components of a vector input to $N$. The vector output of $N$ has for its components the outputs of the various $M_k$. Theorem: $I$ and $O$ can be extended so that $N$ is a mechanism between its vector inputs and outputs, with latency $\lambda$. A motivation is the possibility of application to nerve fiber networks. (Received November 4, 1942.)

### 69. W. H. Roever: *A new formula for the deviation in range of a projectile due to the earth's rotation.*

On page 68 in his monograph entitled, *The weight field of force of the earth*, published in the Washington University Studies, September, 1940, the author derives for the range of a projectile, a formula [second part of (129)] which by a simple trigonometric transformation can be put in the new form $x=(v_0^2/g_1)\sin 2\beta+\Delta x$ where $\Delta x=-(4v_0^3/3g_1^2)\ \omega\cos\phi_1\sin 3\beta\sin\alpha$, in which $\omega$ is the angular velocity of the earth's rotation, $g_1$ is the acceleration, due to weight, at the position of the gun, $\phi_1$ is the astronomical latitude of the position of the gun, $\alpha$ is the azimuth (measured from the south through the west) of the direction of fire, $\beta$ is the angle of elevation of the gun, $v_0$ is the muzzle velocity of the projectile, he points out particularly that for fixed values of $\alpha$ and $\phi_1$, $\Delta x$ changes sign when $\beta=60°$. (Received November 23, 1942.)

## GEOMETRY

### 70. John DeCicco: *Conformal geometry of second order differential equations.*

Kasner in his fundamental paper, *Conformal geometry*, Prcoeedings of the International Congress of Mathematicians, 1912, initiated the conformal study of sets of analytic curves. In previous work, Kasner (with the author) studied the conformal geometry of velocity systems of curves $y''=(1+y'^2)[\phi(x,y)+y'\psi(x,y)]$. This class of velocity systems characterizes the conformal group. Any velocity system possesses six absolute conformal differential covariants of second order. In this paper, it is shown that a system of $\infty^2$ curves, not of the velocity type, possesses three absolute conformal differential covariants of third order. Moreover any other conformal covariant is a function of these and their partial derivatives. Geometric interpretations of these covariants are also obtained. (Received November 21, 1942 )

### 71. Jesse Douglas: *Point transformations and isothermal families of curves.*

A classification is made of all point transformations $U=f(u, v)$, $V=g(u, v)$ $(u=x+iy$, $v=x-iy)$ with respect to the isothermal families of curves $(\phi(u)+\psi(v)=\text{const.})$ which they convert again into isothermal families. An incidental result is a very simple proof of the theorem that only conformal transformations, $U=f(u)$, $V=g(v)$, convert all isothermal families into isothermal families. (A proof of a more general theorem, involving arbitrary lineal element transformations, has been given recently by Kasner.) Canonical transformations found are (1) $U=u$, $V=u+v$, which preserves the isothermal character of $V+\phi(U)=\text{const.}$ ($\phi$ arbitrary); (2) $U=u+v$, $V=u-v$, preserving the isothermality of all curve families equivalent by translation and similitude to that defined by $\text{sn}\ U\,dU+\text{sn}\ V\,dV=0$. Here sn is Jacobi's elliptic function with any modulus $k$. Interesting special cases (trigonometric, algebraic) are obtained for particular values of $k$. The given canonical transformations may be pre- and post-multiplied by arbitrary conformal transformations. The problem may be rendered topological by referring it to the determination of the common diagonal curves of two given nets. (Received November 23, 1942.)

### 72. Edward Kasner: *The close packing of spheres.*

If space is packed with equal spheres the volume occupied is about seventy-five per cent and the porosity about twenty-five per cent. By using many spheres of unequal radii, it is proved that the porosity can be made as near zero as desired. An analogous theorem holds for circles in a plane, or for hyperspheres in any space. (Received October 9, 1942.)

### 73. Edward Kasner and John DeCicco: *Bi-isothermal systems in pseudo-conformal geometry.*

In this paper, generalizations are obtained of Kasner's theorems on biharmonic functions in the paper, *Biharmonic functions and certain generalizations*, Amer. J. Math. vol. 58 (1936) pp. 377–390. A bi-isothermal system of hypersurfaces in euclidean four-dimensional space is pseudo-conformally equivalent to $\infty^1$ parallel hyperplanes. Similarly a bi-isothermal system of curves is pseudo-conformally equivalent to $\infty^3$ parallel lines. Any bi-isothermal system of hypersurfaces is intersected by a conformal surface in an isothermal system of curves. If every bi-isothermal system of hypersurfaces intersects a given surface $s$ is an isothermal system, then $s$ is conformal. If a system $S$ of $\infty^1$ hypersurfaces is intersected by every conformal surface in an isothermal system, then $S$ is bi-isothermal. Kasner's pseudo-angle between any two bi-isothermal systems of hypersurfaces and curves is a biharmonic function. The complete group of point transformations preserving the class of biharmonic systems of hypersurfaces (or curves) is the mixed pseudo-conformal group. (Received November 21, 1942.)

### 74. Mary E. Ladue: *Conformal geometry of horn angles of higher order.*

The conformal measures of horn angles (curvilinear angles of zero magnitude) between curves having third, fourth and fifth order contact are obtained. The conformal geometry of the third order horn set (the set of all curves having third order contact with each other at a given point) is studied by making correspond to each curve of the horn set a point in a four-dimensional space $K_4$ in which the metric is given by the conformal measure of a horn angle of the horn set, and the transformation is that induced on the points of the $K_4$-space by a conformal transformation on the curves of the horn set. The invariants of the geometry of this $K_4$-space are then obtained. Finally the equilong measure (see Kasner, *Equilong invariants and convergence proofs*, Bull. Amer. Math. Soc. vol. 23 (1917) pp. 341–347) of horn angles of third order contact is obtained and the resulting equilong geometry of the horn set is shown to be isomorphic to the conformal geometry of the corresponding horn set. Similar results for the equilong geometries of horn sets of fourth and fifth order are to be presented soon. (Received November 20, 1942.)

### 75. Karl Menger: *Statistical generalizations of metric geometry.*

A probability function $\Pi(x; p, q)$ instead of a number is associated with each ordered pair of elements $p, q$ of a set $S$ and interpreted as the probability that the point $p$ has a distance at most $x$ from the point $q$. Assume as postulates $\Pi(0; p, q) = 1$ or $\neq 1$; according to whether $p = q$ or $p \neq q$; $\Pi(x; p, q) = \Pi(x; q, p)$; and a triangular inequality $\Gamma(\alpha, \beta) \leqq \gamma$ connecting $\alpha = \Pi(x; p, q)$, $\beta = \Pi(y; q, r)$, and $\gamma = \Pi(x+y; p, r)$ where $0 \leqq \Gamma(\alpha, \beta) = \Gamma(\beta, \alpha) \leqq 1$ and $\Gamma$ is a non-decreasing function such that $\Gamma(1, 1) = 1$ and $\Gamma(\alpha, 1) > 0$ if $\alpha > 0$. Define $q$ to be between $p$ and $r$ and write $pqr$ if $\Gamma(1-\alpha, 1-\beta) \leqq 1-\gamma$. Then it can be shown that (1) $pqr$ implies $rqp$; (2) if $q$ is distant from $r$, that is, if for some $y > 0$, $\Pi(y; q, r) = 0$, then $pqr$ and $prq$ are incompatible; (3) under certain conditions, $pqr$ and $prs$ imply $pqs$ and $qrs$. If $S$ contains more than four elements it can be ordered by means of the above defined betweenness relation. (Received October 28, 1942.)

### 76. P. M. Pepper: *A new method for imbedding theorems.*

Let $\Sigma$ be a system of sets with a congruence relation for pairs of elements of $\Sigma$, and let $E$ be an element of $\Sigma$ such that each $n+4$ or more point element of $\Sigma$ which is not congruent to a subset of $E$ has at least $m+1$ of its $n+3$ point subsets not congruent to subsets of $E$. Then for $k \geqq 1$, each $n+3+k$ or more point element of $\Sigma$ which is not congruent to a subset of $E$ has (in terms of the binomial coefficients $C_{n,r}$) at least $1+mk+(m-1)C_{k,2}+ \cdots +C_{k,m}$ of its $n+3$ point subsets not congruent to subsets of $E$. If $\Sigma$ is the class of all semimetric spaces, the above theorem, in conjunction with a finite covering theorem, permits a new proof of a sharp imbedding theorem for the euclidean $n$-space under weakened hypotheses and a comparable new result for a convex $n$-sphere. (Received October 29, 1942.)

### 77. E. J. Purcell: *Flat space congruences of order one in* $[n]$.

By an $[n-k]$-congruence of order one in $[n]$, $(k<n)$, is meant an algebraic $\infty^k$-system of $[n-k]$'s in $n$-dimensional projective space such that one and only one $[n-k]$ of the system passes through an arbitrary point of $[n]$. This paper defines, classifies, and studies $2^{k-1}$ types of $[n-k]$-congruences of order one in $[n]$, for which the fundamental loci on a generic $[n-k]$ are all distinct. (Received November 2, 1942.)

### 78. W. H. Roever: *The axonometric method of representing the points of space on a plane.*

After stating that axonometry is one of the modern methods of descriptive geometry which is but little known in this country, the author describes his method of representing space upon the plane as follows:—Adjoin to the object to be represented its shadow on some fixed plane, such as the ground, and then project this augmented figure on the picture plane, thus obtaining thereon for each point $P$ of the object its projection $P^*$ and also the projection $P'^*$ of its shadow. He thus obtains for a point $P$ of space the two related picture-plane points $P^*$ and $P'^*$ which together furnish enough information to obtain therefrom definitely and uniquely the position of $P$ in space. He also shows that this method yields the same result as that given by him in his monograph entitled *Fundamental theorems of orthographic axonometry and their value in picturisation* (Washington University Studies, n.s., Science and Technology, no. 12, St. Louis, 1941). (Received November 19, 1942.)

### 79. Domina E. Spencer: *The tensor representation of the figures of Study's "Geometrie der Dynamen."*

In an earlier paper (*Geometric figures in affine space*, Bull. Amer. Math. Soc. abstract 48-9-281) the affine ancestors of the Study figures were investigated and the foundations were laid for the tensor representation of the figures themselves. The present paper continues the work by the introduction of a metric and the detailed study of the various Study figures. This systematic approach to the subject will make possible important applications which were obscured by the complexity of the original treatment of Study. (Received November 18, 1942.)

### 80. C. E. Springer: *Dual geodesics on a surface.*

In this paper a dual geodesic is defined to be a curve on a metric surface with the property that its ray-point corresponding to every point $P$ on the curve lies on the line $l_2$ which is in Green's Relation $R$ to the normal line $l_1$ to the surface at $P$. The differential equation of the dual geodesics is derived. The directions of Segrè are characterized as the directions in which the geodesics and dual geodesics coincide. Finally, the cubic curve, which is the locus of the ray-points of geodesics through a point of the surface, is studied. (Received October 27, 1942.)

### 81. S. M. Ulam: *On the length of curves, the surface area and the isoperimetric problem under a general Minkowski metric.* Preliminary report.

Given a metric in Euclidean space, defined through a symmetric convex unit gauge, one is led to the notion of length of curves through the usual process of polygonal approximation. The isoperimetric problem in the plane has a solution. A notion of surface area, invariant with respect to congruence under the given metric, is introduced. (Received November 23, 1942.)

### 82. J. E. Wilkins: *The first canonical pencil.*

It is the purpose of this paper to give several geometric definitions for a general canonical line of the first kind. Each such canonical line except the first axis of Čech may be defined in terms of the cusp-axes of the two families of hypergeodesics which are extremals of the integrals $\int \beta^n \gamma^{1-n} \varphi'^{2-3n} du$, $\int \gamma^n \beta^{1-n} \varphi'^{3n-1} du$, where $n$ is constant, or as the cusp-axis of a cone of class three which is defined by means of the osculating planes of these hypergeodesics. If at each point of the surface is introduced the triple of directions $\overline{D}_k$ conjugate to the directions $D_k$ considered by Bell (Duke Math. J.

vol. 5 (1939) pp. 784–788), the general canonical line as the line of intersection of the osculating planes of the projective geodesics in the directions $\overline{D}_k$ results. (Received October 2, 1942.)

### 83. Y. C. Wong: *Some Einstein spaces with conformally separable fundamental tensors.*

When the fundamental tensor ${}^*g_{\alpha\beta}$ of a Riemannian $m$-space is of the form ${}^*g_{ij}=[\rho(x^\alpha)]^{-2}g_{ij}(x^k)$, ${}^*g_{ip}=0$, ${}^*g_{pq}=[\sigma(x^\alpha)]^{-2}g_{pq}(x^r)$, $\alpha,\beta=1,\cdots,m$; $i,j,k=1,\cdots,n$; $p,q,r=n+1,\cdots,m$, it is said to be conformally separable; ${}^*g_{ij}$ and ${}^*g_{pq}$, with $x^r$ and $x^k$, respectively, as parameters, are called its component tensors. The author studies in this paper the conformally separable tensor which is the fundamental tensor of an Einstein $m$-space and each of whose component tensors either is of dimension less than three or is a family of fundamental tensors of Einstein spaces. It is found that the constructions of such a conformally separable tensor is invariably reduced to the construction of the fundamental tensor $g_{ij}$ of an Einstein $n$-space or a Riemannian 2-space for which the equation $y_{,ij}=-Ig_{ij}$ admits a non-constant solution for $y$, where the comma denotes covariant differentiation with respect to $g_{ij}$ and $I$ is an unspecified scalar. The author is content with this result, because the latter problem has already been considered in detail by H. W. Brinkmann in his study of Einstein spaces which are conformal to each other. (This paper will be published in the Trans. Amer. Math. Soc.) (Received October 2, 1942.)

## NUMERICAL COMPUTATION

### 84. H. E. Salzer and Abraham Hillman: *Exact values of the first 120 factorials.*

Due to their fundamental importance, the exact values of the first 120 factorials were computed and checked. 120! contains 199 digits. 100! agreed with Uhler's value (Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) p. 61). When these values were compared with Potin's table of the first 50 factorials (*Formules et tables numériques*, p. 836) errors were found in Potin's values for 18!, 38!, 45!, and 50!. (Received November 11, 1942.)

## STATISTICS AND PROBABILITY

### 85. J. H. Curtiss: *A note on the theory of moment generating functions.*

The moment generating function (m.g.f.) of a variate $X$ is defined as the mean value of exp $(\alpha X)$, the characteristic function (c.f.) as the mean value of exp $(itX)$, where $\alpha$ and $t$ are real. The purpose of this note is to place on record careful statements and proofs of the appropriate analogues for the m.g.f. of the well known uniqueness and limit theorems for the c.f For example, Levy's continuity theorem assumes the following form: Let $F_n(x)$ and $G_n(\alpha)$ be, respectively, the d.f. and m.g.f. of a variate $X_n$. If $G_n(\alpha)$ exists for $|\alpha|<\alpha_1$ and for all $n\geqq n_0$, and if there exists a function $G(\alpha)$ defined for $|\alpha|\leqq\alpha_2<\alpha_1$, $\alpha_2>0$, such that $\lim_{n\to\infty} G_n(\alpha)=G(\alpha)$ uniformly, $|\alpha|\leqq\alpha_2$, then there exists a variate $X$ with d.f. $F(x)$ such that $\lim_{n\to\infty} F_n(x)=F(x)$ uniformly in each finite interval of continuity of $F(x)$. The m.g.f. of $X$ exists for $|\alpha|\leqq\alpha_2$ and is equal to $G(\alpha)$ in that interval. (Received October 9, 1942.)

### 86. W. K. Feller: *On a probability limit theorem of H. Cramér.*

Let $\{X_k\}$ be mutually independent random variables whose first moments vanish and whose second moments are $\sigma_k^2$; let $s_n^2=\sigma_1^2+\cdots+\sigma_n^2$. In various applications one is concerned with the distribution function $Pr\{X_1+\cdots+X_n>xs_n\}$, where $x\rightarrow\infty$ as $n\rightarrow\infty$. The simplest binomial case has been studied by A. Khintchine, P. Lévy, Smirnoff and others. H. Cramér found a complete description of the asymptotic behavior of the above sums in the case where all $X_k$ have the same distribution function. This theorem is generalized to the case of unequal components. The theorem is to serve as a base for a solution of the "problem of the iterated logarithm" in the general case. (Received November 21, 1942.)

### 87. W. K. Feller: *On the general form of the so-called law of the iterated logarithm.*

Let $\{X_k\}$ be a sequence of mutually independent random variables whose first moments vanish and whose second moments are $\sigma_k^2$; let $s_n^2=\sigma_1^2+\cdots+\sigma_n^2$. A sequence of numbers $\phi_k\uparrow\infty$ is said to be of upper (lower) class if the probability that $X_1+\cdots+X_n>s_{n_k}\phi_{n_k}$ for infinitely many $k$ is one (zero); any sequence $\{\phi_k\}$ is either of upper or of lower class. A n.a.s. condition is found for a sequence $\{\phi_k\}$ to belong to the upper class. It generalizes the condition found by Kolmogoroff and Erdos in the special case where $X_k$ assumes the values $\pm 1$ only, each with probability 1/2, however, it is different in form. The new theorem also contains a result of Marcinkiewicz and Zygmund on the necessity of the condition imposed by Kolmogoroff on the $X_k$ in his proof of the law of the iterated logarithm. (Received November 21, 1942.)

### 88. P. G. Hoel: *On indices of dispersion.*

The sampling distribution of the index of dispersion for binomial and Poisson distributions is investigated by means of semi-invariants. Approximations to terms of order $N^{-3}$ are obtained for the descriptive moments of the distribution, by means of which the accuracy of the chi-square approximation can be determined. (Received October 30, 1942.)

## TOPOLOGY

### 89. R. F. Arens: *Homeomorphism groups of a space.* Preliminary report.

Let $A$ be a locally bicompact, locally connected Hausdorff space, and let $G$ be a group of homeomorphisms of $A$. Then there is a certain topology $N$ making $G$ into a topological group (Pontrjagin, *Topological groups*, Princeton, 1939). This topology $N$ is the weakest admissible topology that can be introduced into $G$, in this sense: Sets in $G$ open by $N$ are open by any other admissible topology $M$. A topology $M$ for a group of homeomorphisms, $G$, of a space $A$ is called admissible if by using that topology for $G$ the two functions $g(a)$ and $g^{-1}(a)$, where $g\in G$ and $a\in A$, become continuous functions of both arguments $g$ and $a$ simultaneously. The topology $N$ is determined by the following system of neighborhoods of the identity in $G$: Select in $A$ an open set $W$ whose closure is bicompact, and another open set whose closure $K$ is contained in $W$. Then the set $U$ of all $g\in G$ which transform $K$ into $W$ is defined to be a neighborhood of the identity. The set of all such $U$ together with all their finite intersections form a complete system of neighborhoods for the identity homeomorphism in $G$. (Received November 16, 1942.)

### 90. G. D. Birkhoff: *Measure-preserving transformations of a planar ring without planar periodic points.*

Let there be given a planar ring $R: a \leqq r \leqq b$ and a one-to-one continuous measure-preserving (that is, conservative) direct transformation $T$ of $R$ into itself, which has the same rotation number $\alpha$ along the circular boundaries $C_a$ and $C_b$, so that it is not possible to infer the existence of periodic point groups by the aid of Poincaré's last geometric theorem. The note discusses the conclusions which may be drawn if in fact there are no periodic points whatsoever, in the special case $T = RU$ ($R^2 = U^2 =$ identity); this special hypothesis as to the form of $T$ is satisfied in the transformation $T$ associated with the restricted problem of three bodies. Among other things, it is proved that there will then exist a qualitative integral $\Theta(P) =$ constant such that $\Theta[T(P)] = \Theta(P) + \alpha$ where $\Theta$ is continuous in the polar coordinates $(r, \theta)$ of $P$ and increases by $2\pi$ when $P$ makes a positive circuit of the ring $R$. (Received December 1, 1942.)

### 91. D. G. Bourgin: *On a theorem of Goldstine's.*

A new, topological proof is given for Goldstine's theorem (Duke Math. J. vol. 4 (1938), pp. 125–131) relating $w(E)$ completeness and reflexiveness in a Banach space. (Received November 24, 1942.)

### 92. D. G. Bourgin: *Quasi norms in linear topological spaces.*

It is proved that a continuous quasi norm can always be defined on a locally bounded linear topological space. This answers a question of Hyers (Revista de Ciencias vol. 4 (1939) pp. 558–574) in the affirmative. The multiplier of the quasi norm is not uniquely determined by the l.t.s., and it is shown by means of an example that there need be no "best" multiplier. (Received November 24, 1942.)

### 93. D. E. Christie: *Net homotopy for compacta.*

Let $R$ be a compactum embedded in a parallelotope $P$. Two mappings $f$ and $g$ of a topological space $S$ into $R$ are said to be neighborhood-homotopic if they are homotopic in every neighborhood of $R$ in $P$. The notion of neighborhood-homotopy leads to corresponding homotopy groups. These groups are independent of the particular embedding of $R$ in $P$ since they are equivalent to similar groups defined in terms of the net (sense of Lefschetz) of all open coverings of $R$. This equivalence is demonstrated by means of a study of the Kuratowski mappings of neighborhoods of $R$ into their nerves. The point of view adopted for homotopies is next applied to mappings. Both net and neighborhood definitions are developed; the former lead to easier proofs and wider applications, but the latter are more intuitive. The homotopy groups obtained from these new considerations are significant even for non-arcwise-connected spaces; they are suitably related to Čech homology even for spaces subject to no restrictions about local connectedness; yet they agree with the Hurewicz homotopy groups for absolute neighborhood-retracts. (Received November 23, 1942.)

### 94. H. S. M. Coxeter: *The map-coloring of unorientable surfaces.*

Heawood proved that every map on an unbounded surface of characteristic $K < 2$

can be colored with $[F_K]$ colors, where $F_K = (1/2)(7+(49-24K)^{1/2})$, and he conjectured that this number of colors is necessary as well as sufficient. Franklin disproved this conjecture in the case of the unorientable surface with $K=0$. On the other hand, the necessity of $[F_K]$ colors has been verified in many other cases by considering a map of this number of regions, each touching all the others; for example, Bose discovered a "balanced incomplete block design" which can be interpreted as a map of ten enneagons ($K=-5$). In the present paper this map is shown to be related to two of the stellated icosahedra in ordinary space; and a map of nine octagons ($K=-3$) is derived from a four-dimensional polytope. (Received November 5, 1942.)

### 95. Samuel Eilenberg and Saunders MacLane: *On a group construction by Hopf.*

In studying the influence of the fundamental group of a polyhedron on its second homology group H. Hopf (Comment. Math. Helv. vol. 14 (1942) pp. 257–309) constructs for each discrete group $G$ an abelian group $G_1^*$ and shows that when $G$ is the fundamental group, $G_1^*$ is a suitable quotient group of the second homology group. Hopf's definition of $G_1^*$ uses a representation of $G$ by means of generators and relations. In order to make the definition intrinsic the authors consider the central extensions of the group $P$ of reals reduced mod 1 by the group $G$. The equivalence classes of such extensions form a compact abelian group Ext Cent $\{P, G\}$ whose character group is shown to be naturally isomorphic with $G_1^*$. The result is based on a generalization of a previous result of the authors concerning abelian group extensions (see Bull. Amer. Math. Soc. abstract 48-1-94). (Received November 23, 1942.)

### 96. L. M. Graves: *Metrisation of weak convergence in Banach spaces.*

In the topology of weak convergence in a Banach space, derived sets need not be closed, so that weak convergence cannot in general be expressed in terms of a metric. In this paper a class of cases is characterized in which a simple explicit solution can be given for the problem of the metrisation of weak convergence in certain subsets of a Banach space or of its adjoint. (Received October 29, 1942.)

### 97. Karl Menger and S. G. Reed: *On a surface not intersecting the set $R_3^1$.*

By the sum theorem of dimension theory $R_3^1$, the set of all points of the 3-space with exactly one rational coordinate, is 0-dimensional. As O. Schreier remarked, $R_3^1$ intersects each surface $z=f(x, y)$ where $f$ is a continuous function defined on a square in the $x$, $y$-plane. The boundary of a neighborhood not intersecting $R_3^1$ can be constructed by topping a cube with steppyramids and iterating this process on each of the cubes of which the step consists. (Received October 28, 1942.)

### 98. A. N. Milgram: *A topologically invariant metric property of simple closed curves.*

Recently L. M. Blumenthal raised the question as to whether each simple closed curve contains three points which are vertices of an equilateral triangle. This question seems heretofore to have been unanswered even for closed plane curves. In this paper it is shown that: 1. Each simple closed plane curve contains the vertices of a triangle similar to any prescribed triangle. 2. Each closed polygon in euclidean $n$-space has the same property. (Received October 28, 1942.)

### 99. C. E. Rickart: *Decomposition of additive set functions.*

An additive set function $x(e)$ defined in a $\sigma$-field $\mathfrak{M}$ to a Banach space is said to satisfy property (A) provided $\lim_{n\to\infty} x(e_n)=0$, for any disjoint sequence $\{e_n\}\subset\mathfrak{M}$. If $x(e)$ satisfies (A) and $\Gamma$ is any $\sigma$-ideal contained in $\mathfrak{M}$, then there exists a unique decomposition of $x(e)$ of the form $x(e)=x_1(e)+x_2(e)$, where there exists $e_0\in\Gamma$ such that $x_1(e)=x_1(ee_0)$ and $x_2(e)=0$ for every $e\in\Gamma$. A consequence of this result is that, for an arbitrary outer measure $\mu(e)$ defined over $\mathfrak{M}$, every completely additive function $x(e)$ can be decomposed uniquely into the form $x(e)=x_1(e)+x_2(e)$, where $x_1(e)=x_1(ee_0)$, $\mu(e_0)=0$, and $x_2(e)$ is absolutely continuous relative to $\mu(e)$ (that is, $\lim_{n\to\infty}\mu(e_n)=0$ implies $\lim_{n\to\infty}x_2(e_n)=0$). (Received November 19, 1942.)

### 100. R. L. Swain: *Approximate isometries in bounded spaces.*

Approximate isometries have been studied by S. M. Ulam and D. H. Hyers (abstract 47-9-427). The present paper exhibits an example to show that results of the type which they obtained cannot generally be obtained in bounded spaces. Also the following theorem is proved: Let $M$ be a subset of a compact metric space $S$ and let $\eta$ be a positive number. Then there exists a positive number $\epsilon$ such that if $T$ is any $\epsilon$-isometry of $M$ (within $S$), there is an isometry $U$ of $M$ such that for each point $x$ of $M$, the distance between $T(x)$ and $U(x)$ is less than $\eta$. (Received October 30, 1942.)

### 101. A. D. Wallace: *On non-alternating transformations.*

The $H$-spaces $M$ and $N$ are assumed to be compact and connected and $\Delta$ will denote a continuous transformation from $M$ onto $N$. If $\Delta$ is non-alternating it is known that (a) for each $p$-chain $Y$ there exists a unique $p$-chain $X$ such that $Y\subset\Delta X$; (b) if $X$ is a chain then so also is $\Delta X$. To this may be added, if $M$ is locally connected (c) if $A$ and $B$ are intersecting chains then $\Delta(AB)=(\Delta A)(\Delta B)$. This is no longer true if the hypothesis of local connectivity is deleted. It may be shown ($\Delta$ non-alternating) that (d) if $a\sim b$ and a continuum meets both $\Delta^{-1}a$ and $\Delta^{-1}b$ then the continuum contains points $a_0$, $b_0$ with $a=\Delta a_0$, $b=\Delta b_0$ such that $a_0\sim b_0$; (e) if $K$ is a continuum and $\Delta^{-1}K=A+B$ is a separation then there are points $a_1$, $b_1$ in $A$ and $B$ such that $a_1\sim b_1$. If it is assumed that $\Delta$ satisfies (b), (d) and (e) then (a) holds even if $\Delta$ alternates. The proof of (c) may be based on the following characterization of a chain: In order that the closed set $C$ be a chain it is n.a.s. that it satisfy both (i) if $a$, $b$ are distinct points of $C$ and $a\sim x\sim b$ then $x$ is in $C$ and (ii) if $p$ separates the points $x$ and $y$ of $C$ in the space, then $p$ is in $C$. (Received October 26, 1942.)

### 102. G. S. Young: *Sets of axioms for the plane.* Preliminary report.

R. L. Moore has given several sets of axioms characterizing the plane, of which the Axioms 0–8 of his *Foundations of point set theory* (Amer. Math. Soc. Colloquium Publications vol. 13, 1932), are probably the most satisfactory. The purpose of this paper is to present several other sets of axioms also defining the plane, which postulate such properties as the two-sidedness of arcs, the non-existence of primitive skew curves, separation by open curves, non-separation by arcs, and so on. Axioms 0–2 of *Foundations* have been assumed throughout. The axiom sets considered divide into two classes: those in which the primary intent has been to state axioms equivalent to Moore's Axioms 3–5; and those in which the purpose was to furnish substitutes for Axioms 5–8. It is thought that these alternative approaches to the plane may prove useful in, for example, imbedding theorems. (Received October 28, 1942.)

# ON THE CONVERGENCE OF SEQUENCES OF LINEAR OPERATIONS

R. P. BAILEY

1. **Introduction.** Let $S$ be a sequence $\{U_n(x)\}(n=1, 2, \cdots)$ of linear operations defined over the elements $\{x\}$ of a Banach space[1] $E$ with values lying in a second $B$ space $E_1$. The fundamental convergence theorem associated with such sequences may be stated as follows:

BANACH-STEINHAUS THEOREM. *For the convergence of $S$ over $E$ it is necessary and sufficient that $S$ converge over a set $H$ dense in a sphere $K$ of $E$ and that the set of norms $\{|U_n|\}$ be bounded.*

The solution of convergence problems involving sequences of positive functionals defined on certain function-spaces[2] led the author to investigate a different form of this convergence criterion which seems to have a more direct practical application in some special cases of importance. The development hinges on the fact that from a certain point of view every linear operation has domains over which it is "positive."

2. **A new convergence criterion.** Let us make the following definition:

DEFINITION. *A set of points $P$ in a normed vector space will be called a p-set provided that, for every pair of points $z$ and $z'$ of $P$, $\|z-z'\| \leqq \|z+z'\|$.*

Such a set has many of the characteristic properties of a set of real numbers all of the same sign. In effect, we have extended the notion of "positiveness" to the elements of an abstract space.

In terms of this definition we may now state the following theorem:

THEOREM 1. *For the convergence of $S$ over $E$, it is sufficient that there exist a sphere $K$ of $E$ such that* (a) *$S$ converges over a set $H$ dense in $K$ and* (b) *for all fixed $n$ sufficiently large, the set of transforms $\{U_n(x)\}$ $(x \subset K)$, is a p-set, $P_n$.*

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[1] A space of type B. Cf. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 53. In general we follow the terminology of this treatise.

[2] R. P. Bailey, *Convergence of sequences of positive linear functional operations*, Duke Math. J. vol. 2 (1936) pp. 287–303.

Assuming the conditions of Theorem 1 are satisfied, it is possible to determine a sphere $K'\subset K$ of center $\xi\subset H$ and radius $r>0$. Let $\xi'=\xi/r$. Then, if $\|x\|\leqq 1$,

$$\|\xi - r(\xi' \pm x)\| = \|rx\| \leqq r,$$

so that $r(\xi'\pm x)\subset K'\subset K$. Hence, by our hypothesis, there exists a positive integer $N$, such that, for $n\geqq N$ and $\|x\|\leqq 1$,

$$P_n \supset U_n[r(\xi' \pm x)] = [U_n(r\xi') \pm U_n(rx)].$$

Setting $z=U_n(r\xi')+U_n(rx)$ and $z'=U_n(r\xi')-U_n(rx)$, since $\|z-z'\| \leqq \|z+z'\|$, we may conclude

$$\|U_n(x)\| \leqq \|U_n(\xi')\|, \qquad \|x\| \leqq 1,\ n \geqq N.$$

Thus $|U_n| \leqq \|U_n(\xi')\|$ $(n\geqq N)$.

But the sequence $S$ converges at $\xi$ and hence at $\xi'$. It follows that the set $\{\|U_n(\xi')\|\}$ $(n=1, 2, \cdots)$ is bounded, and therefore the set of norms $\{|U_n|\}$ is bounded. Thus the Banach-Steinhaus conditions are satisfied, and $S$ converges at all points of $E$. This completes the proof of Theorem 1.

As an illustration consider the sequences of so-called "positive" linear functional operations, that is, linear operations which associate real numbers $U(x)$ with the elements $x=x(t)$ of a function-space in such a way that $U(x)\geqq 0$ whenever $x(t)\geqq 0$. For instance let $\{\alpha_n(t)\}$ $(n=1, 2, \cdots)$ be a set of monotone non-decreasing functions defined on $(0, 1)$. The functionals $U_n(x)=\int_0^1 x(t)d\alpha_n(t)$ $(n=1, 2, \cdots)$ are positive linear functionals over the $B$ space $C$ consisting of the continuous functions $\{x(t)\}$ defined on $(0, 1)$ with norm $\|x\|=\max_{0\leqq t\leqq 1}|x(t)|$, and therefore obviously transform spheres of non-negative functions into the $p$-set consisting of the positive real numbers. It follows that the sequence $\{U_n(x)\}$ cannot converge over a set dense in a sphere of $C$ without converging over the whole of $C$.

It should not be assumed, however, that the only practical value of the criterion lies in its application to sequences of positive functionals. All of the better known Banach spaces contain easily recognized $p$-sets of a very extensive character. We may point, for instance, to the set of non-negative functions of $C$ space, the set of points $x=(x_1, x_2, \cdots, x_n)$ of Euclidean $n$-space such that $x_i\geqq 0$ $(i=1, 2, \cdots, n)$, and the set of functions $\{x(t)\}$ Lebesgue square integrable on $(0, 1)$ with the norm $\|x\| = [\int_0^1 x^2dt]^{1/2}$ which consists of those functions having non-negative Fourier coefficients. In fact, we shall prove in what follows that the sphere with arbitrary center $z_0$ and radius $\|z_0\|/2$ in any Banach space is a $p$-set.

In other words, the real value of the criterion lies in the fact that it is frequently obvious at a glance that certain spheres in $E$ transform into $p$-sets in $E_1$, while the boundedness of the norms may be difficult to establish.

3. **Necessity of the criterion.** In an abstract of this paper[3] the author made an error in stating that the conditions of Theorem 1 were necessary for convergence, as well as sufficient. The question of necessity is clarified in this section. We shall assume the Banach-Steinhaus conditions are satisfied, and ask under what circumstances it is necessary that some sphere of $E$ transform into $p$-sets in $E_1$.

By hypothesis there exists a number $M>0$ such that $|U_n|\leqq M$ for all $n$, and $S$ converges at all points of $E$ to a linear operation $U(x)$. It is convenient to divide the discussion into two parts, depending upon whether or not $U(x)$ vanishes identically over $E$. Denote by $\theta_1$ the zero element of $E_1$.

Case 1. ($U(x)\not\equiv\theta_1$.) Choose $\xi\subset E$ such that $U(\xi)\neq\theta_1$. Since $\|U(\xi)\|>0$, positive numbers $\epsilon>0$, $r>0$ may be chosen such that $\epsilon+rM\leqq\|U(\xi)\|/2$. If $x$ is any point of the sphere $K$ defined by the condition $\|\xi-x\|\leqq r$, and $N$ be so chosen that $\|U(\xi)-U_n(\xi)\|\leqq\epsilon$ $(n\geqq N)$, it follows that for $n\geqq N$ and $\|\xi-x\|\leqq r$,

$$\begin{aligned}\|U(\xi)-U_n(x)\| &\leqq \|U(\xi)-U_n(\xi)\|+\|U_n(\xi-x)\|\\ &\leqq \epsilon+rM\\ &\leqq \|U(\xi)\|/2.\end{aligned}$$

That is, for $n\geqq N$, and $\|\xi-x\|\leqq r$, all transforms $\{z=U_n(x)\}$ lie in a sphere of $E_1$ defined by the condition

$$\|U(\xi)-z\|\leqq\|U(\xi)\|/2.$$

Moreover this spherical set is a $p$-set, for if $z$ and $z'$ are any two of its points,

$$\begin{aligned}\|z-z'\| &\leqq \|z-U(\xi)\|+\|U(\xi)-z'\|\\ &\leqq \|U(\xi)\|,\end{aligned}$$

while

$$\begin{aligned}\|z+z'\| &= \|z-U(\xi)+z'-U(\xi)+2U(\xi)\|\\ &\geqq 2\|U(\xi)\|-\|z-U(\xi)\|-\|z'-U(\xi)\|\\ &\geqq 2\|U(\xi)\|-\|U(\xi)\|\\ &\geqq \|U(\xi)\|.\end{aligned}$$

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[3] Bull. Amer. Math. Soc. abstract 43-3-97.

That is, the criterion of Theorem 1 is satisfied. We have proved this theorem:

THEOREM 2. *The conditions of Theorem* 1 *are necessary for the convergence of* $S$ *over* $E$ *to a linear operation* $U(x)$ *which does not vanish identically.*

Case 2. ($U(x)\equiv\theta_1$.) An example will be furnished to show that no necessity exists in this case. Let $E$ consist of the set of continuous functions $\{x(t)\}$ defined on $(0, 1)$ with $x(0)=0$. We use the customary norm $\|x\| = \max_{0\leq t\leq 1}|x(t)|$. If $U_n(x) = n\int_0^{1/n}x(t)dt$ $(n=1, 2, \cdots)$, clearly $\lim_{n\to\infty} U_n(x)=0$ at all points of $E$. Nevertheless, it is impossible to choose $N>0$, and a sphere $K$ of $E$, such that for fixed $n\geq N$ and $x\subset K$, the transforms $\{U_n(x)\}$ constitute $p$-sets $P_n$. For, suppose $K$ be any arbitrary sphere $\|x_0-x\|\leq r$ with center $x_0$. Since $x_0(0)=0$, it is possible to determine a point $t_0$ $(0<t_0<1)$ such that $\max|x_0(t)|$ $(0\leq t\leq t_0)$ is arbitrarily small. Clearly, if $t_0$ is sufficiently small, there exist functions $x_1(t)$ and $x_2(t)$ having the following properties:

$$\text{(i)}\qquad x_1(t)>0,\qquad x_2(t)<0,\qquad 0<t\leq t_0,$$

$$\text{(ii)}\qquad |x_0(t)-x_1(t)|\leq r,\qquad |x_0(t)-x_2(t)|\leq r,\qquad 0\leq t\leq 1.$$

But now, if $n\geq 1/t_0$, $U_n(x_1)>0$, while $U_n(x_2)<0$. The set of transforms $\{U_n(x)\}$ $(x\subset K)$ therefore cannot form a $p$-set, no matter what sphere $K$ is selected, if $n$ is sufficiently large.

4. **Convergence on a locally compact** $B$ **space.** We have shown in the preceding section that the conditions of Theorem 1 need not be satisfied in all cases to obtain convergence. However, it is possible to establish the following criterion:

THEOREM 3. *For the convergence of* $S$ *over a* $B$ *space* $E$ *which is locally compact, the conditions of Theorem* 1 *are necessary and sufficient.*[4]

We have only to demonstrate necessity in the case $U(x)\equiv\theta_1$. The remainder follows at once from Theorems 1 and 2. We shall need the following lemma.

LEMMA. *A sequence* $\{F_n(x)\}$ $(n=1, 2, \cdots)$ *of linear operations defined over a locally compact* $B$ *space* $E$ *may not possess both of the following characteristics:*

$$\text{(i)}\qquad |F_n|=1,\qquad n=1, 2, \cdots,$$

$$\text{(ii)}\qquad \liminf_{n\to\infty}\|F_n(x)\|=0\qquad \textit{for all } x \textit{ of } E.^{5}$$

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[4] We may mention in passing that the property of local compactness holds if, and only if, the space has a finite number of dimensions.

[5] Note that the operations $U_n(x)=n\int_0^{1/n}x(t)dt$ defined in §3 do possess both properties.

Suppose $\{G_n(x)\}$ $(n=1, 2, \cdots)$ is any infinite selection of the operations $\{F_n(x)\}$. Let $\epsilon>0$ be any arbitrarily specified real positive number. For each $n=1, 2, \cdots$ there exists a point $\xi_n \subset E$ such that $\|\xi_n\| \leqq 1$ and $\|G_n(\xi_n)\| \geqq 1-\epsilon$. Since the unit sphere is compact, it is possible to extract from the sequence $\{\xi_n\}$ a subsequence $\{\xi_{n_k}\}$ which converges to a point $\xi$ of $E$. For $n_k$ sufficiently large, $\|\xi-\xi_{n_k}\| \leqq \epsilon$. Hence, for all $n_k$ sufficiently large,

$$\begin{aligned} \|G_{n_k}(\xi)\| &= \|G_{n_k}(\xi_{n_k}) + G_{n_k}(\xi - \xi_{n_k})\| \\ &\geqq \|G_{n_k}(\xi_{n_k})\| - \|G_{n_k}(\xi - \xi_{n_k})\| \\ &\geqq 1 - 2\epsilon, \end{aligned}$$

and therefore $\limsup_{n\to\infty} \|G_n(\xi)\| \geqq 1$. Since $\{G_n(x)\}$ is any arbitrary infinite selection of the operations $\{F_n(x)\}$, this contradicts (ii).

Now consider any arbitrary sequence $\{U_n(x)\}$ defined over a locally compact space $E$, such that $\lim_{n\to\infty} U_n(x) = U(x)$ vanishes at all points of $E$. We wish to show that there exists a sphere $K$ of $E$ such that $\{U_n(x)\}$ $(x \subset K)$ constitutes a $p$-set $P_n$, for all $n$ sufficiently large.

The contradomain of an operation $U_n(x) \equiv \theta_1$ $(x \subset E)$ is obviously a $p$-set. Hence consider only the subsequence $\{U_{n_k}(x)\}$ of $\{U_n(x)\}$ consisting of those operations which do not vanish identically over[4] $E$. Since $|U_{n_k}| > 0$, we may set

$$F_{n_k}(x) = |U_{n_k}|^{-1} \cdot U_{n_k}(x), \qquad x \subset E.$$

Since $|F_{n_k}| = 1$ $(k=1, 2, \cdots)$, it is impossible that $\liminf_{k\to\infty} \|F_{n_k}(x)\| = 0$ at all points of $E$, by the previously proved lemma. Hence there exists a point $\xi \neq \theta$, and a number $r>0$, such that

$$\liminf_{k\to\infty} |U_{n_k}|^{-1} \cdot \|U_{n_k}(\xi)\| > 2r > 0.$$

Therefore it is possible to fix $N$ so large, that for $n_k > N$,

$$|U_{n_k}|^{-1} \cdot \|U_{n_k}(\xi)\| \geqq 2r,$$

or

$$r|U_{n_k}| \leqq \|U_{n_k}(\xi)\|/2.$$

Now, if $\|x-\xi\| \leqq r$, and $n_k > N$,

$$\|U_{n_k}(x) - U_{n_k}(\xi)\| \leqq r|U_{n_k}| \leqq \|U_{n_k}(\xi)\|/2.$$

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[4] We assume the subsequence is infinite; in the contrary case nothing remains to be proved.

Since $\|U_{n_k}(\xi)\|>0$, the sphere $\|z-U_{n_k}(\xi)\|\leqq\|U_{n_k}(\xi)\|/2$, is nonvacuous. That such a sphere is a $p$-set was demonstrated in §3. The sphere $K$ we were required to construct has therefore been shown to exist, and Theorem 3 is proved.

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# ON THE APPROXIMATION OF FUNCTIONS BY SUMS OF ORTHONORMAL FUNCTIONS

EDWIN N. OBERG

1. **Introduction.** The main object of this paper is to derive, in a simple manner, upper bounds for the norms of the derivative of

$$\sum_{i=0}^{n} a_i\phi_i(x) \tag{1}$$

in $C$ and $L^2$ spaces, where the $a_i$ are arbitrary constants, and $\{\phi_i(x)\}$ is any set of functions on a given finite or infinite interval $(a, b)$. We apply our method, properly modified, first to the case where the $\phi_i(x)$ are characteristic solutions of conjugate sets of integral equations, then to other classes of functions whose first derivatives $\{\phi_i'(x)\}$ are orthogonal with respect to a weight function $\sigma(x)$. Finally, we apply our results to the question of convergence of sums[1] of type (1) that minimize

$$\int_a^b \rho(x)\left|f(x)-\sum_{i=0}^{n} a_i\phi_i(x)\right|^m dx, \qquad m>0.$$

The leading results of our investigation may be summarized briefly as follows:

$$\left|\frac{d}{dx}\sum_{i=0}^{n} a_i\phi_i(x)\right| \leqq \lambda_n k(x)\left(\int_a^b \Big[\sum_{i=0}^{n} a_i\phi_i(x)\Big]^2 dx\right)^{1/2} \tag{A}$$

where $\lambda_n$ is a positive number that increases with $n$ and $k(x)$ is a func-

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[1] For the specialized cases when the approximating functions are trigonometric sums or polynomials, see D. Jackson, *The theory of approximation*, Amer. Math. Soc. Colloquium Publications vol. 11, 1930, pp. 86–89, 96–101.

tion, independent of $n$, that is less than or equal to a bound $M$ almost everywhere.

$$\left(\int_a^b \sigma(x)\left[\frac{d}{dx}\sum_{i=0}^{n} a_i\phi_i(x)\right]^2 dx\right)^{1/2} \leq N_n\left(\int_a^b \left[\sum_{i=0}^{n} a_i\psi_i(x)\right]^2 dx\right)^{1/2} \tag{B}$$

where $\{\psi_i(x)\}$ may be *any* set of orthonormal functions on $(a, b)$, and $N_n$, for a given $n$ and arbitrary coefficients $a_i$, is the least number that relates the left- and right-hand sides of (B).

Somewhat similar investigations have been made by Hille, Szegö, and Tamarkin,[2] McEwen,[3] and Shohat[4] but in each case with more restricted functions and by methods that are different from ours.

**2. Rough bounds for the derivative of sums of characteristic solutions.** Let $K(x, t)$ be a given continuous function on an interval $a \leq x \leq b$, $a \leq \tau \leq b$, and let

$$\phi(x) = \lambda\int_a^b K(x, t)\psi(t)dt, \qquad \psi(x) = \lambda\int_a^b K(t, x)\phi(t)dt \tag{2}$$

be conjugate integral equations which have $K(x, t)$ as kernel. It is well known[5] that, except for the special case when $K(x, t) = \sum_{i=0}^{n}\phi_i(x)\psi_i(t)/\lambda_i$, there exists an infinite set of characteristic values for $\lambda$, and corresponding sets of characteristic functions,

$$\phi_0(x),\ \phi_1(x),\ \cdots,\ \phi_n(x),\ \cdots,\ \psi_0(x),\ \psi_1(x),\ \cdots,\ \psi_n(x),\ \cdots$$

that satisfy these equations. The set of functions $\{\psi_i(x)\}$ can be adjusted so that all of the characteristic numbers are positive and it can be assumed without loss of generality that $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots$. Moreover, the sets $\{\phi_i(x)\}$ and $\{\psi_i(x)\}$ can be regarded as separately orthonormal on $(a, b)$. In addition to the hypotheses of continuity on $K(x, t)$ we assume differentiability with respect to $x$ under the integral sign of $\int_a^b K(x, t)dt$ and that

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[2] Hille, Szego, and Tamarkin, *On some generalizations of a theorem of A. Markoff*, Duke Math. J. vol. 3 (1937) pp. 729–739.

[3] W. H. McEwen, *A note on an extension of Bernstein's theorem*. Amer. J. Math. vol. 60 (1938) pp. 309–319.

[4] J. Shohat. *On a general formula in the theory of Tschebycheff polynomials and its application*, Trans. Amer. Math. Soc. vol. 29 (1927), p. 569.

[5] See E. Schmidt. *Zur theorie der linearen und nichtlinearen Integralgleichungen*, Math. Ann. vol. 63 (1907) pp. 459–463.

$$\int_a^b \left[\frac{\partial}{\partial x} K(x, t)\right]^2 dt$$

as a function of $x$ exists almost everywhere on $(a, b)$.

Let $a_0, a_1, \cdots, a_n$ be any set of coefficients and let

$$s_n(x) = \sum_{i=0}^{n} a_i\phi_i(x), \qquad \mathfrak{s}_n(x) = \sum_{i=0}^{n} a_i\psi_i(x), \qquad \mathfrak{s}_{n\lambda}(x) = \sum_{i=0}^{n} a_i\lambda_i\psi_i(x).$$

From (2) it follows that

$$\frac{d}{dx} s_n(x) = \int_a^b \frac{\partial}{\partial x} K(x, t)\mathfrak{s}_{n\lambda}(t)dt$$

whence, as a consequence of Schwarz's inequality, we have

$$\left|\frac{d}{dx} s_n(x)\right| \leqq \left(\int_a^b \left[\frac{\partial}{\partial x} K(x, t)\right]^2 dt\right)^{1/2} \left(\int_a^b \mathfrak{s}_{n\lambda}^2(t)dt\right)^{1/2}.$$

By the property of orthonormality of the set $\{\psi_i(x)\}$, and the fact that the $\lambda$'s increase with $n$,

$$\int_a^b \mathfrak{s}_{n\lambda}^2(t)dt = \sum_{i=0}^{n} a_i^2\lambda_i^2 \leqq \lambda_n^2 \sum_{i=0}^{n} a_i^2 = \lambda_n^2 \int_a^b \mathfrak{s}_n^2(t)dt.$$

The above relation, in view of the fact that the $a$'s are arbitrary, is the best possible bound for $\int_a^b \mathfrak{s}_{n\lambda}^2(t)dt$ in terms of $\int_a^b \mathfrak{s}_n^2(t)dt$ since the equality holds when $a_n$ is the only nonzero coefficient. However, the last integral may be replaced by the integral of the square of a sum of any $n+1$ orthonormal functions on the interval $(a, b)$ if the coefficients of this sum are $a_0, a_1, \cdots, a_n$. In particular, $\int_a^b \mathfrak{s}_n^2(t)dt$ may be replaced by $\int_a^b s_n^2(t)dt$.

The previous discussion may be summarized as follows.

THEOREM A. *If the kernel $K(x, t)$ satisfies the hypotheses of the preceding discussion, and if $s_n(x)$ is any arbitrary sum of $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$, then*

$$\left|\frac{d}{dx} s_n(x)\right| \leqq M\lambda_n \left(\int_a^b s_n^2(t)dt\right)^{1/2}$$

*where $M$ is a constant, over the parts of the interval $(a, b)$ for which $\int_a^b [\partial K(x, t)/\partial x]^2 dt$ is bounded. Moreover, if $(a, b)$ is finite and $u_n$ is the maximum of $|s_n(x)|$ on $(a, b)$, then*

$$\left|\frac{d}{dx} s_n(x)\right| \leqq M'\lambda_n u_n, \qquad M' = M(b - a)^{1/2}. \tag{3}$$

For special sets of characteristic functions and kernels, closer bounds then (3) are obtainable by other means[6] than ours. Thus, for example, if we consider the particular kernel[7]

$$
\begin{aligned}
K(x, t) &= x(1 - t), & 0 \leqq x \leqq t, \\
&= t(1 - x), & t \leqq x \leqq 1,
\end{aligned}
$$

on the interval $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, the integral equation $u(x) = \lambda \int_0^1 K(x, t) u(t) dt$ has as characteristic solutions and corresponding characteristic numbers the sets $\{\sin n\pi x\}$ and $\{n^2\pi^2\}$, respectively. The application of (3) gives a bound of order $n^2 u_n$ for the derivative of a sum of sine functions, which is rough when compared with the well known Bernstein's theorem for trigonometric sums.[8] However, the bound given by (3) is of interest since it applies to a very wide class of functions for which it seems impossible to obtain better results unless more restrictive hypotheses are assumed on the kernel or on the characteristic functions themselves.

3. **Bernstein's theorem in $L^2$ space for a sum of functions whose derivatives are orthogonal with respect to a weight function.** Hille, Szegö, and Tamarkin[9] have established the corresponding Markoff's theorem for $L^m$ spaces when $m \geqq 1$, but only when the sums involved are polynomials or trigonometric sums. Our method of proof is entirely different from theirs and we are not able to obtain the results of this section from their paper.

Let $\{\phi_i(x)\}$ be any set of functions, not necessarily orthonormal, but let the set of first derivatives of these functions be orthogonal with respect to a positive weight function $\sigma(x)$ on the interval $(a, b)$. Assume that the normalization factor $N_i = (\int_a^b \sigma(x) [\phi_i'(x)]^2 dx)^{1/2}$ increases with the subscript $i$. Let $K_n(x, t) = \sum_{i=0}^n \phi_i(x) \psi_i(t)$ where $\{\psi_i(x)\}$ is any set of orthonormal functions on $(a, b)$. Let $s_n(x)$ denote any arbitrary sum of the functions $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$, and let $s_n'(x)$ be its first derivative. Let $\mathfrak{s}_n(x) = \sum_{i=0}^n a_i \psi_i(x)$ where $a_0, a_1, \cdots, a_n$ are the same coefficients as in $s_n(x)$.

Since

$$
s_n'(x) = \int_a^b \frac{\partial}{\partial x} K_n(x, t) \mathfrak{s}_n(t) dt
$$

---

[6] See McEwen, op. cit. pp. 295–296.

[7] The kernel $K(x, t)$ in this instance is the Green's function of the system $d^2u/dx^2 + \lambda u = 0$, $u(0) = u(1) = 0$, and thus highly restricted when compared with any kernel of a system of form (2).

[8] See Jackson, op. cit. p. 80.

[9] Hille, Szegö, and Tamarkin, op. cit. Footnote 2.

it follows from the hypotheses placed on the sets $\{\phi_i'(x)\}$, $\{N_i\}$, and $\{\psi_i(x)\}$ that if this equation is multiplied by $\sigma(x)s_n'(x)$ and integrated over $(a, b)$, then

$$\begin{aligned}\int_a^b \sigma(x)\,[s_n'(x)]^2dx &= \int_a^b \left[\int_a^b \sigma(x)s_n'(x)K_n'(x,t)dx\right]\mathfrak{s}_n(t)dt\\ &= \int_a^b \left[\sum_{i=0}^n a_iN_i^2\psi_i(t)\right]\left[\sum_{i=0}^n a_i\psi_i(t)\right]dt\\ &= \sum_{i=0}^n a_i^2N_i^2 \leqq N_n^2\sum_{i=0}^n a_i^2 = N_n^2\int_a^b \mathfrak{s}_n^2(t)dt.\end{aligned}$$

The factor $N_n^2$ is obviously the least number that relates the right- and left-hand members of the above inequality since the equality holds when all but $a_n$ of the coefficients $a_1, a_2, \cdots, a_n$ are equal to zero. As a result we have this theorem.

THEOREM B. *If* $\{\phi_i(x)\}$ *is a given set of functions, and if there exists a positive weight function* $\sigma(x)$ *such that the set of functions*

$$\sigma^{1/2}(x)\phi_0'(x),\ \sigma^{1/2}(x)\phi_1'(x),\ \cdots,\ \sigma^{1/2}(x)\phi_n'(x)$$

*are orthogonal on* $(a, b)$ *and if* $s_n(x)=\sum_{i=0}^n a_i\phi_i(x)$ *is any arbitrary sum of the set* $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$ *and* $\mathfrak{s}_n(x)=\sum_{i=0}^n a_i\psi_i(x)$ *is the sum of any arbitrary set of orthonormal functions on the interval* $(a, b)$, *the coefficients* $a_0, a_1, \cdots, a_n$ *being the same in both* $s_n(x)$ *and* $\mathfrak{s}_n(x)$, *then*

$$\left(\int_a^b \sigma(x)\,[s_n'(x)]^2dx\right)^{1/2} \leqq N_n\left(\int_a^b [\mathfrak{s}_n(x)]^2dx\right)^{1/2} \tag{4}$$

*gives the closest relation between the integrals*

$$\left(\int_a^b \sigma(x)\,[s_n'(x)]^2dx\right)^{1/2} \quad \textit{and} \quad \left(\int_a^b [\mathfrak{s}_n(x)]^2dx\right)^{1/2} \textit{when}$$

$$N_n = \left(\int_a^b \sigma(x)\,[\phi_n'(x)]^2dx\right)^{1/2}.$$

As a first application of the above theorem consider the case when both of the sets $\{\phi_i(x)\}$ and $\{\psi_i(x)\}$ are the normalized Legendre[10] polynomials $\{(2i+1/2)^{1/2}P_i(x)\}$. The set of first derivatives of these functions are orthogonal[11] with respect to the weight function $1-x^2$

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[10] See, for example, D. Jackson, *Fourier series and orthogonal polynomials*, The Carus Monograph no. 6, pp. 45–68.

[11] See Jackson op. cit. Footnote 10 p. 149.

on the interval $(-1, 1)$ so that the hypothesis on $\{\phi_i'(x)\}$ holds in accord with the above theorem. Since

$$N_n^2 = \int_{-1}^{+1}(1-x^2)[P_n'(x)]^2dx = \frac{2n+1}{2}\,\frac{2(n+1)!}{(n-1)!}\cdot\frac{1}{2n+1} = n(n+1)$$

we have[12]

$$\left(\int_{-1}^{+1}(1-x^2)[s_n'(x)]^2dx\right)^{1/2} \leqq (n(n+1))^{1/2}\left(\int_{-1}^{+1}s_n^2(x)dx\right)^{1/2}$$

$$= O(n)\left(\int_{-1}^{+1}s_n^2(x)dx\right)^{1/2}.$$

Similar results can be obtained with other Jacobi, the Laguerre, and the Hermite polynomials.[13] In the last instance we take $\{\phi_i(x)\} = \{H_i(x)/(2\pi)^{1/4}(i!)^{1/2}\}$ and $\{\psi_i(x)\} = \{e^{-x^2/4}\ H_i(x)/(2\pi)^{1/4}(i!)^{1/2}\}$. Since $H_i'(x) = iH_{i-1}(x)$, then $\sigma(x) = e^{-x^2/2}$ makes the set of functions $\{\sigma^{1/2}(x)\phi_i'(x)\}$ orthogonal on $(-\infty, +\infty)$. The normalization factor $N_n = (\int_a^b\sigma(x)[\phi_n'(x)]^2dx)^{1/2} = n^{1/2}$ so that

$$\left(\int_{-\infty}^{+\infty}e^{-x^2/2}[s_n'(x)]^2dx\right)^{1/2} \leq n^{1/2}\left(\int_{-\infty}^{+\infty}e^{-x^2/2}[s_n(x)]^2dx\right)^{1/2}.$$

Another example is of interest in that it gives an application of the above theorem to functions which do not involve polynomials. Let $J_0(\mu_i x)$ denote the Bessel functions of zero order where $\{\mu_i\}$ are the roots[14] of the Bessel functions of the first order, $J_1(\mu) = 0$. Let $\phi_i(x) = J_0(\mu_i x)/k_i$ and $\psi_i(x) = x^{1/2}J_0(\mu_i x)/k_i$, where $k_i^2 = \int_0^1 xJ_0^2(\mu_i x)dx = J_0^2(\mu_i)/2$. Since $J_0'(\mu_i x) = -\mu_i J_1(\mu_i x)$ it follows that the set of derivatives, $\{\phi_i'(x)\}$, are orthogonal on the interval $(0, 1)$ with respect to the weight function $\sigma(x) = x$. Thus

$$\int_0^1 x\phi_i'(x)\phi_j'(x)dx = (1/k_ik_j)\int_0^1 xJ_0'(\mu_i x)J_0'(\mu_j x)dx$$

$$= (\mu_i\mu_j/k_ik_j)\int_0^1 xJ_1(\mu_i x)J_1(\mu_j x)dx = 0$$

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[12] It is of interest to compare this result with Bernstein's theorem for polynomials in $C$ space. See D. Jackson, *Bernstein's theorem and trigonometric approximation*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 225–226.

[13] See Jackson op. cit. Footnote 10 p. 225 Problem 5; p. 227 Problem 7; p. 180 Formula 6.

[14] See, for example, Watson, *Theory of Bessel functions*, pp. 477–521, 576–596. We are making use of the well known formulas, $\int_0^1 xJ_k^2(\mu x)dx = (1/2)[J_k^2(\mu)+J_{k+1}^2(\mu)] - (k/\mu)[J_k(\mu)J_{k+1}(\mu)]$ and $\int_0^1 xJ_k(\mu_i x)J_k(\mu_j x)dx = [J_k(\mu_j)J_k'(\mu_i) - J_k(\mu_i)J_k'(\mu_j)]/(\mu_j^2-\mu_i^2)$ when $k=0$ and $k=1$.

when $i\neq j$. The normalization factor of $x^{1/2}J_1(\mu_n x)$ is $|J_2(\mu_n)|/2^{1/2}$ so that $N_n^2=\int_0^1 x[\phi_n'(x)]^2dx=\mu_n^2J_2^2(\mu_n)/J_0^2(\mu_n)$. But by the well known[15] identity, $(2/x)J_1(x)=J_0(x)+J_2(x)$, it is seen that when $\mu_n$ is a root of $J_1(\mu)=0$ then $J_0(\mu_n)=-J_2(\mu_n)$ so that the right-hand side of the above equation is $\mu_n^2$ which is known[16] to be of the order $n^2$. Hence if

$$s_n(x)=\sum_{i=0}^{n}a_i\phi_i(x)=\sum_{i=0}^{n}a_i\frac{J_0(\mu_i x)}{k_i},$$

it follows that

$$\left(\int_0^1 x[s_n'(x)]^2dx\right)^{1/2}\leqq\mu_n\left(\int_0^1 x[s_n(x)]^2dx\right)^{1/2}$$
$$=O(n)\left(\int_0^1 x[s_n(x)]^2dx\right)^{1/2}.$$

4. **Degree of convergence of sums of characteristic solutions.** Let $\{\phi_i(x)\}$ and $\{\psi_i(x)\}$ be closed infinite sets of orthonormal solutions of a system of integral equations of form (2) in which the kernel $K(x,t)$ is continuous[17] on an interval $a\leqq x\leqq b$, $a\leqq t\leqq b$. Let the corresponding characteristic numbers be positive and arranged as a nondecreasing sequence with respect to the subscript $i$. Let $f(x)$ be a continuous function on $a\leqq x\leqq b$ and let

$$\sum_{i=1}^{\infty}\left(\lambda_i^p\int_a^b f(x)\phi_i(x)dx\right)^2$$

be convergent for some value of $p\geqq 1$. Then sufficient hypotheses have been assumed so that the equation of the first kind

$$f(x)=\int_a^b K(x,t)u(t)dt \tag{5}$$

has a solution $u(x)$ in $L^2$ space.[18]

Let $b_0, b_1, \cdots, b_n$ be arbitrary coefficients and let

$$\tau_{n\lambda}(x)=\frac{b_0\phi_0(x)}{\lambda_0}+\frac{b_1\phi_1(x)}{\lambda_1}+\cdots+\frac{b_n\phi_n(x)}{\lambda_n},$$
$$\bar{\tau}_n(x)=b_0\psi_0(x)+b_1\psi_1(x)+\cdots+b_n\psi_n(x).$$

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[15] See Watson op. cit. p. 17. Take $n=1$.

[16] See Watson op. cit. p. 506. Take $\nu=0$, $\alpha=0$.

[17] This hypotheses on $K(x,t)$ can be lightened to include kernels that are summable and of summable square. See Footnote 18.

[18] See E. Picard, *Sur un théorème général relatif aux équations intégrales de première espèce et sur quelques problèmes de physiques mathématique*, Rend. Circ. Math. Palermo vol. 29 (1930) pp. 79–83.

By (2) it follows that $\tau_{n\lambda}(x) = \int_a^b K(x, t)\bar{\tau}_n(t)dt$ so that by subtracting $\tau_{n\lambda}(x)$ from $f(x)$ we have with the aid of Schwarz's inequality,

$$(6) \qquad |f(x) - \tau_{n\lambda}(x)| \leqq \left(\int_a^b K^2(x, t)dt\right)^{1/2}\left(\int_a^b [u(t) - \bar{\tau}_n(t)]^2 dt\right)^{1/2}.$$

The above expression shows that any set of the $b$'s that will make $\bar{\tau}_n(t)$ converge in the mean to $u(t)$ will make $\tau_{n\lambda}(x)$ converge uniformly to $f(x)$ as $n$ becomes infinite. In particular, if $b_i = \int_a^b u(t)\psi_i(t)dt$, then

$$\int_a^b [u(t) - \bar{\tau}_n(t)]^2 dt = \sum_{i=n+1}^{\infty}\left(\int_a^b u(t)\psi_i(t)dt\right)^2$$
$$= \sum_{i=n+1}^{\infty}\left(\lambda_i \int_a^b f(t)\phi_i(t)dt\right)^2.$$

But this in turn is at most equal to

$$(7) \qquad \frac{1}{\lambda_{n+1}^{2p-2}} \sum_{i=n+1}^{\infty}\left(\lambda_i^p \int_a^b f(t)\phi_i(t)dt\right)^2$$

for any $p \geqq 1$ when $n$ is sufficiently large so that $\lambda_{n+1} > 1$.

From (6) and (7) we have the following consequence.

THEOREM C. *If* $\{\phi_i(x)\}$ *and* $\{\psi_i(x)\}$ *are closed infinite sets of orthonormal solutions of the system of equations*

$$\phi(x) = \lambda \int_a^b K(x, t)\psi(t)dt, \qquad \psi(x) = \lambda \int_a^b K(t, x)\phi(t)dt$$

*corresponding to a set of positive characteristic numbers* $\lambda_0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_n \leqq \cdots$, *and if* $K(x, t)$ *is continuous on* $a \leqq x \leqq b$, $a \leqq t \leqq b$, *then, if* $f(x)$ *is continuous on* $a \leqq x \leqq b$ *and*

$$\sum_{i=0}^{\infty}\left(\lambda_i^p \int_a^b f(x)\phi_i(x)dx\right)^2$$

*is convergent for some* $p \geqq 1$, *there exist sums* $q_n(x)$ *of the set* $\{\phi_i(x)\}$ *such that*

$$|f(x) - q_n(x)| \leqq \frac{I_1 \nu_n}{\lambda_n^{p-1}}$$

*where* $I_1$ *is an absolute constant and* $\lim_{n\to\infty} \nu_n = 0$.

In connection with Theorem C a further comment is of interest if in (5) we can differentiate $f(x)$ and $\int_a^b K(x, t)dt$ $s$ times, the latter under the integral sign, and if the set of functions $\{\phi^{(s)}(x)\}$ are orthogonal with respect to a positive weight function $\sigma(x)$ on $(a, b)$. In that case it follows if $f^{(s)}(x)$ is multiplied by $\sigma(x)\phi_i^{(s)}(x)$ and integrated over $(a, b)$,

$$\int_a^b \sigma(x)f^{(s)}(x)\phi_i^{(s)}(x)dx = \int_a^b \left[\int_a^b \sigma(x)K^{(s)}(x, t)\phi_i^{(s)}(x)dx\right]u(t)dt$$

$$= \frac{\overline{N}_i^2}{\lambda_i}\int_a^b \psi_i(t)u(t)dt$$

where $\overline{N}_i$ is the normalizing factor of $\sigma^{1/2}(x)\phi_i^{(s)}(x)$. The series $\sum_{i=0}^{\infty}((\overline{N}_i/\lambda_i)\int_a^b\psi_i(t)u(t)dt)^2$ converges since

$$\sum_{i=0}^{\infty}\left(\int_a^b \sigma^{1/2}(x)f^{(s)}(x)\sigma^{1/2}(x)\ \frac{\phi_i^{(s)}(x)}{\overline{N}_i}\,dx\right)^2$$

is convergent by virtue of Bessel's inequality. But if $\lim_{i\to\infty} \overline{N}_i/\lambda_i$ is not zero, then $\sum_{i=0}^{\infty}(\int_a^b\psi_i(t)u(t)dt)^2$ converges and as a consequence $\sum_{i=0}^{\infty}(\lambda_i^p\int_a^b f(x)\phi_i(x)dx)^2$ is convergent for some $p \geqq 1$.

5. **Convergence of the minimizing sums of the integral**[19] $g_n = \int_a^b\rho(x)|f(x)-Q_n(x)|^m dx$, $m>0$ **when** $(a, b)$ **is a finite interval.** Let $f(x)$ be a continuous function on $(a, b)$, and let the weight function $\rho(x)$ be summable and have a positive lower bound $v$ on $(a, b)$. Let $\{\phi_i(x)\}$ be a set of characteristic solutions of conjugate sets of integral equations of form (2). Let these solutions correspond to a set of characteristic numbers $\{\lambda_i\}$ that are in the order $0<\lambda_0\leqq\lambda_1 \leqq \cdots \leqq\lambda_n$. Let $\int_a^b K(x, t)dt$ be differentiable with respect to $x$ under the integral sign and let $\int_a^b[\partial K(x, t)/\partial x]^2dt$ as a function of $x$ exist almost everywhere. Let $Q_n(x)=\sum_{i=0}^{n}a_i\phi_i(x)$ be any sum of the first $n+1$ functions of the set $\{\phi_i(x)\}$, and let $\Phi_n(x)$ denote any one[20] of these that makes $g_n$ the least. Let $\epsilon_n$ be the maximum for $|f(x)-Q_n(x)|$ on $(a, b)$. By following the steps Jackson has taken to investigate convergence of minimizing trigonometric sums and polynomials[21] we are led, by virtue of (3) of Theorem A to the following result.

THEOREM D. *If* $Q_n(x)$ *is an arbitrary sum of the first* $n+1$ *functions of the set* $\{\phi_i(x)\}$, *and* $\Phi_n(x)$ *is a corresponding minimizing sum of*

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[19] For a proof of the existence of minimizing sums see D. Jackson, *On functions of closest approximation*, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 117–128. See also vol. 25 (1923) pp. 333–337.

[20] If $m\geqq 1$, the minimizing function is unique. See Jackson op. cit. Footnote 19.

[21] See Jackson op. cit. Footnote 1.

*$g_n$ there exists a positive factor $L$ independent of the characteristic numbers $\{\lambda_i\}$ and $\epsilon_n$ such that*

$$|f(x) - \Phi_n(x)| \leqq L\lambda_n^{1/m}\epsilon_n.$$

*Moreover, since $L$ depends upon $(\int_a^b [\partial K(x, t)/\partial x]^2 dt)^{1/2}$, it is bounded on $a \leqq x \leqq b$ except perhaps on a set of points of measure zero.*

Thus, since $\epsilon_n$ is the maximum of $|f(x) - Q_n(x)|$ on $(a, b)$, the question of convergence of $\Phi_n(x)$ to $f(x)$ on the intervals of $a \leqq x \leqq b$, where $L$ exists, is made to depend upon whether sums $Q_n(x)$ exist for an infinite sequence of values of $n$ such that

$$\lim_{n\to\infty} \lambda_n^{1/m}\epsilon_n = 0.$$

This is insured by Theorem C if the series

$$\sum_{i=0}^{\infty}\left(\lambda_i^p \int_a^b f(x)\phi_i(x)dx\right)^2$$

is convergent for some $p \geqq (1/m) + 1$.

Theorem D can be strengthened[12] considerably through the application of Theorem B if in addition to the property of orthonormality on the set $\{\phi_i(x)\}$, the set of derivatives $\{\phi_i'(x)\}$ are orthogonal on $(a, b)$ with respect to a continuous non-vanishing weight function $\sigma(x)$. The essential point is that $\lambda_n$ is usually large in comparison with the normalizing factor $N_n$, as for example in the integral equation that was cited earlier (p. 71). Here $\lambda_n = O(n^2)$ while $N_n$ of the set of derivatives of $\{\sin nx\}$, that is $\{n \cos nx\}$, is of order $n$.

In the subsequent discussion we shall also assume that $\int_a^b |d\sigma^{1/2}(x)/dx| dx$ exists. Using the same definitions and notations as in the preceding discussion, we let $\pi_n(x) = \Phi_n(x) - Q_n(x)$ and $r_n(x) = f(x) - Q_n(x)$. Let $\eta$ be the maximum of $|\pi_n(x)|$ on $(a, b)$ and let $x_0$ be the abscissa at which this maximum is attained. Since

$$\begin{aligned}\sigma^{1/2}(x)\pi_n(x) &- \sigma^{1/2}(x_0)\pi_n(x_0)\\ &= \int_{x_0}^{x} \frac{d}{dx}[\sigma^{1/2}(x)\pi_n(x)]dx = \int_{x_0}^{x}\sigma^{1/2}(x)\pi_n'(x)dx + \int_{x_0}^{x}\pi_n(x)\frac{d}{dx}\sigma^{1/2}(x)dx\end{aligned}$$

for any $x$ on the interval $(a, b)$, we have with the aid of Schwarz's inequality,

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[12] In this section of the paper we can extend our discussion to the case when the interval $(a, b)$ is infinite. The treatment requires further hypotheses on the functions $\sigma(x)$, $\{\phi_i(x)\}$ and $\{\phi_i'(x)\}$. In general the details are similar and are therefore omitted.

$$\begin{aligned} &|\sigma^{1/2}(x)\pi_n(x) - \sigma^{1/2}(x_0)\pi_n(x_0)| \\ &\qquad \leqq |x - x_0|^{1/2}\left(\int_{x_0}^{x} \sigma(x)[\pi_n'(x)]^2 |dx|\right)^{1/2} \\ &\qquad\quad + \int_{x_0}^{x} |\pi_n(x)| \left|\frac{d}{dx}\sigma^{1/2}(x)\right| |dx|. \end{aligned}$$

But since $x_0$ and $x$ are on $(a, b)$ by Theorem B it follows that

$$\begin{aligned} \left(\int_{x_0}^{x} \sigma(x)[\pi_n'(x)]^2 |dx|\right)^{1/2} &\leqq \left(\int_{a}^{b} \sigma(x)[\pi_n'(x)]^2 dx\right)^{1/2} \\ &\leqq N_n\left(\int_{a}^{b} \pi_n^2(x) dx\right)^{1/2} \end{aligned}$$

so that

$$\begin{aligned} &|x - x_0|^{1/2}\left(\int_{x_0}^{x} \sigma(x)[\pi_n'(x)]^2 |dx|\right)^{1/2} \\ &\qquad + \int_{x_0}^{x} |\pi_n(x)| \left|\frac{d}{dx}\sigma^{1/2}(x)\right| |dx| \\ &\qquad\qquad \leqq |x - x_0|^{1/2} N_n(b - a)^{1/2}\eta + |x - x_0|\,\eta H \end{aligned} \tag{8}$$

where $H$ is a constant independent of $x$ and $N_n$. If $N_n$ is large and

$$|x - x_0| < \frac{\sigma(x_0)}{16N_n^2(b - a)}$$

then

$$\begin{aligned} &|\sigma^{1/2}(x)\pi_n(x) - \sigma^{1/2}(x_0)\pi_n(x_0)| \\ &\qquad \leqq \frac{\eta}{4}\sigma^{1/2}(x_0) + \frac{\eta H\sigma(x_0)}{16N_n^2(b - a)} < \frac{\eta}{2}\sigma^{1/2}(x_0), \end{aligned}$$

whence $|\sigma^{1/2}(x)\pi_n(x)| > (\eta/2)\sigma^{1/2}(x_0)$. If $w$ is the maximum of $\sigma^{1/2}(x)$ on $(a, b)$, then $|\pi_n(x)| > (\eta/2)\sigma^{1/2}(x_0)/w$.

Let $\gamma_n$ be the minimum of $g_n$. For a value of $x$ on $(a, b)$ at a distance not greater than $\sigma(x_0)/16N_n^2(b-a)$ from $x_0$,

$$\gamma_n > v|\pi_n(x) - r_n(x)|^m \frac{\sigma(x_0)}{16N_n^2(b - a)}.$$

Thus, for the case when $(\eta/4)(\sigma^{1/2}(x_0)/w) > \epsilon_n$ we have, since $\gamma_n < I_2\epsilon_n^m$, $I_2 = \int_a^b \rho(x)dx$,

$$\eta < \frac{4w}{\sigma^{1/2}(x_0)}\left(\frac{16[b-a]}{v}\gamma_n N_n^2\right)^{1/m} < s_1 N_n^{2/m}\epsilon_n \tag{9}$$

where $s_1$ is an absolute constant.

On the other hand, if $(\eta/4)(\sigma^{1/2}(x_0)/w) < \epsilon_n$, we have

$$\eta < \frac{4w}{\sigma^{1/2}(x_0)}\epsilon_n$$

so that for an $N_n$ sufficiently large, (9) is a universal bound for $\eta$.

Hence, from the inequality

$$\begin{aligned}|f(x) - \Phi_n(x)| &\leqq |f(x) - Q_n(x)| + |\Phi_n(x) - Q_n(x)| \\ &= |r_n(x)| + |\pi_n(x)|\end{aligned}$$

we have the following result.

THEOREM E. *If in addition to the hypotheses placed on* $\rho(x), f(x)$, *and the set* $\{\phi_i(x)\}$ *in Theorem* D *it is assumed that the set of functions* $\{\phi_i'(x)\}$ *are orthogonal with respect to a continuous positive weight function* $\sigma(x)$, *and if* $\int_a^b |d\sigma^{1/2}(x)/dx|dx$ *exists, then there exists a constant* $s_2$ *independent of* $N_n$, $\epsilon_n$, *and also of* $x$ *such that*

$$|f(x) - \Phi_n(x)| \leqq s_2 N^{2/m}\epsilon_n.$$

Theorem E is subject to further refinement if certain further hypotheses are assumed on the set $\{\phi_i(x)\}$. If the set of functions $\{\phi_i'(x)\}$ are continuous on the interval $(a, b)$, (the condition of orthogonality only requires that the set $\{\phi_i'(x)\}$ with respect to $\sigma(x)$ be integrable $L^2$), and if at the maximum of $|\pi_n(x)|$, $d\pi_n(x)/dx = 0$, then by a law of the mean[22] for integrals it is seen that

$$\frac{\int_{x_0}^{x}\sigma(x)[\pi_n'(x)]^2|dx|}{|x - x_0|} \leqq \frac{\int_a^b\sigma(x)[\pi_n'(x)]^2 dx}{b-a}$$

when $x$ is sufficiently close to $x_0$. Thus the first integral on the left-hand side of (8) can be replaced by

$$\frac{|x - x_0|^{1/2}}{(b-a)^{1/2}}\left(\int_a^b \sigma(x)[\pi_n'(x)]^2 dx\right)^{1/2}$$

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[22] By the law of the mean, there exist points $\xi$ and $c$ on the intervals $|x, x_0|$ and $(a, b)$, respectively, such that $\sigma(\xi)[\pi_n'(\xi)]^2 = (\int_{x_0}^{x}\sigma(x)[\pi_n'(x)]^2|dx|)/|x-x_0|$ and $\sigma(c)[\pi_n'(c)]^2 = (\int_a^b\sigma(x)[\pi_n'(x)]^2dx)/(b-a)$. Since $\pi'(x)$ is continuous and $\pi_n'(x_0) = 0$ it follows, since $\sigma(x)$ is positive and bounded on $(a, b)$, that $\sigma(\xi)[\pi_n'(\xi)]^2$ can be made smaller than $\sigma(c)[\pi_n'(c)]^2$ by taking $x$ sufficiently close to $x_0$.

and this leads to a bound of order $N^{1/m}\epsilon_n$ for $|f(x)-\Phi_n(x)|$ in contrast with $N^{2/m}\epsilon_n$ of Theorem E. As an example, a relative maximum of $|\pi_n(x)|$ is assured if the following boundary conditions[24] are assumed on $\Phi_n(x)$: $\Phi_n(a)=Q_n(a)$, and $\Phi_n(b)=Q_n(b)$. In that case $\pi_n(a)=\pi_n(b)=0$ and as a consequence of the hypotheses on the functions $\{\phi_i'(x)\}$, $\pi_n'(x)=0$ at the maximum of $|\pi_n(x)|$.

The hypothesis on $\sigma(x)$ may be lightened to include weight functions that vanish at some points along the interval. The only restriction required by the proof is that $\sigma(x)$ shall not vanish at the point where $|\pi_n(x)|$ takes on its maximum, and this in turn is assured by appropriate boundary conditions on the minimizing function $\Phi_n(x)$, such as were mentioned in the last paragraph. This last observation is of importance in working out details for the convergence of sums in terms of the Bessel, Legendre, Laguerre, and the Hermite functions, among a great number of others that involve weight functions. The details of these discussions follow the above general outline and are therefore omitted.

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[24] The placing of a boundary condition on $\Phi_n(x)$ does not affect the existence of minimizing sums. See for example, D. Jackson, *Problems of approximation with integral boundary conditions*, Amer. J. Math. vol. 55 (1933) pp. 153–166.

# DOUBLE COSET MATRICES AND GROUP CHARACTERS

J. S. FRAME

1. **Introduction.** The principal theorem of this paper extends to the characters of the irreducible representations of an intransitive group a theorem proved in an earlier paper by the author[1] for the degrees of the irreducible representations of a transitive group. A by-product of the development is the theorem that the sum of the traces of the permutations of any subgroup of a permutation group is not less than the corresponding sum for one of its cosets.

Every finite permutation group $G$, of order $g$ and degree $n$, can be written as a group of permutation matrices. The $n \times n$ matrix $R(\gamma)$ which corresponds to the element $\gamma$ of $G$ can be written as a direct sum of submatrices $R^t(\gamma)$ of $n^t$ dimensions corresponding to the $n^t$ symbols of a transitive constituent $C^t$ of $G$.[2] Associated with such a transitive constituent $C^t$ is a class of conjugate subgroups, $H_\tau^t = (\gamma_\tau^t)^{-1}H^t\gamma_\tau^t$, each of order $h^t$, of which $H^t$ shall be the subgroup leaving fixed the first symbol of $C^t$, and $H_\tau^t$ the subgroup leaving fixed the $\tau$th symbol. If $\gamma_\alpha$ is any element of $G$, then in the set of $h^s h^t$ group elements $H^s\gamma_\alpha H^t$, each element will appear $k_\alpha^{st}$ times, where $k_\alpha^{st}$ is the order of the cross-cut of the subgroups $H_\alpha^s = \gamma_\alpha^{-1}H^s\gamma_\alpha$ and $H^t$.

Counting each element of the set just once, we define the double coset $H_\alpha^{st}$ by the formula

$$H_\alpha^{st} = H^s\gamma_\alpha H^t / k_\alpha^{st}. \tag{1.1}$$

Any element from a double coset can be chosen as the defining element $\gamma_\alpha$. The inverses of the elements of a double coset $H_\alpha^{st}$ themselves form a double coset which we call the inverse double coset and denote by $H_{\alpha'}^{ts}$. The product of two double cosets is a linear combination of double cosets. By considering $H_\alpha^{rs}$ as a sum of $h^r/k_\alpha^{rs}$ left cosets of $H^s$, and $H_{\beta'}^{st}$ as a sum of $h^t/k_{\beta'}^{st}$ right cosets of $H^s$, and noting that $H^sH^s = h^sH^s$, it is apparent that each element in the product $H_\alpha^{rs}H_{\beta'}^{st}$ occurs a multiple of $h^s$ times. We define the positive integers $c_{\alpha\beta\gamma}^{rst}$ by the formula

$$H_\alpha^{rs}H_{\beta'}^{st}/h^s = \sum_\gamma c_{\alpha\beta\gamma}^{rst}H_\gamma^{rt}. \tag{1.2}$$

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[1] J. S. Frame, *The double cosets of a finite group*, Bull. Amer. Math. Soc. vol. 47 (1941) p. 459.

[2] Throughout this paper the superscripts will refer to the transitive constituents.

Let $\mu^{st}$ be the number of double cosets $H_\alpha^{st}$ for fixed $s$, $t$. Then for fixed $\eta$ the constants $c_{\alpha\beta\eta}^{rst}$ are elements of a $\mu^{rs}$- by $\mu^{st}$-dimensional rectangular matrix $M^r(H_\eta^{rt})$, and the transposed matrix is the matrix $M^s(H_{\eta'}^{tr})$ for the inverse double coset. For $r=t$ these matrices are square. Their sum, with parameters $b_\eta^t$ as coefficients, defines the important matrix

$$K^{st} = \sum_\eta b_\eta^t M^s(H_\eta^{tt}), \tag{1.3}$$

each of whose elements are linear combinations of the parameters $b_\eta^t$.

The principal theorem of this paper states that the determinant of the matrix $n^t K^{st}$ splits into factors $\chi_\iota$ which are linear in the parameters $b_\eta^t$, that these factors $\chi_\iota$ are essentially the characters of the irreducible representations $\Gamma_\iota$ of $G$, and that the multiplicity $m_\iota$ of the linear factor $\chi_\iota$ is $\mu_\iota^s\mu_\iota^t$, where $\mu_\iota^t$ is the multiplicity with which $\Gamma_\iota$ occurs in the reduction of the transitive group of matrices $R^t(\gamma)$, and where $\sum\mu_\iota^s\mu_\iota^t=\mu^{st}$. Finally the numerical coefficient which appears is an integer divisible by $n^s n^t$. Written as a formula, the theorem reads

(1.4) THEOREM.

$$(n^t)^m \left| K^{st} \right| = n^s n^t A^{st} \prod_\iota (\chi_\iota)^{m_\iota},$$

*where* $m_\iota=\mu_\iota^s\mu_\iota^t$, $m=\sum m_\iota=\mu^{st}$, $A^{st}=$ *an integer.*

A more precise definition of $\chi_\iota$ is obtained after writing

$$b_\eta^t = \sum_\lambda k_{\eta\lambda}^t a_\lambda, \tag{1.5}$$

where $a_\lambda$ are new arbitrary parameters and $k_{\eta\lambda}^t$ is the number of elements of the double coset $H_\eta^{tt}$ which belong to the class of conjugates $C_\lambda$. Then $\chi_\iota=\chi_\iota(\sum a_\lambda C_\lambda)$ is defined as the character of the ring element $\sum a_\lambda C_\lambda$ in the representation $\Gamma_\iota$. The integer $A^{st}$ enters in the proof as the squared absolute value of the determinant of a certain transformation matrix. It is not necessarily a rational square.

In the special case of this theorem which appeared in the earlier paper already cited,[3] all parameters $a_\lambda$ except $a_1$ were zero, $s$ was equal to $t$, $\chi_\iota$ was the degree $n_\iota$, and the matrix $K^{st}$ was a diagonal matrix with elements there denoted by $k_\iota$.

In §2 the matrices of an intransitive group are broken up into blocks corresponding to the transitive constituents, and it is proved that the sum of the traces of the matrices of a subgroup is not less than the corresponding sum for a coset. In §3 a set of "double coset matrices" $V^{ts}_\alpha$, closely related to the double cosets $H^{ts}_\alpha$, are found to form a basis for the matrices $V$ permutable with a group of permutation matrices. Their products are expressible in terms of the structure constants $c^{rst}_{\alpha\beta\gamma}$ already defined in (1.2). The matrix $R(a)$ corresponding to the ring element $a=\sum a_\lambda C_\lambda$ is expressed in terms of the double coset matrices, and used in §4 to build a matrix of traces $\chi(V^{tu}_{\alpha'}V^{us}_\beta R(a))$ which is shown to be precisely the matrix $n^tK^{st}$ defined by (1.3). A unitary change of coordinates in §5 induces a change of basis from the matrices $V^{ts}_\alpha$ to new basis matrices $E^{ts}_\eta$ by means of a transformation $(\rho^{st}_{\alpha\eta})$ whose determinant enters in the final factoring of the determinant $n^tK^{st}$ from which the group characters are obtained. An example to show the application of the main theorem is given in §6.

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[3] J S Frame, loc cit.

2. **Cosets and submatrices.** The $n$ symbols of the intransitive group $G$ will be assumed to be collected together into transitive constituents $C^t$ of degree $n^t$, and the $n$ right cosets $H^t\gamma^t_r$ will be arranged, in an order which preserves this grouping, so as to form the elements of a basis vector $b$ of $n$ rows and one column. Multiplication of $b$ on the right by an element $\gamma$ of $G$ permutes these cosets among themselves within each transitive constituent, so a permutation matrix $R(\gamma)$ is defined by the equation

$$b\gamma = R(\gamma)b. \tag{2.1}$$

Similarly if $b'$ and $R'(\gamma)$ are the transposes of $b$ and $R(\gamma)$, and if $E$ is the $n\times n$ unit matrix, we have

$$b'(\gamma E) = b'R'(\gamma) = b'R(\gamma^{-1}). \tag{2.2}$$

Let us define the idempotent matrix $E^t$ to be the $n\times n$ matrix which is the unit matrix for the symbols of the constituent $C^t$ and the zero matrix for the other symbols, and let us define a "completely symmetric" $n\times n$ matrix $W$ each of whose $n^2$ elements is a 1. Then

$$E = \sum_t E^t, \qquad E^tE^t = E^t, \qquad E^tE^s = 0 \quad \text{if} \quad t\neq s. \tag{2.3}$$

Let us further define the *block* $C^{ts}$ to be the $n^t\times n^s$ rectangular submatrix of an $n\times n$ matrix, whose rows correspond to the symbols of $C^t$ and whose columns correspond to the symbols of $C^s$. We see that the matrix

$$W^{ts} = E^tWE^s \tag{2.4}$$

consists of 1's in the block $C^{ts}$ and 0's elsewhere.

The general matrix $R(\gamma)$ of the intransitive permutation group $G$

may be broken up into transitive components $R^\iota(\gamma)$ in the diagonal blocks $C^{\iota\iota}$. We define

$$R^\iota(\gamma) = E^\iota R(\gamma) = R(\gamma)E^\iota, \tag{2.5}$$

and we then have

$$R(\gamma) = \sum_\iota R^\iota(\gamma). \tag{2.6}$$

The fact that the group $G$ is transitive in the symbols of the constituent $C^\iota$ shows that the sum of the matrices $R^\iota(\gamma)$ for all $\gamma$ in $G$, which we denote by $R^\iota(G)$, is a multiple of the matrix $W^{\iota\iota}$. A simple count proves the following relation:

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$$R^\iota(G) = h^\iota W^{\iota\iota}. \tag{2.7}$$

A theorem on the traces of subgroups and cosets, suggested to the author by G. Polya as having been experimentally verified but not theoretically proved, is a consequence of this lemma.

(2.8) THEOREM. *The sum of the numbers of symbols left fixed by the individual permutations of a subgroup of any permutation group is greater than or equal to the corresponding sum for the permutations of any coset of that subgroup.*

To prove this theorem we note first that each specified sum is the sum of traces of the elements of a subgroup $H$ or of a coset $H\gamma$, and second that if the theorem holds for every transitive constituent of an intransitive group $G$, then it holds, by addition, for $G$ as well. Focusing attention on a constituent $C^\iota$ which is transitive for $G$, we further subdivide it into sets $C^{\iota u}$ which are transitive for the given subgroup $H$. To the corresponding matrices $R^{\iota u}(H)$ we apply the lemma, showing that each is a square block of dimension $n^{\iota u}$, each of whose elements is equal to $h/n^{\iota u}$, and whose trace is equal to $h$. Multiplication of the matrix $R^\iota(H)$ by a matrix $R(\gamma)$ has the effect of permuting the columns and replacing the diagonal elements either by the same positive integer as before, or else by 0. Thus the trace of $R^\iota(H\gamma)$ is less than or equal to the trace of $R^\iota(H)$, which was to be proved.

3. **Double coset matrices and structure constants.** To describe the structure of the general $n$-dimensional matrix $V$ which is permutable with every permutation matrix $R(\gamma)$, of an intransitive permutation group $G$, we define

$$V^{ts} = E^t V E^s. \tag{3.1}$$

Then the permutability condition

$$VR(\gamma) = R(\gamma)V \tag{3.2}$$

together with (2.5) implies the relations

$$V^{ts}R^s(\gamma) = E^t V R(\gamma) E^s = E^t R(\gamma) V E^s = R^t(\gamma) V^{ts}, \tag{3.3}$$

which may be expressed in words by saying that the rectangular submatrix $V^{ts}$ of $V$ intertwines the matrices $R^s(\gamma)$ and $R^t(\gamma)$. From (3.3) we obtain the equation

$$V^{ts} = R^t(\gamma) V^{ts} R^s(\gamma^{-1}), \tag{3.4}$$

which states that the rectangular matrix $V^{ts}$ is unchanged if its rows are permuted by the matrix $R^t(\gamma)$ while its columns undergo simultaneously the transposed permutation $[R^s(\gamma)]'$. It follows in general that many of the coefficients in the matrix $V^{ts}$ are necessarily equal "by symmetry." If $E^{ts}_{\tau\sigma}$ is defined to be a matrix which is zero except for a 1 in the $\tau$th row and the $\sigma$th column of the block $C^{ts}$, if $H^t$, $\gamma^t_\tau$, $H^s$, $\gamma^s_\sigma$ are defined as in §1, and $\theta^t$ and $\theta^s$ are elements of $H^t$ and $H^s$, respectively, then

$$E^{ts}_{\tau\sigma} = R^t(\theta^t\gamma^t_\tau)^{-1} E^{ts}_{11} R^s(\theta^s\gamma^s_\sigma). \tag{3.5}$$

To obtain an invariant matrix we take the sum of all the transforms of (3.5) by all $\gamma$ in $G$. We get

$$\begin{aligned} \sum_{\gamma\in G} R^t(\gamma^{-1}) E^{ts}_{\tau\sigma} R^s(\gamma) &= \sum_{\gamma} R^t(\theta^t\gamma^t_\tau\gamma)^{-1} E^{ts}_{11} R^s(\theta^s\gamma^s_\sigma\gamma) \\ &= \sum_{\delta} R^t(\delta^{-1})\, E^{ts}_{11} R^s\left(\theta^s\gamma^s_\sigma(\gamma^t_\tau)^{-1}(\theta^t)^{-1}\right)\, R^s(\delta) \\ &= \sum_{\gamma} R^t(\gamma^{-1})\, E^{ts}_{11} R^s(\gamma_\alpha) R^s(\gamma) \end{aligned} \tag{3.6}$$

where $\gamma_\alpha$ is any element chosen from the double coset $H^{st}_\alpha$ which contains $\gamma^s_\sigma(\gamma^t_\tau)^{-1}$. We define the double coset matrix $V^{ts}_\alpha$ by the formula

$$V^{ts}_\alpha = (1/h^{st}_\alpha) \sum_{\gamma\in G} R^t(\gamma^{-1}) E^{ts}_{11} R^s(\gamma_\alpha) R^s(\gamma), \tag{3.7}$$

so that its coefficients are all 0's and 1's. In this matrix there are $h^t/h^{st}_\alpha$ 1's in each row and $h^s/h^{st}_\alpha$ 1's in each column of the block $C^{ts}$, and 0's elsewhere. Furthermore, by averaging equation (3.4) for all $\gamma$ in $G$, it is readily seen that every $V^{ts}$ is a linear combination of these basis matrices $V^{ts}_\alpha$, and in particular that

$$\sum_{\alpha} V^{ts}_{\alpha} = W^{ts}. \tag{3.8}$$

From this discussion it also follows that

$$W^{ts}V^{sr}_{\alpha} = W^{tr}h^{r}/h^{sr}_{\alpha}. \tag{3.9}$$

The number of independent matrices $V^{ts}_{\alpha}$ in the block $C^{ts}$ is equal to the number of double cosets $H^{st}_{\alpha}$, which was denoted by $\mu^{st}$. Since the transposed matrix $(V^{ts}_{\alpha})' = V^{st}_{\alpha'}$ corresponds to the inverse double coset $(H^{st}_{\alpha})^{-1} = H^{ts}_{\alpha'} = H^{t}\gamma_{\alpha}^{-1}H^{s}/h^{st}_{\alpha}$, it follows that $\mu^{st} = \mu^{ts}$.

The product of two double coset matrices $V^{ts}_{\beta'}$ and $V^{sr}_{\alpha}$ is given in terms of the coefficients $c^{rst}_{\alpha\beta\eta}$ of (1.2) as follows:

$$V^{ts}_{\beta'}V^{sr}_{\alpha} = \sum_{\eta} c^{rst}_{\alpha\beta\eta}V^{tr}_{\eta}. \tag{3.10}$$

To prove this we work from the definition (3.7) and apply (1.2)

$$\begin{aligned}
V^{ts}_{\beta'}V^{sr}_{\alpha} &= (1/h^{rs}_{\alpha}h^{st}_{\beta})\sum_{\gamma,\delta} R^{t}(\gamma^{-1})E^{ts}_{11}R^{s}(\gamma_{\beta'})R^{s}(\gamma)R^{s}(\delta^{-1})E^{sr}_{11}R^{r}(\gamma_{\alpha})R^{r}(\delta)\\
&= (1/h^{rs}_{\alpha}h^{st}_{\beta})\sum_{\gamma} R^{t}(\gamma^{-1})E^{ts}_{11}E^{sr}_{11}R^{r}(\gamma_{\alpha}H^{s}\gamma_{\beta'}\gamma)\\
&= (1/h^{r}h^{s}h^{t})\sum_{\gamma} R^{t}(\gamma^{-1})E^{tr}_{11}R^{r}(H^{rs}_{\alpha}H^{st}_{\beta'})R^{r}(\gamma)\\
&= \sum_{\eta} c^{rst}_{\alpha\beta\eta}\sum_{\gamma} R^{t}(\gamma^{-1})E^{ts}_{11}R^{r}(H^{rt}_{\eta}/h^{r}h^{t})R^{r}(\gamma)\\
&= \sum_{\eta} c^{rst}_{\alpha\beta\eta}V^{tr}_{\eta}.
\end{aligned}$$

By summing equation (3.10) first for $\beta$ and then for $\alpha$, and applying (3.9) we obtain

$$\begin{aligned}
W^{ts}V^{sr}_{\alpha} &= \sum_{\beta\eta} c^{rst}_{\alpha\beta\eta}V^{tr}_{\eta} = W^{tr}(h^{r}/h^{sr}_{\alpha}),\\
V^{ts}_{\beta'}W^{sr} &= \sum_{\alpha\eta} c^{rst}_{\alpha\beta\eta}V^{tr}_{\eta} = (h^{t}/h^{ts}_{\beta})W^{tr}.
\end{aligned} \tag{3.11}$$

Hence

$$\sum_{\beta} c^{rst}_{\alpha\beta\eta} = h^{r}/h^{sr}_{\alpha}; \qquad \sum_{\alpha} c^{rst}_{\alpha\beta\eta} = h^{t}/h^{st}_{\beta}; \qquad \sum_{\alpha,\beta} c^{rst}_{\alpha\beta\eta} = n^{s}. \tag{3.12}$$

These sums are the sums for a column, or row, or all elements of the matrix $M^{s}(H^{rt}_{\eta})$ defined in §1.

**4. Conjugate classes and the trace matrix.** The matrix $R(C_{\lambda})$, defined to be the sum of the matrices $R(\gamma)$ for all $\gamma$ in the class of conjugate elements $C_{\lambda}$, and likewise the more general matrix $R(a)$, where $a=\sum a_\lambda C_\lambda$ is a general element from the central of the group ring of $G$ and $a_\lambda$ are arbitrary parameters, are both matrices permutable with the group, and as such are expressible in terms of the double coset matrices. They are confined to the diagonal blocks $C^{tt}$, so we have

$$R^t(C_\lambda) = \sum_\delta g^t_{\delta\lambda} V^{tt}_\delta; \qquad R^t(a) = \sum a_\lambda R^t(C_\lambda). \tag{4.1}$$

The coefficient $g^t_{\psi\lambda}$ is the number of elements of $C_\lambda$ which belong to any particular right coset of $H^{st}_\psi$.

We are now ready to define and evaluate the important $\mu^{st}$-dimensional trace matrix $X^{st}_{\alpha\beta}(a)$:

$$\begin{aligned} X^{st}_{\alpha\beta}(a) &\equiv \chi(V^{ts}_{\alpha'} V^{st}_\beta R^t(a)) \\ &= \chi\left( \sum_\psi c^{tst}_{\alpha\beta\psi} V^{tt}_{\psi'} \sum_\lambda a_\lambda R^t(C_\lambda) \right) \\ &= \sum_\lambda a_\lambda \chi\left( \sum_{\psi,\delta} c^{tst}_{\alpha\beta\psi} g^t_{\delta\lambda} V^{tt}_{\psi'} V^{tt}_\delta \right) \\ &= \sum_{\lambda,\psi} a_\lambda c^{tst}_{\alpha\beta\psi} g^t_{\psi\lambda} \chi(V^{tt}_{\psi'} V^{tt}_\psi) \\ &= \sum_{\lambda,\psi} a_\lambda c^{tst}_{\alpha\beta\psi} k^t_{\psi\lambda} n^t \\ &= n^t \sum_\psi c^{tst}_{\alpha\beta\psi} b^t_\psi. \end{aligned} \tag{4.2}$$

The letter $\chi$ is used to denote the trace of a matrix. The new coefficients $k^t_{\psi\lambda}$ and parameters $b^t_\psi$ are defined as follows:

$$k^t_{\psi\lambda} = g^t_{\psi\lambda} h^t / k^{tt}_\psi; \qquad b^t_\psi = \sum_\lambda a_\lambda k^t_{\psi\lambda}. \tag{4.3}$$

The integer $k^t_{\psi\lambda}$ is the number of elements of $C_\lambda$ which occur in the whole double coset $H^{tt}_\psi$. In terms of the matrix $K^{st}$ of (1.3), we have

$$X^{st}(a) = n^t K^{st}(a) = n^t \sum_\psi b^t_\psi M^s(H^{tt}_\psi). \tag{4.4}$$

In applying the theory to a given group, as in §6, we first compute the matrices $M^s$ for each double coset, form the sum with parameters $b^t_\psi$ as coefficients, and factor the resulting determinant into factors linear in the $b^t_\psi$. One of these factors will be $(b^t_1+b^t_2+\cdots)$, corresponding to the identity representation. We then compute the integers $k^t_{\psi\lambda}$ by studying the group directly, and replace the parameters

$b^t_\gamma$ by appropriate linear combinations of the parameters $a_\lambda$. The coefficients of $a_\lambda$ so obtained in any linear factor, divided by the number $g_\lambda$ of elements in $C_\lambda$, are proportional to the traces $\chi_{\iota\lambda}$ of the elements of $C_\lambda$ in the irreducible representation $\Gamma_\iota$ corresponding to that factor.

5. **The determinant of the trace matrix.** A unitary matrix $U$ exists which will simultaneously reduce all the matrices $R(\gamma)$, where $\gamma$ can now be thought of as an arbitrary element of the group ring of $G$. The matrix $U$ can be chosen to be zero outside the diagonal blocks $C^{tt}$, in which we define $U^t = E^t U E^t$. It can further be so chosen that all the equivalent irreducible components in the decomposition of $R(\gamma)$ are actually identical, and we shall assume that this is done, and that the irreducible representation $\Gamma_\iota$ of degree $n_\iota$ occurs with multiplicity $\mu^t_\iota$ in $(U^t)^{-1}R^t(\gamma)U^t$. The intersections of the $n_\iota\mu^t_\iota$ rows and $n_\iota\mu^s_\iota$ columns of the block $C^{ts}$ which belong to the representation $\Gamma_\iota$ will be called a representation *district* $D^{ts}_\iota$. The district itself divides up into $\mu^t_\iota\mu^s_\iota$ squares of $n_\iota$ dimensions. With each square is associated a unit matrix $E^{ts}_{\iota,kl}$, $k=1, \cdots, \mu^t_\iota; l=1, \cdots, \mu^s_\iota$, conveniently denoted by $E^{ts}_\gamma$. Every matrix permutable with all the reduced matrices $U^{-1}R(\gamma)U$ is a linear combination of these matrices $E^{ts}_\gamma$. In particular we have

$$(5.1)\qquad \begin{aligned} U^{-1}V^{st}_\alpha U &= (U^s)^{-1}V^{st}_\alpha U^t = \sum_\gamma \rho^{st}_{\alpha\gamma}E^{st}_\gamma, \\ U^{-1}V^{ts}_{\alpha'}U &= \sum_\gamma \bar\rho^{st}_{\alpha\gamma}E^{ts}_{\gamma'}. \end{aligned}$$

The number of basis matrices $E^{ts}_\gamma$ for fixed $t$, $s$, must equal $\mu^{st}$, the number of $V^{ts}_\alpha$, so we have the relation

$$(5.2)\qquad \mu^{st} = \sum_\iota \mu^s_\iota\mu^t_\iota.$$

Thus the coefficients $(\rho^{st}_{\alpha\gamma})$ defined by (5.1) form a $\mu^{st}$-dimensional square matrix. Its determinant, denoted by $\mathrm{P}^{st}$, is not zero, since both the $U^{-1}V^{ts}U$ and the $E^{ts}$ form a basis for the same set of matrices. That $\mathrm{P}^{st}$ is an algebraic integer (not necessarily rational), divisible by $(n^s n^t)^{1/2}$, can be shown as follows.

A partial reduction of the linear group to split off the identity representation $\Gamma_1$ in each constituent $C^t$ can be effected by first replacing $U^t$ by a unitary matrix whose $(u, v)$th element in the block $C^{tt}$ is $\epsilon^{(u-1)(v-1)}/(n^t)^{1/2}$, ($\epsilon$ an $n^t$th root of unity), and then effecting the reduction in the remaining $n^t-1$ dimensions. The coefficients $\rho^{st}_{\alpha 1}$ are thus readily computed explicitly:

$$\overset{st}{\rho}_{\alpha 1} = (g/\overset{st}{h}_{\alpha})/(\overset{s}{n}\overset{t}{n})^{1/2} = (\overset{s}{h}\overset{t}{h})^{1/2}/\overset{st}{h}_{\alpha}; \tag{5.3}$$

$$\sum_{\alpha} \overset{st}{\rho}_{\alpha 1} = \sum_{\alpha} (\overset{s}{n}\overset{s}{h}/\overset{st}{h}_{\alpha})/(\overset{s}{n}\overset{t}{n})^{1/2} = (\overset{s}{n}\overset{t}{n})^{1/2}. \tag{5.4a}$$

Furthermore, since the matrix $W^{ts}$ of (3.8) is of rank 1 and is represented by 0 except in the district $D^{ts}_1$, we have

$$\sum_{\alpha} \overset{st}{\rho}_{\alpha\eta} = 0, \qquad \eta \neq 1. \tag{5.4b}$$

Adding all rows of $(\overset{st}{\rho}_{\alpha\eta})$ to the first, we find that

$$\overset{st}{\mathrm{P}} = (\overset{s}{n}\overset{t}{n})^{1/2}\overset{st}{\mathrm{P}}_1, \tag{5.5}$$

where $\mathrm{P}^{st}_1$ is the cofactor of $\rho^{st}_{11}$ in $\mathrm{P}^{st}$. Chopping off the first row and column of each, we multiply the conjugate of $(\rho^{st}_{\alpha\eta})$ by the transpose of this same matrix, and obtain a matrix $(A^{st}_{\alpha\beta})$ of $\mu^{st}-1$ dimensions, with determinant $A^{st}$ given by the formula

$$\overset{st}{A} = \overset{st}{\overline{\mathrm{P}}}_1 \overset{st}{\mathrm{P}}_1 = \overset{st}{\overline{\mathrm{P}}}\,\overset{st}{\mathrm{P}}/\overset{s}{n}\overset{t}{n}. \tag{5.6}$$

The element in row $\alpha-1$ and column $\beta-1$ of $(A^{st}_{\alpha\beta})$ is

$$\overset{st}{A}_{\alpha\beta} = \sum_{\eta>1} \overset{st}{\bar{\rho}}_{\alpha\eta}\overset{st}{\rho}_{\beta\eta} = \sum_{\eta>1} \overset{ts}{\rho}_{\alpha'\eta'}\overset{st}{\rho}_{\beta\eta} = \sum_{\iota>1}\sum_{k,l} \overset{ts}{\rho}_{\alpha'\iota,lk}\overset{st}{\rho}_{\beta\iota,kl} \tag{5.7}$$

since the subscripts $\eta$ and $\eta'$ stand for the three subscripts $(\iota, kl)$ and $(\iota, lk)$, respectively. The rule of combination (3.10) which defines the product of $V^{ts}_{\alpha'}$ and $V^{st}_{\beta}$ holds likewise for the representatives of these matrices in the districts corresponding to $\Gamma_{\iota}$, and we have for fixed $\iota$

$$\sum_{k} \overset{ts}{\rho}_{\alpha'\iota,hk}\overset{st}{\rho}_{\beta\iota,kl} = \sum_{\delta} \overset{tst}{c}_{\beta\alpha\delta}\overset{tt}{\rho}_{\delta\iota,hl}. \tag{5.8}$$

The trace $T^t_{\delta\iota}=\sum_\kappa \rho^{\delta}_{\kappa\kappa}$ of the $\mu^t_\iota$-dimensional matrix representing $V^\delta_\iota$ $n_\iota$ times in the district $D^\delta_\iota$ is an algebraic integer. Hence each element of the matrix $(A^{st}_{\alpha\beta})$,

$$\overset{st}{A}_{\alpha\beta} = \sum_{\delta,\iota>1} \overset{tst}{c}_{\beta\alpha\delta}\overset{t}{T}_{\delta\iota} \tag{5.9}$$

is an algebraic integer, and the determinant $A^{st}$ is an integer.

We now use the transformation (5.1) to obtain a new expression for (4.2):

$$\begin{aligned} \overset{st}{X}_{\alpha\beta}(a) &= \chi\Big(\sum_{\zeta,\eta} \overset{st}{\bar{\rho}}_{\alpha\zeta}\overset{ts}{E}_{\zeta'}\overset{st}{\rho}_{\beta\eta}\overset{st}{E}_{\eta}(\overset{t}{U})^{-1}\overset{t}{R}(a)\overset{t}{U}\Big) \\ &= \sum_{\eta} \overset{st}{\rho}_{\beta\eta}\overset{st}{\bar{\rho}}_{\alpha\eta}\chi(\overset{tt}{E}(\overset{t}{U})^{-1}\overset{t}{R}(a)\overset{t}{U}) = \sum_{\eta} \overset{st}{\rho}_{\beta\eta}\overset{st}{\bar{\rho}}_{\alpha\eta}\chi_{\eta}(a). \end{aligned} \tag{5.10}$$

Taking determinants of both sides of (5.10) we obtain

$$|X^{st}_{\alpha\beta}(a)| = \mathrm{P}^{st}\overline{\mathrm{P}}^{st}\prod_{\eta}\chi_{\eta}(a). \tag{5.11}$$

The subscript $\eta$ stands for the three subscripts $i$, $kl$, but since $\chi_\eta(a)$ is a character depending on $i$ alone, we group together the $m_i = \mu_i^t\mu_i^s$ equal factors $\chi_i(a)$, use (5.6), and write

$$|X^{st}_{\alpha\beta}(a)| = n^s n^t A^{st}\prod_{i}[\chi_i(a)]^{m_i}. \tag{5.12}$$

Reference to (4.4) gives us the final formula for the principal theorem of (1.4).

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$$(n^t)^m\,|K^{st}(\textstyle\sum a_\lambda C_\lambda)| = n^s n^t A^{st}\prod_{i}\Bigl(\sum_{\lambda} a_\lambda\chi_i(C_\lambda)\Bigr)^{\mu_i^s\mu_i^t}, \tag{5.13}$$

$$m = \sum \mu_i^s\mu_i^t = \mu^{st}.$$

6. **An illustrative example.** To illustrate the application of this theorem to the determination of one of the characters of a fairly complicated "simple" group, we choose the simple group $G_{25920}$, which is an invariant subgroup of index 2 in the group of isomorphisms of the general cubic surface. It can be represented by a group of permutations $R^1$ on the 45 tritangent planes (or triangles) of the cubic surface, and also by a group of permutations $R^2$ on the 27 lines of the cubic surface. These two transitive groups, when completely reduced, have besides the identity one common irreducible component, whose character we shall compute by our theorem (5.13). Let $H^1$ and $H^2$ be the subgroups of orders $h^1 = g/n^{(1)} = 576$ and $h^2 = 960$, which leave fixed, respectively, a particular triangle $T$ and one of its lines $L$. The common subgroup of order $h_1^{12} = 192$ leaves both fixed. There are just two double cosets $H_\alpha^{12}$, namely $H_1^{12} = H^1H^2/192$, consisting of 2880 elements which transform $T$ into one of the five triangles on $L$, and a second double coset $H_2^{12}$ consisting of 23040 elements which transform $T$ into one of the other 40 triangles. Hence there are just two irreducible components common to $R^1$ and $R^2$, the identity and one other. There are three double cosets $H_\alpha^{11}$, namely $H_1^{11} = H^1$ containing 1 coset leaving $T$ fixed, $H_2^{11}$ containing 12 cosets which take $T$ into one of the 12 "adjacent" triangles having a line in common with it, and $H_3^{11}$ containing 32 cosets taking $T$ into a non-adjacent triangle.

To determine the constants $c^{tst}_{\alpha\beta\eta}$, we first note that

$$H_1^{12}H_1^{21}/960 = 3H_1^{11} + H_2^{11}, \tag{6.1}$$

since just 1/5 of the operations of $H^1H^2$ belong to $H^1$ and the other 4/5 transform $T$ into an adjacent triangle, and since $H^{11}_2$ contains 12 times as many elements as $H^{11}_1$. Then we use the formulas (3.12), which become

$$\begin{aligned}\sum_{\alpha} c^{121}_{\alpha 1\nu} &= \sum_{\alpha} c^{121}_{1\alpha\nu} = h^1/k^{12}_1 = 3,\\ \sum_{\alpha} c^{121}_{\alpha 2\nu} &= \sum_{\alpha} c^{121}_{2\alpha\nu} = h^1/k^{12}_2 = 24.\end{aligned} \tag{6.2}$$

The matrices $M^{\cdot}(H^{11}_{\nu})$ may now be written

$$M^2(H^{11}_1) = \begin{pmatrix}3 & 0\\ 0 & 24\end{pmatrix}, \qquad M^2(H^{11}_2) = \begin{pmatrix}1 & 2\\ 2 & 22\end{pmatrix},$$
$$M^2(H^{11}_3) = \begin{pmatrix}0 & 3\\ 3 & 21\end{pmatrix}. \tag{6.3}$$

In factoring the determinant $|K^{21}|$ of (1.3) it will be convenient to omit the superscripts from $b^1_{\nu}$, $k^1_{\nu\lambda}$, $g^1_{\nu\lambda}$, $A^{21}$. Then

$$\begin{aligned}|K^{21}| &= \begin{vmatrix} 3b_1 + b_2 & 2b_2 + 3b_3 \\ 2b_2 + 3b_3 & 24b_1 + 22b_2 + 21b_3 \end{vmatrix}\\ &= (b_1 + b_2 + b_3)(72b_1 + 18b_2 - 9b_3).\end{aligned} \tag{6.4}$$

The first of these factors corresponds to the identity representation $\Gamma_{\iota}$. The other factor remains to be investigated. Since $n^1 = 45$, equation (5.13) gives

$$(45)^2(72b_1 + 18b_2 - 9b_3) = 27\ 45A \sum_{\lambda} a_{\lambda}\chi(C_{\lambda}) \tag{6.5}$$

where

$$b_{\nu} = \sum_{\lambda} a_{\nu}k_{\nu\lambda}, \qquad A = \mathrm{P}^{21}_1\bar{\mathrm{P}}^{21}_1. \tag{6.6}$$

Equating coefficients of $a_{\lambda}$ in (6.5) after using (6.6) we have

$$A\chi(C_{\lambda}) = (5/3)(72k_{1\lambda} + 18k_{2\lambda} - 9k_{3\lambda}). \tag{6.7}$$

The degree $d$ of this representation is $d = 120/A$, which must be an integer less than 27. Noting that $k_{1\lambda} = g_{1\lambda}$, $k_{2\lambda} = 12g_{2\lambda}$, and $k_{3\lambda} = 32g_{3\lambda}$, we may rewrite (6.7) in the form

$$\begin{aligned}\chi(C_{\lambda}) &= d(g_{1\lambda} + 3g_{2\lambda} - 4g_{3\lambda})\\ &= g_{\lambda}\chi_{\lambda}\end{aligned} \tag{6.8}$$

where $\chi_\lambda$ is the trace of a single element of $C_\lambda$ and $g_\lambda$ is the number of elements in the class $C_\lambda$. To complete the computation it is necessary to find the constants $g_{\nu\lambda}$. Taking the 20 classes of conjugates of the simple group of order 25920 in the order listed in an earlier paper,[4] we calculate $\chi(C_\lambda)$ and $\chi_\lambda$ as follows.

| $g_\lambda$ | $g_{1\lambda}$ | $g_{2\lambda}$ | $g_{3\lambda}$ | $\chi(C_\lambda)$ | $\chi_\lambda$ |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | $d$ | $d = 20$ |
| 45 | 13 | 0 | 1 | $9d$ | $d/5 = 4$ |
| 270 | 30 | 12 | 3 | $54d$ | $d/5 = 4$ |
| 540 | 12 | 12 | 12 | 0 | $0 = 0$ |
| 40 | 8 | 0 | 1 | $4d$ | $d/10 = 2$ |
| 40 | 8 | 0 | 1 | $4d$ | $d/10 = 2$ |
| 360 | 8 | 0 | 11 | $-36d$ | $-d/10 = -2$ |
| 360 | 8 | 0 | 11 | $-36d$ | $-d/10 = -2$ |
| 2160 | 48 | 48 | 48 | 0 | $0 = 0$ |
| 2160 | 48 | 48 | 48 | 0 | $0 = 0$ |
| 480 | 32 | 0 | 14 | $-24d$ | $-d/20 = -1$ |
| 1440 | 32 | 48 | 26 | $72d$ | $d/20 = 1$ |
| 240 | 32 | 12 | 2 | $60d$ | $d/4 = 5$ |
| 2160 | 96 | 60 | 42 | $108d$ | $d/20 = 1$ |
| 720 | 64 | 12 | 16 | $36d$ | $d/20 = 1$ |
| 720 | 64 | 12 | 16 | $36d$ | $d/20 = 1$ |
| 3240 | 72 | 72 | 72 | 0 | $0 = 0$ |
| 5184 | 0 | 144 | 108 | 0 | $0 = 0$ |
| 2880 | 0 | 48 | 72 | $-144d$ | $-d/20 = -1$ |
| 2880 | 0 | 48 | 72 | $-144d$ | $-d/20 = -1$ |

The scalar product of the columns $\chi(C_\lambda)$ and $\chi_\lambda$ is equal to the order 25920 of the group, so we have $1296d^2/20 = 25920$; $d = 20$; $A = 6$. Hence the degree $d$ is 20, and the last column gives the required character of the irreducible representation of this degree. Eliminating the common components of degree 1 and 20 from the permutations on 45 and 27 symbols, we immediately obtain two other irreducible components of degree 24 and 6, respectively. The components of degree 1 and 20 occur in the Kronecker square of the one of degree 6, and an irreducible component of degree 15 is left. Thus the laborious work of obtaining the character of degree 20 bears fruit in obtaining three more characters with almost no work at all. The theorem of this paper gives a method of digging out some of the characters of a group if simpler methods are not available.

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[4] J. S. Frame, *The simple group of order* 25920, Duke Math. J. vol. 2 (1936) p. 483.

# THE ZEROS OF CERTAIN COMPOSITE POLYNOMIALS

MORRIS MARDEN

1. **Introduction.** If $A_0(z)$ is a given $m$th degree polynomial and

$$(1.1)\quad A_k(z) = (\beta_k - z)A'_{k-1}(z) + (\gamma_k - k)A_{k-1}(z), \qquad \gamma_k \neq m + k, \quad k = 1, 2, \cdots, n,$$

we may obtain various theorems on the relative location of the zeros of $A_0(z)$ and $A_n(z)$ by the familiar method of first finding such relations for two successive $A_k(z)$ and then iterating the relations $n$ times.

This method has already been employed in the study of the zeros of sequence (1.1) for the following three cases: (1) for all $k$, $\beta_k = 0$ and $\gamma_k$ is real;[1] (2) for all $k$, $\gamma_k = m+1$—a limiting case leading to Grace's theorem,[2] and (3) the limiting case that for all $k$, as $h \to 0$, $h\beta_k \to \beta_k'$ and $h(\gamma_k - k) \to 1$, in which case $\lim h^k A_k(z)$ is a linear combination of $A_0(z)$ and its first $k$ derivatives.[3]

In the present article we propose to apply the method to the case that *the parameters $\beta_k$ and $\gamma_k$ are complex numbers represented by points within certain given regions of the plane.*

To calculate the $n$th iterate $A_n(z)$ in our case, let us define

$$(1.2)\quad A(z) \equiv A_0(z) \equiv a_0 + a_1 z + \cdots + a_m z^m;$$

$$(1.3)\quad \begin{aligned} B(z) &\equiv (\beta_1 - z)(\beta_2 - z) \cdots (\beta_n - z) \\ &\equiv b_0 + b_1 z + \cdots + b_n z^n, \end{aligned}$$

$$(1.4)\quad C(z) \equiv (\gamma_1 - 1 - z)(\gamma_2 - 2 - z) \cdots (\gamma_n - n - z);$$

$$S(z, k, p) \equiv B(z) \sum \frac{\gamma_{j_1}^{(k+p)} - 1}{\beta_{j_1} - z} \cdot \frac{\gamma_{j_2}^{(k+p)} - 2}{\beta_{j_2} - z} \cdots \frac{\gamma_{j_{n-p}}^{(k+p)} - (n-p)}{\beta_{j_{n-p}} - z},$$

where $[\gamma_j^{(r)} \equiv \gamma_j - r]$ thus $\gamma_j^{(r)} - j$ is a zero of $C(z+r)$, $p < n$, and the sum is formed for all $j$, such that $1 \leq j_1 < j_2 < \cdots < j_{n-p} \leq n$;

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[1] See Laguerre, *Oeuvres*, Paris, 1898, vol. 1 pp. 200–202, and G. Polya, *Ueber einen Satz von Laguerre*, Jber. Deutschen Math. Verein. vol. 38 (1929) pp. 161–168.

[2] See Laguerre, *Oeuvres*, vol 1 p. 49, and G. Szego, *Bemerkungen zu einem Satz von S. H. Grace*, Math. Zeit. vol. 13 (1922) pp. 28–55, p. 33.

[3] See M. Fujiwara, *Eine Bemerkungen uber die elementare Theorie der algebraischen Gleichungen*, Tôhoku Math. J. vol. 9 (1916) pp. 102–108; T. Takagi, *Note on the algebraic equations*, Proceedings of the Physico-Mathematical Society of Japan vol. 3 (1921) pp. 175–179; J. L. Walsh, *On the location of the roots of polynomials*, Bull. Amer. Math. Soc. vol. 30 (1924) p. 52, and M. Marden, *On the zeros of the derivative of a rational function*, Bull. Amer. Math. Soc. vol. 42 (1936) p. 406.

$$S(z, k, n) = B(z) \quad \text{and} \quad S(z, k, p) \equiv 0 \quad \text{for} \quad p > n.$$

Then by repeated use of formula (1.1), we find for

$$D(z) = A_n(z) = d_0 + d_1 z + \cdots + d_m z^m \tag{1.5}$$

the two expressions

$$\begin{aligned} D(z) &= \sum_{p=0}^{n} S(z, 0, p) \frac{d^p A(z)}{dz^p} \\ D(z) &= \sum_{k=0}^{m} \sum_{p=0}^{m-k} \frac{(k+p)!}{k!} S(0, k, p) a_{k+p} z^k. \end{aligned} \tag{1.6}$$

Let us note two special cases of these formulas. First, if $\beta_k = 0$ for all $k$, then

$$S(0, k, p) = 0 \quad \text{for} \quad p \neq 0, \qquad S(0, k, 0) = C(k)$$

and, hence,

$$D(z) = C(0)a_0 + C(1)a_1 z + \cdots + C(m)a_m z^m. \tag{1.7}$$

Secondly, if, for all $k$, $\gamma_k = \gamma + 1$, where $\gamma$ is any constant other than $m, m+1, \cdots, m+n-1$, then

$$\begin{aligned} S(0, k, p) &= (\gamma - k - p)(\gamma - k - p - 1) \cdots (\gamma + 1 - k - n) \sum \beta_n \beta_n \cdots \beta_j \\ &= (-1)^{n-p}(n-p)! C_{\gamma-k-p, n-p} b_{n-p} \end{aligned}$$

where $C_{r,s} = r(r-1) \cdots (r-s+1)/1 \cdot 2 \cdots s$ and, hence, except for the multiplier $n!$,

$$D(z) = \sum_{k=0}^{m} \sum_{p=0}^{m-k} (-1)^{n-p} C_{n,p}^{-1} C_{\gamma-k-p, n-p} C_{k+p, k} a_{k+p} b_{n-p} z^k \tag{1.8}$$

with $b_{n-p} = 0$ for $p > n$.

In what follows it will be convenient to denote by a script capital $\mathcal{F}$ a region containing all the zeros of a given function $F(z)$. Thus, $\mathcal{A}$: $|z| \leq r$ will mean that all the zeros of the polynomial $A(z)$ lie in or on the circle $|z| = r$.

2. **Zeros of two successive $A_k(z)$.** Using the preceding notation, the following lemma may be stated.

LEMMA. *Let $\gamma_j' = \gamma_j - j$ denote the zeros of $C(z)$. Then,*
(a) $\mathcal{A}_k$: $r_1 \leq |z| \leq r_2$ *and* $|\beta_k| \leq \lambda r_1$ *imply*

$$\mathcal{A}_{k+1}\colon\ r_1 \min\left[1, \frac{|\gamma_k'| - m\lambda}{|\gamma_k' - m|}\right] \leqq |z| \leqq r_2 \max\left[1, \frac{|\gamma_k'| + m\lambda}{|\gamma_k' - m|}\right]; \tag{2.1}$$

(b) $\mathcal{A}_k$: $|z| \leqq r$ *and* $|\beta_k| \geqq \lambda r$ *imply*

$$\mathcal{A}_{k+1}\colon\ |z| \leqq r \quad \text{and} \quad |z| \geqq r \max\left[1, \frac{m\lambda - |\gamma_k'|}{|m - \gamma_k'|}\right]; \tag{2.2}$$

(c) $\mathcal{A}_k$: $\omega_1 \leqq \arg z \leqq \omega_2$ *with* $\omega_2 - \omega_1 \leqq \pi$ *and* $\beta_k = 0$ *imply*

$$\mathcal{A}_{k+1}\colon\ \omega_1 + \min\left(0, \arg \frac{\gamma_k'}{\gamma_k' - m}\right) \leqq \arg z \leqq \omega_2 + \max\left(0, \arg \frac{\gamma_k}{\gamma_k' - m}\right). \tag{2.3}$$

This lemma may be deduced from the results of a previous paper[4] or may be proved directly as follows.

Let $\mathcal{A}_k$ be a circular region and let $\zeta$ be any zero of $A_{k+1}(z)$ outside $\mathcal{A}_k$. Then, by Laguerre's theorem,[5] there exists a point $\alpha$ in $\mathcal{A}_k$ such that $[A_k'(\zeta)/A_k(\zeta)] = m/(\zeta - \alpha)$ and, hence, by (1.1)

$$\zeta = \frac{\gamma_k' \alpha - m\beta_k}{\gamma_k' - m}. \tag{2.4}$$

In particular for $|\beta_k| \leqq \lambda r_1$, if $\mathcal{A}_k$: $|z| \leqq r_2$, then[6] we have that $|\zeta| \leqq r_2(|\gamma_k'| + m\lambda)|\gamma_k' - m|^{-1}$, whereas if $\mathcal{A}_k$: $|z| \geqq r_1$, then $|\zeta| \geqq r_1(|\gamma_k'| - m\lambda)|\gamma_k' - m|^{-1}$. Hence, if all the zeros of $A_k(z)$ lie in the ring $r_1 \leqq |z| \leqq r_2$, an arbitrarily chosen zero of $A_{k+1}(z)$ must lie in the ring (2.1).

If $|\beta_k| \geqq \lambda r$ and $\mathcal{A}_k$: $|z| \leqq r$, then $|\zeta| \geqq r(m\lambda - |\gamma_k'|)|\gamma_k' - m|^{-1}$ and hence the zeros of $A_{k+1}(z)$ not satisfying the first inequality (2.2) must satisfy the second inequality (2.2).

Finally, for $\beta_k = 0$, if $\mathcal{A}_k$: $\omega \leqq \arg z \leqq \omega + \pi$, then $\omega + \arg [\gamma_k'(\gamma_k' - m)^{-1}] \leqq \arg \zeta \leqq \omega + \pi + \arg [\gamma_k'(\gamma_k' - m)^{-1}]$. Setting $\omega = \omega_1$ and $\omega = \omega_2 - \pi$ and combining the results, we conclude that, if all the zeros of $A_k(z)$ lie in the sector $\omega_1 \leqq \arg z \leqq \omega_2$, then all the zeros of $A_{k+1}(z)$ lie in the sector (2.3).

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[4] M. Marden, ibid. pp. 400–401. See also J. L. Walsh, *On the location of the roots of certain types of polynomials*, Trans. Amer. Math. Soc. vol. 24 (1922) p. 169, lemma, and Polya-Szegö, *Aufgaben der Analysis*, Berlin 1925 vol. 2 p. 58, problem 117.

[5] Laguerre, *Oeuvres*, vol. 1 p. 49.

[6] See M. Marden, ibid. p. 402.

3. **Zeros of $A_0(z)$ and $A_n(z)$.** We shall now apply part (1) of the lemma to the successive $A_k(z)$ in order to determine the relative location of the zeros of the polynomials $A(z) \equiv A_0(z)$, $B(z)$, $C(z)$ and $D(z) \equiv A_n(z)$. In addition to the notation used hitherto, we shall use the symbol $\mathfrak{C}(z)$ for the polynomial whose zeros are the moduli of the zeros of $C(z)$:

$$\mathfrak{C}(z) = (|\gamma_1'| - z)(|\gamma_2'| - z) \cdots (|\gamma_n| - z).$$

THEOREM I. *Given the positive constants $\rho$ and $\lambda$ ($\lambda < 1$). Then,*

(1) $\mathcal{A}$: $|z| \leqq r$, $\mathcal{B}$: $|z| \leqq \lambda r$ *and* $\mathcal{C}$: $\rho|z-m| \geqq |z| + m\lambda$ *imply* $\mathcal{D}$: $|z| \leqq r \max(1, \rho^n)$;

(2) $\mathcal{A}$: $|z| \leqq r$, $\mathcal{B}$: $|z| \leqq \lambda r$ *and* $\mathcal{C}$: $0 < \rho|z-m| \leqq |z| + m\lambda$ *with* $\rho \geqq 1$ *imply* $\mathcal{D}$: $|z| \leqq r|\mathfrak{C}(-m\lambda)/C(m)|$;

(3) $\mathcal{A}$: $|z| \geqq r$, $\mathcal{B}$: $|z| \leqq \lambda r|\mathfrak{C}(m\lambda)/C(m)|$ *and* $\mathcal{C}$: $\rho|z-m| \geqq |z| - m\lambda > 0$ *with* $\rho \leqq 1$ *imply* $\mathcal{D}$: $|z| \geqq r|\mathfrak{C}(m\lambda)/C(m)|$;

(4) $\mathcal{A}$: $|z| \geqq r$, $\mathcal{B}$: $|z| \leqq \lambda r \min(1, \rho^n)$ *and* $\mathcal{C}$: $0 < \rho|z-m| \leqq |z| - m\lambda$ *imply* $\mathcal{D}$: $|z| \geqq r \min(1, \rho^n)$.

To prove this theorem, let us define

$$\mu_k = |m - \gamma_k'|^{-1}(|\gamma_k'| + m\lambda);$$

$$M_k = \max \mu_1^{\sigma_1} \mu_2^{\sigma_2} \cdots \mu_k^{\sigma_k}, \qquad \text{where } \sigma_j = 0, 1;$$

$$\nu_k = |m - \gamma_k'|^{-1}(|\gamma_k'| - m\lambda) \quad \text{if} \quad |\gamma_k'| > m\lambda \quad \text{and}$$

$$\nu_k = 0 \qquad \text{if} \quad |\gamma_k'| \leqq \lambda m;$$

$$N_k = \min \nu_1^{\sigma_1} \nu_2^{\sigma_2} \cdots \nu_k^{\sigma_k}, \qquad \text{where } \sigma_j = 0, 1.$$

If $\mathcal{A}$: $|z| \leqq r$ and $\mathcal{B}$: $|z| \leqq \lambda r$, then by the right side of (2.1)

$$\mathcal{A}_1: |z| \leqq rM_1, \quad \mathcal{A}_2: |z| \leqq rM_2, \cdots, \mathcal{A}_n: |z| \leqq rM_n.$$

Since in part (1) of Theorem I

$$\mu_k \leqq \rho,$$

$M_n = \max(1, \rho^n)$, and, since in part (2) $\mu_k \geqq 1$,

$$M_n = \mu_1 \mu_2 \cdots \mu_n = |\mathfrak{C}(-m\lambda)/C(m)|.$$

If $\mathcal{A}$: $|z| \geqq r$ and $\mathcal{B}$: $|z| \leqq \lambda r N_n$, then by the left side of (2.1)

$$\mathcal{A}_1: |z| \geqq rN_1, \quad \mathcal{A}_2: |z| \geqq rN_2, \cdots, \mathcal{A}_n: |z| \geqq rN_n.$$

Since in part (3) of Theorem I $0 < \nu_k \leqq \rho \leqq 1$, $N_n = \nu_1 \nu_2 \cdots \nu_n = |\mathfrak{C}(m\lambda)/C(m)|$; whereas since in part (4) $\nu_k \geqq \rho$, $N_n = \min(1, \rho^n)$. We have thus established Theorem I.

It is to be noticed that each region $\mathcal{C}$ of Theorem I is bounded by one of the ovals $\rho|m-z| = |z| \pm m\lambda$ of the cartesian curve[7]

$$(3.1)\quad [(\rho^2-1)(x^2+y^2) - 2m\rho^2 x + m^2(\rho^2-\lambda^2)]^2 = 4m^2\lambda^2(x^2+y^2)$$

having ordinary foci at the three points $z=0$, $z=m$ and $z=m(\rho^2-1)^{-1}(\rho^2-\lambda^2)$ and a singular focus at the point $z=m\rho^2(\rho^2-1)^{-1}$. If $\rho>1$, curve (3.1) consists of two nested ovals both enclosing $z=m$ and both excluding $z=0$; in this case, the region $\mathcal{C}$ of part (1) of the theorem is the exterior of the outer oval, $\mathcal{C}$ of part (2) is the interior of the outer oval exclusive of point $z=m$ and $\mathcal{C}$ of part (4) is the interior of the inner oval exclusive of point $z=m$. If $\rho=1$, curve (3.1) degenerates into the hyperbola with foci at $z=0$ and $z=m$ and transverse axis of $m\lambda$; in this case $\mathcal{C}$ of part (1) is the region left of the left branch of the hyperbola, $\mathcal{C}$ of part (2) is the region right of the left branch not including $z=m$, $\mathcal{C}$ of part (3) is the region common to the exterior of circle $|z|=m\lambda$ and the left of the right branch and $\mathcal{C}$ of part (4) is the interior of the right branch with point $z=m$ omitted. If $\lambda<\rho<1$, curve (3.1) consists of nested ovals, now however both containing $z=0$ and excluding $z=m$; in this case, $\mathcal{C}$ of part (1) is the interior of the inner oval, $\mathcal{C}$ of part (3) is the region common to the exterior of circle $|z|=m\lambda$ and the interior of the outer oval and $\mathcal{C}$ of part (4) is the exterior of the outer oval exclusive of point $z=m$. In the latter case, if $\rho\to\lambda$, the inner oval shrinks to a point and hence, for $\rho<\lambda$, $\mathcal{C}$ of part (1) is a null-set, and the $\mathcal{C}$'s of parts (3) and (4) are those described for $\lambda<\rho<1$.

In the foregoing discussion, we have implied that $\lambda\neq 0$. If $\lambda=0$, curve (3.1) degenerates into the dipolar circle $\rho|z-m| = |z|$ and $D(z)$ is given by formula (1.7). We may thus state the following corollary.

COROLLARY. *If all the zeros of a polynomial* $A(z)=a_0+a_1z+\cdots+a_mz^m$ *lie in the ring* $0\leqq r_1\leqq |z| \leqq r_2\leqq\infty$ *and if all the zeros of an nth degree polynomial* $C(z)$ *lie in the connected region bounded by the circles* $|z|=\rho_1|z-m|$ *and* $|z|=\rho_2|z-m|$ *with* $\rho_1\leqq\rho_2$, *then all the zeros of the polynomial* $D(z)=C(0)a_0+C(1)a_1z+\cdots+C(m)a_mz^m$ *lie in the ring*[8]

$$(3.2)\qquad r_1 \min(1, \rho_1^n) \leqq |z| \leqq r_2 \max(1, \rho_2^n).$$

*If* $\rho_2<1$, *the left side of* (3.2) *may be replaced by the then larger number*

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[7] See G. Loria, *Curve piane speciali*, Milan, 1930, vol. I pp. 212–214.

[8] For the cases (1) $r_1=0$, $\rho_1=0$, $\rho_2=1$; (2) $r_2=\infty$, $\rho_1=1$, $\rho_2=\infty$; and (3) $r_1=r_2$, $\rho_1=\rho_2=1$, see N Obrechkoff, *Sur les zeros des polynômes*, C. R. Acad. Sci. Paris vol 209 (1939) pp. 1270–1272, and L. Weisner, *Roots of certain classes of polynomials*, Bull. Amer. Math. Soc. vol. 48 (1942) p. 283–286.

$r_1|C(0)/C(m)|$ *and, if* $1<\rho_1$, *the right side may be replaced by the then smaller number* $r_2|C(0)/C(m)|$.

So far we have applied part (1) of the lemma to the successive $A_k(z)$. Similarly, if we apply part (3) of the lemma and formula (1.7), we may obtain the following result.

THEOREM II. *If all the zeros of the polynomial* $A(z)=a_0+a_1z+\cdots+a_mz^m$ *are in the sector* $\omega_1\leqq\arg z\leqq\omega_2$ *with* $\omega_2-\omega_1=\omega\leqq\pi$, *and if all the zeros of an nth degree polynomial* $C(z)$ *are in the lune* $\theta_1\leqq\arg[z/(z-m)]\leqq\theta_2$ *with* $|\theta_1|+|\theta_2|\leqq(\pi-\omega)/n$, *then all the zeros of the polynomial* $D(z)=a_0C(0)+a_1C(1)z+\cdots+a_mC(m)z^m$ *lie in the sector*

$$\omega_1+\min(0,n\theta_1)\leqq\arg z\leqq\omega_2+\max(0,n\theta_2). \tag{3.3}$$

*If* $\theta_2<0$, $\min(0,n\theta_1)$ *may be replaced in* (3.3) *by the then larger number* $\arg C(0)/C(m)$ *and, if* $0<\theta_1$, $\max(0,n\theta_2)$ *may be replaced by the then smaller number* $\arg C(0)/C(m)$.

4. **Entire functions.** Theorem II and the corollary to Theorem I may be generalized at once through replacing

$$\begin{aligned} D(z) &= a_0C(0)+a_1C(1)z+\cdots+a_mC(m)z^m \\ &= \delta(\delta_1-z)(\delta_2-z)\cdots(\delta_m-z) \end{aligned}$$

by

$$F(z)=a_0E(0)+a_1E(1)z+\cdots+a_mE(m)z^m,$$

where

$$E(z)=e^{\lambda z}C(z) \quad \text{and} \quad \lambda=\mu+i\nu.$$

In fact, since

$$F(z)=\sum_{k=0}^{m}a_kC(k)e^{\lambda k}z^k=D(e^{\lambda}z)=\delta e^{m\lambda}\prod_{k=1}^{m}(\delta_ke^{-\lambda}-z),$$

the substitution of $E(z)$ and $F(z)$ for $C(z)$ and $D(z)$ would require only the following changes: in the corollary to Theorem I, inequality (3.2) becomes

$$e^{-\mu}r_1\min(1,\rho_1^m)\leqq|z|\leqq e^{-\mu}r_2\max(1,\rho_2^m) \tag{4.1}$$

where $e^{m\mu}|E(0)/E(m)|$ may replace $\min(1,\rho_1^m)$ if $\rho_2\leqq1$ and $\max(1,\rho_2^m)$ if $\rho_1\leqq1$; in Theorem II, inequality (3.3) becomes

$$\omega_1-\nu+\min(0,n\theta_1)\leqq\arg z\leqq\omega_2-\nu+\max(0,n\theta_2) \tag{4.2}$$

where $[m\nu+\arg E(0)/E(m)]$ may replace min $(0, n\theta_1)$ if $\theta_2\leqq 0$ and max $(0, n\theta_2)$ if $0<\theta_1$.

Furthermore, these results may be extended to entire functions $E(z)$ of genus zero or one provided the zeros of $E(z)$ are assumed to lie in infinite regions, determined by taking $\rho_1=1$ or $\rho_2=1$ and $\theta_1=\theta_2=0$.

THEOREM III. *Given the entire functions*

$$A(z) = \sum_{k=0}^{m} a_k z^k, \qquad E(z) = e^{\lambda_0 z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\gamma_k}\right)e^{\lambda_k z},$$

$$F(z) = \sum_{k=0}^{m} a_k E(k) z^k,$$

*where* $\lambda_j=\mu_j+i\nu_j$.

(a) *If all the zeros of* $A(z)$ *lie in the ring* $0\leqq r_1\leqq |z| \leqq r_2\leqq \infty$, *if all the zeros of* $E(z)$ *lie in the region* $\rho_1\leqq |z/(z-m)| \leqq\rho_2$ *with at least one number* $\rho_1$, $\rho_2$ *unity, and if* $\mu_0+\mu_1+\cdots\to\mu$, *then all the zeros of* $F(z)$ *lie in the ring* $K_1e^{-\mu}r_1\leqq |z| \leqq K_2e^{-\mu}r_2$, *where* $K_1=1$ *or* $e^{\mu m}|E(0)/E(m)|$ *according as* $\rho_1=1$ *or* $\rho_1<1$ *and* $K_2=1$ *or* $e^{\mu m}|E(0)/E(m)|$ *according as* $1=\rho_2$ *or* $1<\rho_2$.

(b) *If all the zeros of* $A(z)$ *lie in the sector* $\omega_1\leqq\arg z\leqq\omega_2$ *with* $\omega_2-\omega_1\leqq\pi$, *if all the zeros of* $E(z)$ *lie on the real axis outside of the segment* $(0, m)$ *and if* $\nu_0+\nu_1+\cdots\to\nu$, *then all the zeros of* $F(z)$ *lie in the sector* $\omega_1-\nu\leqq\arg z\leqq\omega_2-\nu$.

Theorem III(b) is a partial generalization of results due to Laguerre and Polya[1] in the case that both $\nu=0$ and all the zeros of $A(z)$ are real. However, it may also be derived from this special case by use of the theorem quoted in problem 153, p. 65, vol. 2 Polya-Szegö's *Aufgaben der Analysis*. For this fact and its following proof, the author is indebted to the referee, Professor Polya.

We may assume without loss of generality that $\nu=0$. Then, according to the Laguerre-Polya results, $\alpha_k=E(k)$ form a set of multipliers such that, if any polynomial $A(z)=a_0+a_1z+\cdots+a_mz^m$ has only positive (negative) zeros, so has also the polynomial $C(z)=\alpha_0a_0+\alpha_1a_1z+\cdots+\alpha_ma_mz^m$. But such multipliers have also the property that, if all the zeros of $A(z)$ lie in the sector $\omega_1\leqq\arg z\leqq\omega_2$ with $\omega_2-\omega_1\leqq\pi$, all the zeros of $C(z)$ also lie in this sector. For, since all the zeros of $(1+z)^m$ are negative, the zeros of polynomial

$$G(z) = \alpha_0 + C_{m,1}\alpha_1 z + C_{m,2}\alpha_2 z^2 + \cdots + \alpha_m z^m$$

are also all negative, and, since the sector is a convex region containing the origin, the theorem from Polya-Szegö may be applied with the $F(z)$ of the theorem taken as $A(z)$. Theorem III(b) then follows immediately.

As an application of Theorem III, let us consider the polynomial $F(z)=\sum_{k=0}^{m} a_k G(k+p) z^k$ where $p>0$ and $G(z)=\Gamma(z)^{-1}=e^{\gamma z}\prod_{n=1}^{\infty}(1+n^{-1}z)e^{-z/n}$, the reciprocal of the gamma function. Since $\nu=0$ and all the zeros of $G(z+p)$ are negative, any sector $\omega_1 \leqq \arg z \leqq \omega_2 \leqq \pi-\omega_1$ containing all the zeros of $A(z)$ will also contain all the zeros of $F(z)$. For example, if $A(z)=(z-2)(z+1-i)$, then $F(z)=0.5z^2-(1+i)z-2+2i$, which has the zeros $(3.058+0.514i)$ and $(-1.058+1.486i)$, both thus being in the sector $0 \leqq \arg z \leqq 135°$ containing the zeros of $A(z)$.

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# ON THE EXTENSION OF A VECTOR FUNCTION SO AS TO PRESERVE A LIPSCHITZ CONDITION

F. A. VALENTINE

1. **Introduction.** Let $V$ be a two-dimensional Euclidean space, and let $x$ be a vector ranging over $V$. The vector function $f(x)$ is to be a vector in $V$ defined over a set $S$ of the space $V$. The Euclidean distance between any two points $x$ and $y$ in the plane is denoted by $|x-y|$. Furthermore $f(x)$ is to satisfy a Lipschitz condition, so that there exists a positive constant $K$ such that

$$|f(x_1)-f(x_2)| \leqq K|x_1-x_2| \tag{1}$$

holds for all pairs $x_1$ and $x_2$ in $S$.

In event $f(x)$ is a real-valued function of a variable $x$ ranging over a set $S$ of a metric space, then the extension of the definition of $f(x)$ to any set $T \supset S$ so as to satisfy the condition (1) has been accomplished.[1] The present paper establishes the result that the *vector* function $f(x)$ can be extended to any set $T \supset S$ so as to satisfy the Lipschitz condition with the same constant $K$. In §3 it is shown how the method used to obtain the above result can be applied to yield an extension for the case considered by McShane.[2] If $f(x)$ has its

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[1] E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 837–842.

[2] Loc. cit.

functional values in a metric space then, in general, it cannot be extended so as to preserve (1). The geometric theorems which arise in forming the extension prove to be of as much interest as the extension itself. In the following section it is shown that this extension is a consequence of the following theorem about sets of circles in the plane. A circle with center $x_i$ and radius $r_i$ is the two-dimensional set of points $x$ for which $|x-x_i| \leqq r_i$.

THEOREM 1. *Consider in the plane two sets of circles $M$ and $M'$ such that to each circle in $M$ there corresponds a circle in $M'$. Let $C_i' \in M'$, having center $x_i'$ and radius $r_i'$, correspond to $C_i \in M$, having center $y_i$ and radius $r_i$.*

*Suppose that*

$$r_i' = r_i$$

*for all corresponding circles $C_i$ and $C_i'$. Furthermore suppose that*

$$|x_i' - x_j'| \leqq |y_i - y_j| \tag{2}$$

*for all corresponding pairs of circles $(C_i, C_j)$ and $(C_i', C_j')$. Finally suppose that the intersection of all the circles in $M$ is not empty, so that the product*

$$\prod_M C_i \neq 0. \tag{3}$$

*Then it is true that the intersection of all the circles in $M'$ is such that*

$$\prod_{M'} C_i' \neq 0. \tag{4}$$

2. **The extension.** If the set $S$ consists of a single point $x_1$, the extension of $f(x)$ to a second point $x_2$ is trivial. We will let $S'$ be the set in the $f$-plane which corresponds to the set $S$ in the $x$-plane.

Suppose that the set $S$ consists of two points $x_1$ and $x_2$, and that we wish to extend $f(x)$ to a third point $x_3$. With the points $Kx_j$ as centers $(j=1, 2)$ draw circles $C_j$ with radii $r_j = K|x_j - x_3|$, respectively. Define $x_j'$ to be

$$x_j' = f(x_j), \qquad j = 1, 2. \tag{5}$$

With $x_j'$ as centers draw circles $C_j'$ with radii $r_j' = r_j$, respectively. Since by hypothesis $|x_1' - x_2'| \leqq K|x_1 - x_2|$, and since by construction the product $C_1 \cdot C_2 \neq 0$, it is true that $C_1' \cdot C_2' \neq 0$. To extend $f(x)$ to $x_3$ so as to satisfy (1), choose $x_3' \in C_1' \cdot C_2'$. Since $|x_j' - x_3'| \leqq r_j' = r_j = K|x_j - x_3|$, letting $f(x_3) = x_3'$, the function $f(x)$ has been extended from $S = (x_1, x_2)$ to $T = (x_1, x_2, x_3)$ so as to satisfy the Lipschitz condition (1).

The crucial extension, as we shall see, is from $S=(x_1, x_2, x_3)$ to $T=(x_1, x_2, x_3, x_4)$. In order to construct this extension, the following lemma and theorem are essential. For convenience denote the triangular set determined by three points $y_1, y_2, y_3$ by $\Delta(y_1, y_2, y_3)$.

LEMMA 1. *If three circles $C_i'$ $(i=1, 2, 3)$ with centers $x_i'$, respectively, are such that*

$$C_i' \cdot C_j' \ \Delta(x_1', x_2', x_3') \neq 0, \qquad i, j = 1, 2, 3, \tag{6}$$

$$C_1' \ C_2' \ C_3' \cdot \Delta(x_1', x_2', x_3') = 0, \tag{7}$$

*then it is true that*[3]

$$B(C_i') \cdot B(C_j') \cdot \Delta(x_1', x_2', x_3') \neq 0, \qquad i, j = 1, 2, 3, \tag{8}$$

*where $B(C_i')$ is the boundary of the circle $C_i'$.*

To prove Lemma 1 select three points $x_{ij} \in C_i' \cdot C_j' \ \Delta$ $(i=1, 2; j=2, 3; j>i)$ which exist by virtue of (6). Since $\Delta(x_{12}', x_{13}', x_{23}') \subset \Delta(x_1', x_2', x_3')$, in proving (8) it is sufficient to prove

$$B(C_i') \cdot B(C_j') \cdot \Delta(x_{12}', x_{13}', x_{23}') \neq 0. \tag{9}$$

Denote the line segment joining points $x$ and $y$ by $\overline{xy}$. Since $C_1'$ is convex, $\overline{x_{12}'x_{13}'} \subset C_1'$. Also (7) implies that the point $x_{23}'$ is not contained in $C_1'$. Hence the circle $C_1'$ contains a maximum subsegment $\overline{x_{12}'P_2}$ of $\overline{x_{12}'x_{23}'}$ and a maximum subsegment $\overline{x_{13}'P_3}$ of $\overline{x_{13}'x_{23}'}$, where the points $P_2 \neq P_3 \neq x_{23}'$. Since $\overline{P_2P_3} \subset C_1'$, since $x_{23}' \notin C_1'$, and since $B(C_1')$ cannot intersect the $B[\Delta(P_2, P_3, x_{23}')]$ except at $P_2$ and $P_3$, a simple arc of $B(C_1')$ joins $P_2$ and $P_3$ and is contained in $\Delta(P_2, P_3, x_{23}')$ and hence in $\Delta(x_{12}', x_{13}', x_{23}')$.

In exactly the same way, there exist points $Q_3 \in \overline{x_{13}'x_{23}'}$, $Q_1 \in \overline{x_{12}'x_{13}'}$, with $Q_1 \neq Q_3 \neq x_{13}'$, such that $Q_1$ and $Q_3$ are joined by a simple arc of $B(C_2')$ contained in $\Delta(x_{12}', x_{13}', x_{23}')$. *Suppose* the point $P_3$ lay between $x_{23}'$ and $Q_3$ on $\overline{x_{23}'x_{13}'}$. This supposition, together with the facts $C_2' \supset \overline{x_{23}'Q_3}$, $C_1' \supset \overline{x_{13}'P_3}$, implies that $\overline{x_{23}'x_{13}'} \subset C_1' + C_2'$. But since $\overline{x_{23}'x_{13}'} \subset C_3'$, the previous statement implies that $C_1' \ C_2' \cdot C_3' \cdot \Delta \neq 0$, contrary to assumption (7). Hence $P_3$ is *not* between $x_{23}'$ and $Q_3$ on

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[3] Lemma 1 is stated for circles in order to shorten the proof of Theorem 2; however the lemma can be generalized as follows: *Suppose that each of the sets $C_1$, $C_2$, $C_3$, $\Delta$ and each of the products $C_i \cdot \Delta$, $C_i \cdot C_j \cdot \Delta$ $(i, j=1, 2, 3)$ is a simply connected plane set having a simple closed curve as its boundary. Furthermore suppose $C_1 \cdot C_2 \cdot C_3 = 0$, $C_i \cdot C_j \cdot B(\Delta) \neq 0$, where $B(\Delta)$ is the boundary of $\Delta$. Then it is true that $B(C_i) \cdot B(C_j) \cdot \Delta \neq 0$.*

$\overline{x_{23}'x_{13}'}$, so that $Q_3 \in \overline{x_{23}'x_{13}'} - \overline{x_{13}'P_3}$. This implies that the points $P_2$, $Q_3$, $P_3$, $Q_1$ lie on the boundary of $\Delta(x_{12}', x_{13}', x_{23}')$ in a clockwise order *in the order* just given.

Since $P_2$ and $P_3$ are joined by a simple arc of $B(C_1')$ contained in $\Delta(x_{12}', x_{13}', x_{23}')$, and since $Q_1$ and $Q_3$ are joined by an arc of $B(C_2')$ having the same properties, condition (9) holds with $i=1$, $j=2$. Since we can choose any pair of the circles $C_i'$ and repeat the above argument, condition (9) has been proved, and hence Lemma 1 is established.

THEOREM 2. *Suppose the two triangles* $\Delta(y_1, y_2, y_3)$ *and* $\Delta(x_1', x_2', x_3')$ *are such that*

$$|x_i' - x_j'| \leqq |y_i - y_j|, \qquad i, j = 1, 2, 3. \tag{10}$$

*Then to any point* $y_4$ *in the plane there corresponds a point* $x_4'$ *contained in* $\Delta(x_1', x_2', x_3')$ *such that*

$$|x_4' - x_i'| \leqq |y_4 - y_i|, \qquad i = 1, 2, 3. \tag{11}$$

To prove Theorem 2, with $y_i$ as centers draw circles $C_i$ with radii $r_i = |y_4 - y_i|$, respectively. Similarly with $x_i'$ as centers draw circles $C_i'$ with radii $r_i' = r_i$, respectively. Since $y_4 \in C_i$, the products $C_i \cdot C_j \cdot \Delta(y_1, y_2, y_3) \neq 0$. Hence condition (10), together with the fact $r_i' = r_i$, implies condition (6).

We first prove that condition (10) implies $C_1' \cdot C_2' \cdot C_3' \cdot \Delta(x_1', x_2', x_3') \neq 0$. *Suppose this were not so.* Let $P_{ij}$ be a point of intersection of $B(C_i')$ and $B(C_j')$. Since we suppose (7) holds, Lemma 1 implies that $P_{ij}$ can be chosen so that $P_{ij} \in \Delta(x_1', x_2', x_3')$ $(i = 1, 2; j = 2, 3; j > i)$. Since we supposed $C_1' \cdot C_2' \cdot C_3' \cdot \Delta = 0$, we have

$$|x_2' - P_{12}| < |x_2' - P_{13}|. \tag{12}$$

Denote the angle less than or equal to $\pi$, determined by three points $a_1$, $a_2$, $a_3$, with vertex at $a_2$, by $\angle a_1a_2a_3$. Since by definition $|x_1' - P_{12}| = |x_1' - P_{13}| = r_1$, condition (12) implies by the law of cosines that

$$\angle\, x_2' x_1' P_{12} < \angle\, x_2' x_1' P_{13}. \tag{13}$$

Condition (13), together with the fact $P_{13} \in \Delta(x_1', x_2', x_3')$, implies the first of the inequalities

$$\begin{aligned} &\angle\, x_2' x_1' P_{12} + \angle\, x_3' x_1' P_{13} < \angle\, x_2' x_1' x_3', \\ &\angle\, x_1' x_3' P_{13} + \angle\, x_2' x_3' P_{23} < \angle\, x_1' x_3' x_2', \\ &\angle\, x_3' x_2' P_{23} + \angle\, x_1' x_2' P_{12} < \angle\, x_3' x_2' x_1'. \end{aligned} \tag{14}$$

The last two inequalities are proved exactly as the first.

The sum of all the angles in the three triangles $\Delta(x_i', P_{ij}, x_j')$ $(i=1, 2; j=2, 3; j>i)$ is $3\pi$. By adding the inequalities in (14), we see that in these three triangles $\Delta(x_i', P_{ij}, x_j')$ the *sum* of those six angles, each of which has its vertex at one of the points $x_1'$, $x_2'$, $x_3'$ is less than $\pi$. Hence the sum of the remaining three angles in $\Delta(x_i', P_{ij}, x_j')$ satisfies the condition

$$(15) \qquad \sum_{i,j} \angle\, x_i' P_{ij} x_j' > 2\pi, \qquad i = 1, 2; j = 2, 3; j > i.$$

Furthermore for the $\Delta(y_1, y_2, y_3)$ of Theorem 2 we have

$$(16) \qquad \sum_{i,j} \angle\, y_i y_4 y_j \leqq 2\pi, \qquad i = 1, 2; j = 2, 3; j > i,$$

the equality holding if $y_4 \in \Delta(y_1, y_2, y_3)$; otherwise the inequality holds. Consequently *at least one* of the three angles in (15) is greater than the corresponding angle in (16). Without loss of generality renumber the angles so that

$$(17) \qquad \angle\, x_1' P_{12} x_2' > \angle\, y_1 y_4 y_2.$$

By construction $|x_i' - P_{12}| = |y_i - y_4|$ $(i=1, 2)$. Condition (17) implies by the law of cosines that in the triangles $\Delta(x_1', P_{12}, x_2')$ and $\Delta(y_1', y_4', y_2)$

$$(18) \qquad |x_1' - x_2'| > |y_2 - y_1|.$$

This contradicts assumption (10). Hence the assumption that (7) holds is false. Thus choosing a point $x_4' \in C_1' \cdot C_2' \cdot C_3' \cdot \Delta \neq 0$, we have $|x_4' - x_i'| \leqq r_i' = r_i = |y_i - y_4|$ $(i=1, 2, 3)$ and Theorem 2 has been established.

We now readily establish the following theorem.

THEOREM 3. *If the plane vector $f(x)$ is defined on a plane set $S=(x_1, x_2, x_3)$ so as to satisfy the Lipschitz condition (1), then it can be extended to any plane set $T=(x_1, x_2, x_3, x_4)$ so as to be Lipschitz preserving. The extension $f(x_4)$ can be defined so as to be contained in the triangle formed by the points $f(x_i)$ $(i=1, 2, 3)$.*

PROOF. In Theorem 2, let

$$y_i = Kx_i, \qquad x_i' = f(x_i), \qquad y_4 = Kx_4, \qquad i = 1, 2, 3.$$

Then the point $x_4'$ in Theorem 2 is the desired extension $f(x_4) = x_4'$, since conditions (10) and (11) imply that

$$|f(x_j) - f(x_k)| \leqq K|x_j - x_k|, \qquad j, k = 1, \cdots, 4.$$

In order to extend $f(x)$ from an arbitrary set $S$ to any set $T \supset S$ *we first prove Theorem* 1. This can be accomplished by means of Theorem 2 and by a theorem of Helly[4] as follows. Choose an arbitrary but fixed set of three circles $C_i$ $(i=1, 2, 3)$ in the set of circles $M$ of Theorem 1. Since in Theorem 1 the hypotheses include the fact that $\prod_M C_i \neq 0$, choose the arbitrary point $y_4$ in Theorem 2 so that $y_4 \in \prod_{i=1}^3 C_i$. Then Theorem 2 implies that there exists a point $x_4'$ such that $|x_4' - x_i'| \leqq |y_4 - y_i|$. Since $y_4 \in \prod_{i=1}^3 C_i$, it is true that $|y_4 - y_i| \leqq r_i$, whence $|x_4' - x_i'| \leqq r_i$. Since $r_i' = r_i$ in Theorem 1, the above inequalities imply that *each set of three circles in $M'$ has a point common to all three circles. Now by Helly's theorem, with $n=2$, it is true that all the circles in $M'$ have a point in common.* Thus Theorem 1 is established.

Now we are in a position to prove the following theorem.

THEOREM 4. *Suppose the plane vector function $f(x)$ is defined and satisfies the Lipschitz condition* (1) *on a set $S$ of the plane.*

*Then if $T$ is any set containing $S$, it is true that $f(x)$ can be extended to $T$ so as to preserve the Lipschitz condition* (1). *The extension of $f(x)$ can be defined so that the set of points $(f(x)$, for $x \in T)$ is contained in any prescribed closed convex set containing $S'$, where $S' \equiv (f(x)$, for $x \in S)$.*

PROOF. First, we prove that if $U$ is any set on which $f(x)$ satisfies the Lipschitz condition (1), and if $x_0$ is an arbitrary point exterior to $U$, then $f(x)$ can be extended from $U$ to $U + x_0$ so as to satisfy (1). To do this let $x_i$ be an *arbitrary* point in $U$, and let

$$x_i' = f(x_i)$$

be the corresponding point in $U' = [f(x), x \in U]$. With $y_i = Kx_i$ as center and with radius $r_i = K|x_0 - x_i|$ draw a circle $C_i$. As $x_i$ ranges over $U$, denote the set of circles thus defined by $M$. Similarly with $x_i'$ as center and with radius $r_i' = r_i$ draw a circle $C_i'$, and denote the set of all circles, as $x_i'$ ranges over $U'$, by $M'$. Since condition (1) holds on $U$, condition (2) in Theorem 1 holds on $U$. Furthermore since by construction $r_i = K|x_0 - x_i|$ we have $\prod_M C_i \neq 0$. *Hence Theorem* 1 *implies that* $\prod_{M'} C_i' \neq 0$. Consequently there exists a point $x_0'$, with $x_0' \in \prod_{M'} C_i'$, which implies that

$$|x_0' - x_i'| \leqq r_i = K|x_0 - x_i|$$

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[4] E. Helly, Jber. Deutschen Math. Verein vol. 32 (1923) pp. 175–176. The theorem states: *If each $n+1$ sets of a family of closed, bounded, convex sets of the $n$-dimensional Euclidean space intersect, then there is a point common to all the sets.*

for all corresponding pairs $x_i$ and $x_i'$ in $U$ and $U'$, respectively. Hence letting $f(x_0)=x_0'$, we have extended $f(x)$ from $U$ to $U+x_0$ so that the Lipschitz condition (1) is preserved. *For convenience denote the above extension from an arbitrary set $U$ to the set $U+x_0$, where $x_0$ is arbitrary, as extension* E.

The extension from a set $S$ to a region $T\supset S$ of Theorem 4 can be accomplished as follows. Let $R$ be the set of all the points in the plane which have rational coordinates. Extension E of the preceding paragraph implies by *ordinary induction* that $f(x)$ can be extended from the set $S$ to the set $S+R$ so as to satisfy the Lipschitz condition (1). Let $\bar{x}$ be any other point in the plane. In order to effect the extension to $\bar{x}$ let $x_i$ be any sequence of points with rational coordinates such that $\lim_{i\to\infty} x_i=\bar{x}$. If we define $U=S+\sum x_i$, $x_0=\bar{x}$, then extension E of the preceding paragraph yields an extension $f(\bar{x})$ so that $f(x)$ satisfies (1) on $U+x_0$. Condition (1) implies that $\lim_{i\to\infty} f(x_i)=f(\bar{x})$. The function $f(x)$ is defined *uniquely* at $\bar{x}$, for let $y_i$ be any other sequence of rational points such that $\lim_{i\to\infty} y_i=\bar{x}$. Letting $U=S+\sum y_i$, and $x_0=\bar{x}$, extension E yields an extension $f^*(\bar{x})$ such that $f(x)$ satisfies (1) on $U+x_0$. Since

$$|f(y_i)-f(x_i)|\leqq K|y_i-x_i|,$$

and since $\lim_{i\to\infty} f(x_i)=f(\bar{x})$, it follows that

$$\lim_{i\to\infty} f(y_i)=\lim_{i\to\infty} f(x_i)=f^*(\bar{x})=f(\bar{x}). \tag{19}$$

The function $f(x)$ thus defined satisfies the Lipschitz condition (1) on the whole plane. To prove this statement, let $x$ and $y$ be any two points in the plane, and let $x_i$ and $y_i$ be two sequences of rational points such that $\lim_{i\to\infty} x_i=x$, $\lim_{i\to\infty} y_i=y$. Then the inequalities

$$\begin{aligned}|f(x)-f(y)| &\leqq |f(x)-f(x_i)|+|f(x_i)-f(y_i)|+|f(y_i)-f(y)|\\ &\leqq K|x-x_i|+K|x_i-y_i|+K|y_i-y|\end{aligned}$$

imply, by passage to the limit, that

$$|f(x)-f(y)|\leqq K|x-y|.$$

Thus the first part of Theorem 4 is proved since if we can extend $f(x)$ to the whole plane we can surely extend it to $T\supset S$.

To prove the last sentence of Theorem 4, let $L$ be any closed convex set such that $L\supset S'$. Consider the sets of circles $M$ and $M'$ defined in the first paragraph following Theorem 4, however with $U=S$ and $U'=S'$. Consider *any* two circles $C_1'$ and $C_2'$ in $M'$. Since the straight line joining the centers of these circles is in $L$, and since (1) implies that $C_1' \cdot C_2' \neq 0$, we have $C_1' \cdot C_2' \cdot L \neq 0$. Furthermore Theorem 1 implies that $\prod_{M'} C_i' \neq 0$. Hence the theorem of Helly[5] implies that the product

$$L \cdot \prod_{M'} C_i' \neq 0.$$

Hence in extension E one can choose $x_0' \in L$, whence $S' + x_0' \subset L$. Now by ordinary induction the function $f(x)$ can be extended to the set $S+R$ (where $R$ is the set of points with rational coordinates), so that the set $S'+R' = [f(x), x$ ranges over $S+R] \subset L$. Now the extension of $f(x)$ to the whole plane so that $f(x) \subset L$, $x$ arbitrary, can be constructed. For consider the point $\bar{x}$ in equation (19), where $x_i \subset R$. Since by (19) $\lim_{i\to\infty} f(x_i) = f(\bar{x})$, since $f(x_i) \in L$, and since $L$ is closed, it follows that $f(\bar{x}) \in L$. Thus the proof of Theorem 4 has been completed.

3. **Generalizations.** The functions studied by McShane[6] can be easily extended by the method of the preceding section. Let the *one-dimensional real-valued* function $f(x)$ be defined over a set $S$ of a metric space. Furthermore suppose there exists a real-valued function $w(t) \geqq 0$ defined for $0 \leqq t$, and satisfying the conditions

$$w(a) + w(b) \geqq w(a+b), \tag{20}$$

$$w(b) \geqq w(a), \qquad \text{for } b \geqq a. \tag{21}$$

*If the function* $f(x)$ *satisfies the condition*

$$|f(x_1) - f(x_2)| \leqq w(\|x_1, x_2\|) \tag{22}$$

*for all pairs* $(x_1, x_2)$ *in* $S$, *then* $f(x)$ *can be extended to the entire space so as to preserve condition* (22).

It should be noted that condition (20) does *not* imply that $w(t)$ be concave downward; however, any function $w(t) \geqq 0$, $t \geqq 0$, which is concave downward must satisfy condition (20). The function $w(t)$ is less restricted than that used by McShane.[7]

To prove the above theorem, let $S = (x_1, x_2)$ and suppose $x_3$ is a third point to which we wish to extend $f(x)$. With $f(x_i)$ $(i=1, 2)$ as centers draw circles $C_i'$ with radii $r_i' = w(\|x_i, x_3\|)$, respectively. These circles are linear intervals. Suppose $C_1' \cdot C_2' = 0$. Then by (20) and (21) we would have

$$\begin{aligned} |f(x_1) - f(x_2)| &> w(\|x_1, x_3\|) + w(\|x_2, x_3\|) \\ &\geqq w(\|x_1, x_3\| + \|x_3, x_2\|) \geqq w(\|x_1, x_2\|), \end{aligned}$$

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[5] Loc. cit.

[6] Loc. cit.

[7] Ibid.

which contradicts the hypothesis $|f(x_1)-f(x_2)| \leq w(\|x_1, x_2\|)$. Thus $C_1'\ C_2' \neq 0$, and any point $x_3' \in C_1'\ C_2'$ serves as an extension $f(x_3)$. In order to extend $f(x)$ from an arbitrary set $U$ on which (22) holds, to a set $U+x_0$, with $f(x_i)$ as centers, $x_i \in U$, draw circles $C_i'$ with radii $r_i' = w(\|x_i, x_0\|)$, respectively. Since each *pair* of the circles $C_i'$ intersect, *Helly's*[8] *theorem, with* $n=1$, implies that $\prod C_i' \neq 0$. This implies that $f(x)$ can be extended from $U$ to $U+x_0$ so as to preserve condition (22). The extension of $f(x)$ from $S$ to the whole space now follows by transfinite induction.

An advantage of the above procedure arises from the fact that *if* the extension were impossible, the method would reveal it. Counterexamples exist which show that if $f(x)$ has its values in a metric space, then, in general, $f(x)$ cannot be extended so as to preserve conditions (1) and (22). For example, let $x$ range over a two-dimensional metric space in which all the "unit spheres" are congruent squares having corresponding sides *parallel*. Let $f(x)$ have its values in the two-dimensional Euclidean space. *In Theorem* 1, let $C_i$ $(i=1, 2, 3)$ consist of three unit squares such that $C_1 \cdot C_2 \cdot C_3 = p$ consists of one and only one vertex from each square. Also choose $C_i'$ so that $r_i' = r_i = 1$, $x_1' = (1, 0)$, $x_2' = (-1, 0)$, $x_3' = (0, 1+\epsilon)$, $\epsilon > 0$. The constant $\epsilon$ can be chosen so that condition (2) holds. Although condition (3) holds, conclusion (4) fails. Hence one cannot extend $f(x)$ from $S=(x_1, x_2, x_3)$ to $T=(x_1, x_2, x_3, p)$ so as to preserve condition (1).

The first generalization of the material in §§1 and 2 would be to the $n$-dimensional case, and thence to a Hilbert space. In the $n$-dimensional case the crux of this generalization lies in the generalization of Theorem 1, since that of Helly applies to the $n$-dimensional case. In the case of a Hilbert space one would also need a generalization of the theorem of Helly. These matters are still open questions. The author wishes to express his appreciation to Professor Max Zorn and to Dr. W. T. Puckett with whom he has had stimulating conversation concerning these topics.

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[8] Loc. cit.

# A FAMILY OF FUNCTIONS AND ITS THEORY OF CONTACT[1]

J. F. RITT

**Introduction.** If $p_1, \cdots, p_n$ are fixed positive integers and $a_1, \cdots, a_n$ arbitrary constants, it is possible so to choose the $a_i$ as to make the function

$$y(x) = \prod_{i=1}^{n} (x - a_i)^{p_i} \tag{1}$$

and its first $p_1 + \cdots + p_n - 1$ derivatives equal to zero for any single value $x_0$ of $x$. This is accomplished by taking each $a_i$ equal to $x_0$. One might say, on this basis, that *the family of polynomials* (1) *has contact of order* $p_1 + \cdots + p_n - 1$, *for every value of* $x$, *with* $y = 0$.

A more interesting situation is met when we allow the $p_i$ to be any fixed positive numbers, not necessarily integral. In that case $y(x)$ may be a function of many branches, with the quotient of any two branches equal to a constant of modulus unity. For our purposes it suffices to consider the value zero of $x$. If no $a_i$ is zero, each branch of $y(x)$ will be analytic at $x = 0$, with an expansion

$$c_0 + c_1 x + \cdots + c_s x^s + \cdots$$

where the $c_j$ depend on the $a_i$. The question which we examine is: *What is the greatest value of* $s$ *such that, by suitably varying the* $a_i$, *the coefficients* $c_0, \cdots, c_s$ *can be made to approach zero simultaneously*? Such a greatest value of $s$ exists, and will be called, below, *the order of contact of the family* (1) *with* $y = 0$. Denoting the greatest value of $s$ by $r$, we shall prove that

$$r \leqq q + n - 1 \tag{2}$$

where $q$ is the greatest integer less than $p_1 + \cdots + p_n$. When no proper subset of the $p_i$ has an integral sum, the equality sign holds in (2). For $n = 2$, (2) can be an inequality only when $p_1$ and $p_2$ are both integers. For $n \geqq 3$, (2) will certainly be an inequality if some integral power of $y(x)$ is a polynomial of degree not exceeding $q + n - 1$; thus the order of contact of the family

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[1] The problem of this note was suggested by the considerations of our paper *On the singular solutions of algebraic differential equations*, Ann. of Math. (2) vol. 37 (1936) p. 552. See also, W. C. Strodt, Trans. Amer. Math. Soc. vol. 45 (1939) p. 276.

$$y(x) = (x - a_1)^{1/2}(x - a_2)^{1/2}(x - a_3)^{1/2}$$

is two rather than three. Whether this describes all exceptional cases for $n \geqq 3$ is not decided here.

1. **The family of functions.** In what follows, the $p_i$ in (1) will be any fixed positive numbers. A few words are necessary to make clear the meaning of the second member of (1) for given $a_i$. If the $a_i$ are distinct from one another, we may take any simply connected area containing no $a_i$ and form the product, in the area, of any selection of branches of the $n$ functions $(x - a_i)^{p_i}$. The various products obtainable in this way are continuations of one another and are all branches of a single analytic function, which we consider the second member of (1) to represent. If two or more $a_i$ coincide, two distinct products, as just described, need not be branches of the same analytic function. There may thus be more than one, possibly even a countable infinitude of interpretations of the second member of (1); every such analytic function will be accepted into the $n$-parameter family of functions (1).

Given any function $y$, as in (1), its values, for any $x$ which is not an $a_i$, are equal in modulus; the same is true for every derivative of $y$.

2. **Order of contact.** Let $\mathcal{J}$ be a family of analytic functions and $f(x)$ a function[2] analytic at a point $x_0$. There may exist non-negative integers $s$ which have the property that, for every $\epsilon > 0$, a $g(x)$ exists in $\mathcal{J}$, with a branch analytic at $x_0$, such that, for this branch of $g(x)$, $g(x) - f(x)$ and its first $s$ derivatives are less than $\epsilon$ in modulus at $x_0$. If such integers $s$ exist, and if the set of them is bounded, we shall represent the greatest of them by $r$ and shall say that $\mathcal{J}$ has *contact of order* $r$ with $f(x)$ at $x_0$. If the $s$ are unbounded, we shall say that $\mathcal{J}$ has contact of infinite order with $f(x)$ at $x_0$.

3. **The bound.** We examine now the functions (1). It is apparent that this family has contact of some order with $y = 0$ at every point. Indeed, because the family is invariant under the addition of any constant to $x$, the contact with $y = 0$ is the same for all values of $x$.

Let $q$ be the greatest integer less than $p_1 + \cdots + p_n$. The order of contact of the family (1) with $y = 0$ is not less than $q$. This is seen by taking all $a_i$ equal to zero. We prove the theorem:

THEOREM. *The order of contact which the family* (1) *has with* $y = 0$, *for every* $x$, *does not exceed* $q + n - 1$.

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[2] Not necessarily in $\mathcal{J}$.

The theorem is readily seen to be true for $n=1$; we employ induction with respect to $n$. We examine the theorem for $n=r>1$, assuming that it has been established for every $n$ less than $r$.

We suppose the theorem false for $n=r$. Then, for $n=r$, and for certain positive numbers $p_1, \cdots, p_r$ which stay fixed during our proof, the family (1) has contact with $y=0$, for $x=0$, of order greater than $q+r-1$. Thus, if we denote the $j$th derivative of $y$ by $y_j$,[3] we can, for every $\epsilon>0$, fix the $a_i$ in (1) at values distinct from 0 so as to have[4]

$$|y_i(0)| < \epsilon, \qquad i = 0, 1, \cdots, q + r. \tag{3}$$

Let us show that, if $\epsilon$ is sufficiently small, each $a_i$, as just fixed, will have a modulus less than unity. Suppose, for instance, that for some very small $\epsilon$, $|a_1| \geqq 1$. Then $y(x)/(x-a_1)^{p_1}$ will be very small, together with its first $q+r$ derivatives, at $x=0$. This, by the case of $n=r-1$, is impossible.

We now put

$$\alpha(x) = (x - a_1) \cdots (x - a_r);$$

$$\beta(x) = \alpha(x)\left[\frac{p_1}{x - a_1} + \cdots + \frac{p_r}{x - a_r}\right]. \tag{4}$$

We have

$$\alpha(x)y_1 - \beta(x)y = 0. \tag{5}$$

The polynomial $\beta$ is of degree $r-1$. Its $(r-1)$st derivative is

$$(r - 1)!(p_1 + \cdots + p_r). \tag{6}$$

We differentiate (5) $j-1$ times, where $j \geqq 1$. Indicating derivatives of $\alpha$ and $\beta$ by subscripts, we find that

$$\begin{aligned} \alpha y_j + [(j - 1)\alpha_1 - \beta]y_{j-1} + \left[\frac{(j - 1)(j - 2)}{2!}\alpha_2 - (j - 1)\beta_1\right]y_{j-2}& \\ + \cdots - \beta_{j-1}y = 0.& \end{aligned} \tag{7}$$

For $j \geqq r$, (7) becomes, because of the degrees of $\alpha$ and $\beta$,

$$\alpha y_j + \cdots + (j - 1)!\left[\frac{\alpha_r}{r!(j - r - 1)!} - \frac{\beta_{r-1}}{(r-1)!(j-r)!}\right]y_{j-r} = 0. \tag{8}$$

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[3] $y_0 = y$.

[4] If $y$ is analytic at $x=0$ when certain $k$ of the $a_i$, say $a_1, \cdots, a_k$, are zero while no other $a_i$ vanish, it must be that $p_1 + \cdots + p_k$ is integral. Thus, if $a_1, \cdots, a_k$ are changed to a common value slightly different from zero, $y$ and any specified finite set of its derivatives will undergo only a slight change at $x=0$.

The coefficient of $y_{j-r}$ in (8) is a constant, which, if we have regard to (6) and notice that $\alpha_r = r!$, is seen to be zero if and only if

$$(9) \qquad p_1 + \cdots + p_r = j - r.$$

Let $p$ represent $p_1 + \cdots + p_r$. If, in (1), the $a_i$ are all multiplied by a number $m$, the values of $y_j(0)$ are multiplied by $m^{p-j}$. If $|m| > 1$, each $y_j(0)$ with $j > q$ will be multiplied by a number of modulus not greater than unity.

We consider a $y(x)$, (with definite $a_i$), which satisfies (3) for some very small $\epsilon$. Let $m$ be such that the greatest of the quantities $|ma_i|$, $i = 1, \cdots, r$, has unity for modulus. Then, by what follows (3), $|m| > 1$. Let

$$\bar{y}(x) = \prod_{i=1}^{r} (x - ma_i)^{p_i}.$$

We inspect the relation (8) as formed for $\bar{y}$. First we let $j = q+r$. In that case, (9) cannot hold. Every $|\bar{y}_i(0)|$ with $q < i \leqq q+r$ is small. Furthermore, because $|ma_i| \leqq 1$, $i = 1, \cdots, r$, there are bounds, independent of $\epsilon$, for the values of the coefficients in (8) at $x = 0$. We infer that $|\bar{y}_q(0)|$ is small. Now, supposing that $q > 0$, let $j = q+r-1$. We find from (8) that $|\bar{y}_{q-1}(0)|$ is small. Continuing, we find that every $|\bar{y}_i(0)|$ with $i \leqq q+r$ is small.

Let $g$ be such that $|ma_g| = 1$. Then the function

$$(10) \qquad \bar{y}(x)/(x - ma_g)^{p_g}$$

is small, together with its first $q+r$ derivatives, for $x = 0$. It is clear that we can use a single $g$ and obtain a sequence of functions (10) which is such that the values at $x = 0$ of the $k$th function of the sequence and its first $q+r$ derivatives tend toward zero as $k$ increases. By the case of $n = r-1$, this is impossible. The theorem is proved.

4. **Attainment of bound.** We prove, for $n > 1$, the theorem:

THEOREM. *If no proper subset of the $p_i$ has an integral sum, the family* (1) *has, for every $x$, contact with $y = 0$ of order precisely* $q+n-1$.

It suffices to show that, when the $p_i$ satisfy the hypothesis, there are values of the $a_i$ distinct from zero such that $y_j(0) = 0$, $j = q+1, \cdots, q+n-1$. Such $a_i$ being found, we can multiply them by a small $m$ distinct from zero and obtain a function (1) which is small, for $x = 0$, together with its first $q+n-1$ derivatives.

The existence of $a_i$ as just described will be established if we can prove that there are numbers $b_1, \cdots, b_n$, distinct from zero, such that the function

$$s = \prod_{i=1}^{n} (1 + b_i x)^{p_i}$$

has derivatives, from the $(q+1)$st to the $(q+n-1)$st inclusive which vanish for $x=0$. The $n-1$ derivatives in question, which we represent by $Z_{q+1}, \cdots, Z_{q+n-1}$, are homogeneous polynomials in the $n$ letters $b_i$. When the $Z_q$ are equated to zero, they determine a nonvacuous algebraic manifold each of whose essential irreducible components is of dimension not less than unity.[8] Thus there is at least one set of numbers $b_1, \cdots, b_n$ which annul the $Z_j$ and are not all zero. We assume in what follows that there is such a set in which the $b_i$ are not all distinct from zero, and prove that some proper subset of the $p_i$ has an integral sum.

We may now work under the assumption that, for some integer $t$ with $0<t<n$, there exist numbers $c_1, \cdots, c_t$, all distinct from zero, such that the function

$$u = \prod_{i=1}^{t} (1 + c_i x)^{p_i}$$

has derivatives from the $(q+1)$st to the $(q+n-1)$st inclusive which vanish for $x=0$. If we put $d_i = -1/c_i$, we find that the function

$$v = \prod_{i=1}^{t} (x - d_i)^{p_i} \tag{11}$$

has derivatives from order $q+1$ through order $q+n-1$ which vanish for $x=0$. For the derivatives $v_j$ of $v$, there exists a relation, analogous to (7), which expresses each $v_j$ in terms of the derivatives which precede it if $j \leq t$, and in terms of the $t$ derivatives which precede it if $j>t$. In this relation, the coefficient of $v_j$ is $(-1)^t d_1 \cdots d_t$ when $x=0$. Thus, as $v_{q+1}, \cdots, v_{q+n-1}$ vanish for $x=0$, and as they include the $t$ derivatives which precede $v_{q+n}$, $v_{q+n}$ and, then, all the derivatives which follow it, vanish for $x=0$. In other words, $v$ is a polynomial. Thus $p_1 + \cdots + p_t$ is integral and the theorem is proved.

When the $p_i$ are not all integers, $Z_{q+1}$ consists of at least two terms. It is then possible to annul $Z_{q+1}$ with $b_i$ which are all distinct from zero, so that, by what precedes, the order of contact is at least $q+1$. In particular, when $n=2$, the order of contact is $q+1$ except when $p_1$ and $p_2$ are both integers.

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[8] van der Waerden, *Einführung in die algebraische Geometrie*, Berlin, 1939, §41.

# A CONVERGENCE THEOREM FOR CERTAIN LAGRANGE INTERPOLATION POLYNOMIALS

M. S. WEBSTER

In the Lagrange interpolation polynomial $L_n[f;\theta]$ where

$$\begin{aligned} &L_n[f;\theta] \equiv \sum_{k=1}^{n} f(x_k)l_k[\theta], \\ (1)\qquad &l_k[\theta] \equiv l_k^{(n)}[\theta] \equiv l_k(x) \equiv \frac{\phi_n(x)}{\phi_n'(x_k)(x-x_k)}, \\ &\phi_n(x) \equiv \prod_{k=1}^{n}(x-x_k), \end{aligned}$$

$$x = \cos\theta;\ -1 < x_k < 1;\ k = 1, 2, \cdots, n;\ n = 1, 2, \cdots,$$

and $f(x)$ is a continuous function defined in $(-1, 1)$, we suppose that

$$(2)\qquad x_k \equiv x_k^{(n)} = \cos\theta_k = \cos k\pi/(n+1).$$

Then [1],[1] we have

$$\begin{aligned} &\phi_n(x) = \frac{\sin(n+1)\theta}{2^n \sin\theta}, \qquad x = \cos\theta, \\ (3)\qquad &l_k[\theta] = \frac{(-1)^{k+1}\sin^2\theta_k \sin(n+1)\theta}{(n+1)\sin\theta(\cos\theta-\cos\theta_k)}. \end{aligned}$$

We introduce the following notations:

$$\begin{aligned} &t_n \equiv t \equiv \theta_1/2 \equiv \pi/2(n+1), \qquad M = \max_{-1\leq x\leq 1} |f(x)|, \\ (4)\qquad &S_k[\theta] \equiv \{l_k[\theta-t] + l_k[\theta+t]\}/2. \end{aligned}$$

We shall prove the following theorem which was suggested by a similar theorem of Grünwald [2].

THEOREM. *Let $f(x)$ be a continuous function in the interval* $-1 \leq x \leq 1$. *Then*

$$(5)\qquad \lim_{n\to\infty}(1/2)\{L_n[f;\theta-t_n] + L_n[f;\theta+t_n]\} = f(\cos\theta), \quad 0 < \theta < \pi,$$

*and the convergence is uniform in the interval* $0 < \alpha \leq \theta \leq \pi - \alpha$ (*$\alpha$ arbi-*

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[1] The numbers in brackets refer to the bibliography.

*rary, fixed constant). In general, convergence does not hold for* $\theta=0$ *or* $\theta=\pi$.

We shall prove first that there is a constant $D$ for which

$$\sum_{k=1}^{n}|S_k[\theta]|<D,\qquad 0<\alpha\leqq\theta\leqq\pi-\alpha;\ n=1,2,\cdots. \tag{6}$$

If $\theta\neq\theta_k\pm t$, it follows from (3) and (4), by the use of trigonometric addition formulas, that

$$\begin{aligned}S_k[\theta]&=\frac{(-1)^{k+1}\sin t\sin^2\theta_k\cos(n+1)\theta}{4(n+1)\sin(\theta-t)\sin(\theta+t)}\\ &\quad\cdot\frac{\cos\theta\cos\theta_k-\cos 2\theta\cos t}{\sin(\theta+\theta_k-t)/2\sin(\theta-\theta_k-t)/2\sin(\theta+\theta_k+t)/2\sin(\theta-\theta_k+t)/2}.\end{aligned}\tag{7}$$

If $\theta$ is restricted to the interval $0<\alpha\leqq\theta\leqq\pi/2$, and if $n$ is large so that $t\leqq\alpha/2$, it is easily seen (assuming $\alpha\leqq\pi/3$) that

$$\left|\frac{\sin^2\theta_k}{\sin(1/2)(\theta+\theta_k-t)\sin(1/2)(\theta+\theta_k+t)}\right|<4,$$

$$|S_k[\theta]|<\frac{\pi^2C}{(n+1)^2|\theta-\theta_k-t|\cdot|\theta-\theta_k+t|}, \tag{8}$$

$$|\csc(\theta-\lambda)\csc(\theta+\lambda)|\leqq\frac{2}{\cos\alpha-\cos 2\alpha}=C,$$

$$\alpha\leqq\theta\leqq\pi/2,\ 0\leqq\lambda\leqq\alpha/2.$$

For a given $\theta$, there are at most two values of $k$ for which $|\theta-\theta_k|<\pi/(n+1)$. Since [1] $|l_k(x)|<2$ $(-1\leqq x\leqq 1;\ k=1,2,\cdots,n;\ n=1,2,\cdots)$, from (8) we have

$$\begin{aligned}\sum_{k=1}^{n}|S_k[\theta]|&<4+\sum_{1\leqq k\leqq n,\,|\theta-\theta_k|\geqq\pi/(n+1)}|S_k[\theta]|\\ &<4+\frac{\pi^2C}{(n+1)^2}\sum_{1\leqq k\leqq n,\,|\theta-\theta_k|\geqq\pi/(n+1)}\frac{1}{|\theta-\theta_k-t|\cdot|\theta-\theta_k+t|}\\ &<4+\frac{2\pi^2C}{(n+1)^2}\sum_{l=1}^{n}\left[\frac{2(n+1)}{\pi}\right]^2\frac{1}{(2l-1)^2}\\ &<4+8\pi C\sum_{m=1}^{\infty}\frac{1}{m^2}=D',\qquad 0<\alpha\leqq\theta\leqq\pi/2;\ n\geqq n_0(\alpha).\end{aligned}\tag{9}$$

because in the summation

$$|\theta-\theta_k \pm t| \geqq |\theta-\theta_k| - t \geqq l\pi/(n+1) - \pi/2(n+1), \qquad l \geqq 1.$$

By continuity, (9) holds for all $\theta$ in $0<\alpha\leqq\theta\leqq\pi/2$. Since (9) is valid for $n$ sufficiently large, there exists a $D$ for which (6) is valid for all $n$ if $0<\alpha\leqq\theta\leqq\pi/2$.

Since $\theta_k=\pi-\theta_{n-k+1}$, it is found that

$$l_k[\theta-t] = l_{n-k+1}[\theta'+t] \quad\text{and}\quad l_k[\theta+t] = l_{n-k+1}[\theta'-t]$$

where $\theta'=\pi-\theta$. It follows that

$$S_k[\theta] = S_{n-k+1}[\theta'], \qquad \sum_{k=1}^{n} |S_k[\theta']| < D, \qquad \pi/2 \leqq \theta' \leqq \pi-\alpha.$$

This completes the proof of (6).

From (8), if $\delta$ ($>0$) is fixed and if $n$ is sufficiently large (so that $t<\delta/2$), it is seen that

$$\tag{10} \begin{aligned} \sum_{1\leqq k\leqq n,\,|\theta-\theta_k|>\delta} |S_k[\theta]| &< \frac{\pi^3 C}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{(\delta-t)^2} \\ &< \frac{4\pi^3 Cn}{\delta^2(n+1)^2} = O\left(\frac{1}{n}\right). \end{aligned}$$

We are now ready to prove the main part of the theorem. Let $\theta$ be fixed ($0<\alpha\leqq\theta\leqq\pi-\alpha$) and $\epsilon>0$. It is well known that

$$\sum_{k=1}^{n} l_k(x) = 1, \qquad \sum_{k=1}^{n} S_k[\theta] = 1.$$

Since $f(x)$ is continuous, there exists a $\delta>0$ such that

$$|f(\cos\theta) - f(\cos\theta_k)| < \epsilon \quad\text{provided}\quad |\theta-\theta_k| \leqq \delta.$$

Let

$$\begin{aligned} \Delta &= (1/2)\{L_n[f;\theta-t] + L_n[f;\theta+t]\} - f(\cos\theta) \\ &= \sum_{k=1}^{n} \{f(\cos\theta_k) - f(\cos\theta)\}S_k[\theta] \\ &= \sum_{1\leqq k\leqq n,\,|\theta-\theta_k|\leqq\delta} \{f(\cos\theta_k) - f(\cos\theta)\}S_k[\theta] \\ &\quad + \sum_{1\leqq k\leqq n,\,|\theta-\theta_k|>\delta} \{f(\cos\theta_k) - f(\cos\theta)\}S_k[\theta]. \end{aligned}$$

Then, by the use of (6) and (10), for sufficiently large $n$, we find that

$$\begin{aligned} |\Delta| &< \epsilon \sum_{1 \leq k \leq n, |\theta-\theta_k| \leq \delta} |S_k[\theta]| + 2M \sum_{1 \leq k \leq n, |\theta-\theta_k| > \delta} |S_k[\theta]| \\ &< \epsilon D + 2MO\left(\frac{1}{n}\right) < (D+1)\epsilon, \qquad n > N. \end{aligned}$$

As in Rogosinki's theorem for Fourier series, the theorem may be easily extended so that convergence holds at any point $x$ ($\neq \pm 1$) of continuity of a bounded function $f(x)$, convergence being uniform in any closed interval of continuity (excluding $x = \pm 1$). In addition, $t_n$ may be replaced by $p\pi/2(n+1)$ where $p$ is any fixed odd integer.

Since $L_n[f; -t] = L_n[f; t]$, the theorem would involve (for $\theta = 0$) the convergence of $L_n[f; t]$ to $f(\cos 0)$. This convergence does not hold for all continuous $f(x)$ because

$$\lambda_n \equiv \sum_{k=1}^{n} |l_k(1)| = n,$$

and, according to H. Hahn [3], a necessary and sufficient condition for $L_n[f; 0]$ to converge to $f(\cos 0)$ for all continuous $f(x)$ is that $\lambda_n$ be bounded for all $n$. We give an example (compare [4] and [5]) of a continuous function for which convergence does not occur at $x = 1$.

Let $f_n(x)$ be defined for each $n$ ($n = 1, 2, \cdots$) as follows:

$$(11)\quad f_n(x) = \begin{cases} (-1)^{n-1}, & -1 \leq x < x_n, \\ (-1)^k \left[1 - \dfrac{2(x - x_{k+1})}{x_k - x_{k+1}}\right], & x_{k+1} \leq x \leq x_k;\ k = 1, 2, \cdots, n-1, \\ 1, & x_1 < x \leq 1. \end{cases}$$

Now,

$$\begin{aligned} L_n[f_n; t] &= -\sum_{k=1}^{n} (-1)^k l_k[t] \\ &= \frac{2}{(n+1)\sin t} \sum_{k=1}^{n} \frac{\sin^2 kt \cos^2 kt}{\sin (k+1/2)t \sin (k-1/2)t}; \end{aligned}$$

Since

$$\frac{\sin kt}{\sin (k+1/2)t} > \frac{2}{3}, \qquad k = 1, 2, \cdots, n,$$

we have

$$(12)\quad \begin{aligned} |L_n[f_n; t]| &> \frac{4}{3(n+1)\sin t} \sum_{k=1}^{n} \cos^2 kt \\ &> \frac{2}{3(n+1)\sin t} \sum_{k=1}^{n} (1 + \cos \theta_k) > \frac{4n}{3\pi}. \end{aligned}$$

By the Weierstrass approximation theorem, we may approximate $f_n(x)$ by a polynomial $g_n(x)$ of degree $d(n)$ such that in $(-1, 1)$,

$$(13) \qquad |g_n(x)| \leqq 3/2, \qquad |L_n[g_n; t]| > 2n/3\pi.$$

This is possible, in view of (12), because

$$|L_n[f_n; t] - L_n[g_n; t]| = \left| \sum_{k=1}^{n} [f_n(x_k) - g_n(x_k)] l_k[t] \right| \leqq \epsilon'$$

provided

$$|f_n(x) - g_n(x)| \leqq \left(\epsilon' \Big/ \sum_{k=1}^{n} |l_k[t]|\right), \qquad -1 \leqq x \leqq 1.$$

Let

$$(14) \qquad g(x) = \sum_{i=1}^{\infty} c_i g_{n_i}(x), \qquad -1 \leqq x \leqq 1,$$

where $c_1 = n_1 = 1$, and

$$(15) \quad c_{i+1} = \min\left\{\frac{c_i}{4}, \frac{1}{T_i}\right\}, \quad T_i = \sum_{k=1}^{n_i} |l_k^{(n_i)}[t_{n_i}]|, \qquad i = 1, 2, \cdots;$$

$n_i$ is the smallest integer satisfying the conditions

$$(16) \qquad \begin{aligned} &\text{(a)} \quad n_i \geqq d(n_{i-1}) + 1, \\ &\text{(b)} \quad |c_i L_{n_i}[g_{n_i}; t_{n_i}] - 8| > 4^i, \qquad i = 2, 3, \cdots. \end{aligned}$$

Condition (b) is possible because of (13). From (15), it follows that

$$(17) \qquad c_{i+1} \leqq 1/4^i, \qquad i = 0, 1, 2, \cdots,$$

and the series for $g(x)$ converges uniformly so that $g(x)$ is continuous and $|g(x)| \leqq 2$ in $(-1, 1)$. Let $m = n_r$ where $r$ $(\neq 1)$ is a positive integer. Using (15) and (17), we have

$$(18) \qquad \begin{aligned} \left| \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m] \right| &\leqq \sum_{i=r+1}^{\infty} c_i \frac{3}{2} \sum_{k=1}^{m} |l_k^{(m)}[t_m]| \leqq 2, \\ \left| \sum_{i=r}^{\infty} c_i g_{n_i}(\cos t_m) \right| &\leqq 2. \end{aligned}$$

If $w(x)$ is any polynomial of degree less than $m$ in $x = \cos\theta$, then $L_m[w; \theta] = w(x)$. Since

$$
\begin{aligned}
\sum_{i=1}^{\infty} c_i L_m[g_{n_i}; t_m]
&= c_r L_m[g_m; t_m] + \sum_{i=1}^{r-1} c_i L_m[g_{n_i}; t_m] + \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m] \\
&= c_r L_m[g_m; t_m] + \sum_{i=1}^{r-1} c_i g_{n_i}(\cos t_m) + \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m],
\end{aligned}
$$

we find from (16) and (18) that

$$
\begin{aligned}
&\left| L_m[g; t_m] - g(1) \right| \\
&\quad = \left| \{L_m[g; t_m] - g(\cos t_m)\} + \{g(\cos t_m) - g(1)\} \right| \\
&\quad \geqq \left| \sum_{i=1}^{\infty} c_i L_m[g_{n_i}; t_m] - g(\cos t_m) \right| - 4 \\
&\quad \geqq \left| c_r L_m[g_m; t_m] + \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m] - \sum_{i=r}^{\infty} c_i g_{n_i}(\cos t_m) \right| - 4 \\
&\quad > 4^r, \qquad\qquad r = 2, 3, \cdots .
\end{aligned}
$$

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PURDUE UNIVERSITY

# A NOTE ON COMPLEMENTARY SUBSPACES IN A RIEMANNIAN SPACE

YUNG-CHOW WONG[1]

1. **Introduction.**[2] Let

$$(1.1)\qquad x^\kappa = x^\kappa(u^a), \qquad \kappa, \lambda, \mu, \nu = 1, \cdots, n;\ a, b, \cdots, f = 1, \cdots, m,$$

be the equations of a $V_m$ in a $V_n$ with fundamental tensor $g_{\lambda\kappa}$ and let

$$(1.2)\qquad B_a^\kappa = \partial_a x^\kappa \equiv \frac{\partial x^\kappa}{\partial u^a}.$$

Then the fundamental tensor and curvature tensor of $V_m$ in $V_n$ are, respectively,

$$(1.3)\qquad {}'g_{cb} = g_{\lambda\kappa} B_c^\lambda B_b^\kappa,$$

$$(1.4)\qquad H_{cb}^{\cdot\cdot\kappa} = D_c B_b^\kappa \equiv \partial_c B_b^\kappa + \Gamma_{\mu\lambda}^\kappa B_c^\mu B_b^\lambda - {}'\Gamma_{cb}^a B_a^\kappa,$$

where $D$ denotes the generalized covariant differentiation with respect to $V_m$ in $V_n$; and $\Gamma_{\mu\lambda}^\kappa$ and ${}'\Gamma_{cb}^a$ are, respectively, the Christoffel symbols of the second kind for $V_n$ and $V_m$.

By definition a $V_m$ in $V_n$ is said to be *totally semi-umbilical*[3] in $V_n$ if a vector $v_\kappa$ exists such that

$$(1.5)\qquad v_\kappa H_{cb}^{\cdot\cdot\kappa} = {}'g_{cb}$$

is satisfied at every point of $V_m$. In particular, this condition is evidently fulfilled when $H_{cb}^{\cdot\cdot\kappa}$ has the form $H_{cb}^{\cdot\cdot\kappa} = {}'g_{cb} n^\kappa$, $n^\kappa$ being a certain vector; in this case we call $V_m$ *totally umbilical* in $V_n$.

In what follows we shall consider the subspaces $V_m$: $x^p$ = const. in a $V_n$ with fundamental tensor of the form

$$(1.6)\qquad g_{\lambda\kappa} = \begin{pmatrix} g_{cb} & 0 \\ 0 & g_{rq} \end{pmatrix}, \qquad \begin{array}{l} a, b, \cdots, f = 1, \cdots, m, \\ p, q, \cdots, s = m+1, \cdots, n. \end{array}$$

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[1] The author is a Chinese Ying-Keng Funds student. He wishes to thank Professor D. J. Struik for the conversations they had from time to time during his stay at Cambridge, Massachusetts.

[2] For the theory of subspaces $V_m$ in a Riemannian $n$-space $V_n$, see Schouten-Struik, *Einführung in der neuern Methoden der Differentialgeometrie* II, Groningen, 1938 chap. 3.

[3] D. Perepelkine, *Sur la courbure et les espaces normaux d'une $V_m$ dans $R_n$*, Rec. Math. (Mat. Sbornik) N.S. vol. 42 (1935) pp. 81–100.

The two families of subspaces $V_m$: $x^p=$const. and $V_{n-m}$: $x^a=$const. are called *completementary families* of subspaces in $V_n$. Recently, Yano[4] proved that a condition for $V_m$ to be totally umbilical in $V_n$ is that $g_{cb}$ be of the form[5] $g_{cb}=\sigma(x^\kappa)\bar{g}_{cb}(x^a)$. We shall obtain, among other results, a similar condition for $V_m$ to be totally semi-umbilical in $V_n$.

2. **First normal complex.** Let $w^a$ and $y^a$ be two arbitrary vectors in $V_m$. Then the vector $y^c w^b H_{cb}^{\cdot\cdot\kappa}$ spans the first normal complex of $V_m$ in $V_n$, whose dimensionality $m_1$ is therefore equal to the rank of the matrix $[H_{cb}^{\cdot\cdot\kappa}]$. In this matrix, as well as in every matrix appearing hereafter, $\kappa$ or $p$ indicates the column and the combination of $b, c, \cdots$ the row.

Now for the subspaces $V_m$: $x^p=$const. in a $V_n$ with fundamental tensor (1.6), we have

$$(2.1)\qquad B^{\kappa}_{a} = \frac{\partial x^\kappa}{\partial x^a} = \delta^{\kappa}_{a}, \qquad 'g_{cb} = g_{cb},$$

$$(2.2)\qquad H_{cb}^{\cdot\cdot a} = \Gamma^{a}_{cb} - '\Gamma^{a}_{cb}, \qquad H_{cb}^{\cdot\cdot p} = \Gamma^{p}_{cb}.$$

But from (1.6) and the definition of the Christoffel symbols of the second kind

$$\Gamma^{\kappa}_{\mu\lambda} = (1/2)g^{\kappa\nu}(\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\nu g_{\mu\lambda})$$

it follows at once that

$$(2.3)\qquad \begin{aligned} \Gamma^{a}_{cb} &= '\Gamma^{a}_{cb}, & \Gamma^{p}_{cb} &= -(1/2)g^{pq}\partial_q g_{cb},\\ \Gamma^{a}_{cq} &= (1/2)g^{ab}\partial_q g_{cb}, & \Gamma^{p}_{cr} &= (1/2)g^{pq}\partial_c g_{rq}. \end{aligned}$$

Thus (2.2) become

$$(2.4)\qquad H_{cb}^{\cdot\cdot a} = 0, \qquad H_{cb}^{\cdot\cdot p} = -(1/2)g^{pq}\partial_q g_{cb}.$$

And therefore the dimensionality $m_1$ of the first normal complex of $V_m$ in $V_n$ is equal to the rank of the matrix $[g^{pq}\partial_q g_{cb}]$. Since Det $(g^{pq})\neq 0$, $m_1$ is also the rank of the matrix $[\partial_p g_{cb}]$. Hence[6] there exist $m_1$ components of $g_{cb}$, which are functionally independent with regard to $x^p$, such that each component of $g_{cb}$ is expressible in terms of them and $x^a$. Conversely, it is evident from (2.4) that if $g_{cb}$ has this property, the first normal complex of $V_m$ in $V_n$ is of dimension $m_1$. Hence we have this theorem.[7]

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[4] K. Yano, *Conformally separable quadratic differential forms*, Proc. Imp. Acad. Tokyo vol. 16 (1940) pp. 83–86. For $n=m+1$ see L. P. Eisenhart, *Riemannian geometry*, Princeton, 1926 p. 182.

[5] Throughout this paper we denote by $\rho$, $\sigma$, $\theta$, $\phi$ scalar functions of $x^\kappa$.

[6] See, for example, T. Levi-Civita, *The absolute differential calculus*, London, 1927 pp. 9–12.

THEOREM 2.1. *The first normal complex of the subspaces* $x^p=$*const. in a* $V_n$ *with fundamental tensor* (1.6) *is of dimension* $m_1$, *if and only if the matrix* $[\partial_p g_{cb}]$ *is of rank* $m_1$, *that is if* $g_{cb}$ *is of the form* $g_{cb}=g_{cb}(x^a, \rho_1, \cdots, \rho_{m_1})$, *where the* $\rho$*'s are* $m_1$ *functions of* $x^\kappa$ *which are functionally independent with regard to* $x^p$.

Now it follows from (2.4) and Theorem 2.1 that the components of the vector $y^c w^b H_{cb}^{\cdot\cdot\kappa}$, which spans the first normal complex, are

$$y^c w^b H_{cb}^{\cdot\cdot a} = 0,$$

$$y^c w^b H_{cb}^{\cdot\cdot p} = -(1/2) y^c w^b g^{pq}\left(\frac{\partial g_{cb}}{\partial \rho_1}\partial_q\rho_1 + \cdot\ \ \cdot + \frac{\partial g_{cb}}{\partial \rho_{m_1}}\partial_q\rho_{m_1}\right). \tag{2.5}$$

To see the implication of these equations let us consider a certain fixed $V_{n-m}$: $x^a=x_0^a$. Each $V_m$ of the family $x^p=$const. has a point in common with $V_{n-m}$, at which the first normal complex of $V_m$ lies in the tangent space of $V_{n-m}$. Equations (2.5) then show that these first normal complexes are orthogonal to the subspaces

$$\rho_1(x_0^a, x^p) = \text{const.}, \cdot\ \ \cdot, \rho_{m_1}(x_0^a, x^p) = \text{const.}$$

of $V_{n-m}$.

THEOREM 2.2. *If a* $V_n$ *admits two complementary families of* $V_m$ *and* $V_{n-m}$, *then the first normal complexes, dimensionality* $m_1$, *of* $V_m$ *at points of any fixed* $V_{n-m}$ *are orthogonal to a family of subspaces* $V_{n-m-m_1}$ *in* $V_{n-m}$.

The condition for $V_m$ to be minimal in $V_n$ is $'g^{cb}H_{cb}^{\cdot\cdot\kappa}=0$, which, by (2.1) and (2.4), can be written $g^{cb}\partial_p g_{cb}=0$, that is, $\partial_p$ Det $(g_{cb})=0$. Hence this theorem follows.[8]

THEOREM 2.3. *If a* $V_n$ *admits two complementary families of* $V_m$ *and* $V_{n-m}$, *a necessary and sufficient condition for* $V_m$ *to be minimal in* $V_n$ *is that* $V_{n-m}$ *determine a correspondence between them which preserves volume.*

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[7] For $m_1=0$, see Eisenhart, loc. cit. p. 186 Example 13.

[8] For $n=m+1$, see Eisenhart, loc. cit. p. 179.

3. **Totally semi-umbilical** $V_m$. According to (1.5), (2.1) and (2.4), the condition for $x^p=$const. to be totally semi-umbilical in $V_n$ is that a vector $v^p$ in $V_{n-m}$ exists such that $v^p\partial_p g_{cb}=g_{cb}$, that is,

$$(3.1)\qquad v^p\partial_p \log g_{cb} = 1.$$

From this it follows that

$$(3.2)\qquad v^p\partial_p \log (g_{cb}/g_{ed}) = 0.$$

Now if the first normal complex of $V_m$ in $V_n$ is of dimension $m_1$, then by Theorem 2.1 $g_{cb}$ is of the form

$$(3.3)\qquad g_{cb} = g_{cb}(x^a, \rho_1, \cdots, \rho_{m_1}).$$

Consequently, (3.2) gives

$$(3.4)\qquad \theta_1\frac{\partial}{\partial\rho_1} \log (g_{cb}/g_{ed}) + \cdots + \theta_{m_1}\frac{\partial}{\partial\rho_{m_1}} \log (g_{cb}/g_{ed}) = 0,$$

where

$$(3.5)\qquad \theta_1 = v^p\partial_p\rho_1, \cdots, \theta_{m_1} = v^p\partial_p\rho_{m_1}.$$

Conversely, let $m_1$ functions $\theta_1, \cdots, \theta_{m_1}$ exist satisfying (3.4). Then since $\rho_1, \cdots, \rho_{m_1}$ are independent with regard to $x^p$, the matrix $[\partial_p\rho_1, \cdots, \partial_p\rho_{m_1}]$ is of rank $m_1$. Therefore, the system of linear equations (3.5) has solutions for $v^p$; that is, $v^p$ exist satisfying (3.2) and also (3.1). Hence, when (3.3) is true, (3.4) is a necessary and sufficient condition for $V_m$ to be totally semi-umbilical in $V_n$.

On the other hand, by a well known theorem[9] on the essential parameters of a set of functions, equation (3.4) is also the condition that there exist $m_1-1$ functions $\sigma_1, \cdots, \sigma_{m_1-1}$ of $x^a$ and the $\rho$'s (and therefore of $x^\kappa$) such that $g_{cb}/g_{ed}$ is expressible in terms of them and $x^a$; that is, that $g_{cb}$ is of the form

$$(3.6)\qquad g_{cb} = \sigma_{m_1}\bar{g}_{cb}(x^a, \sigma_1, \cdots, \sigma_{m_1-1}).$$

It is seen that $\bar{g}_{cb}$ cannot be expressed in terms of $x^a$ and less than $m_1-1$ independent (with regard to $x^p$) functions $\sigma$'s; otherwise, $g_{cb}$ would be expressible in terms of $x^a$ and less than $m_1$ functions, and consequently by Theorem 2.1, the first normal complex of $V_m$ in $V_n$ would be of dimension less than $m_1$.

THEOREM 3.1. *In a* $V_n$ *with fundamental tensor* (1.6), *each of the subspaces* $x^p=$*const., whose first normal complexes are of dimension*

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[9] See, for example, L. P. Eisenhart, *Continuous groups of transformation*, Princeton, 1933, p. 9.

*$m_1$, is totally semi-umbilical in $V_n$, if and only if $g_{cb}$ is of the form*

$$g_{cb} = \sigma_{m_1}\bar{g}_{cb}(x^a, \sigma_1, \cdots, \sigma_{m_1-1}),$$

*where the $\sigma$'s are $m_1$ functions of $x^\kappa$ which are independent with regard to $x^p$.*

For $m_1=1$, we have Yano's result quoted in §1.

4. **Normal complexes of higher order.** We now return to the end of §2 and consider the matrix

$$(4.1)\qquad \begin{pmatrix} B_a^\kappa \\ H_{cb}^{\cdot\cdot\kappa} \\ D_f H_{cd}^{\cdot\cdot\kappa} \end{pmatrix}.$$

Let $m_1$ and $m_2$ be, respectively, the dimensionalities of the first and second normal complexes of $V_m$ in $V_n$, then the rank of the above matrix is $m+m_1+m_2$. Taking account of (2.3), (2.4) and

$$D_f H_{cd}^{\cdot\cdot\kappa} = \partial_f H_{cd}^{\cdot\cdot\kappa} + \Gamma^\kappa_{\mu\lambda} H_{cd}^{\cdot\cdot\lambda} B_f^\mu - {}'\Gamma^e_{fc} H_{ed}^{\cdot\cdot\kappa} - {}'\Gamma^e_{fd} H_{ce}^{\cdot\cdot\kappa}$$

we can easily prove from (4.1) that the following matrices are all of rank $m_1+m_2$:

$$(4.2)\qquad \begin{pmatrix} H_{cb}^{\cdot\cdot p} \\ D_f H_{cd}^{\cdot\cdot p} \end{pmatrix}, \quad \begin{pmatrix} H_{cb}^{\cdot\cdot p} \\ \partial_f H_{cd}^{\cdot\cdot p} + \Gamma^p_{fq} H_{cd}^{\cdot\cdot q} \end{pmatrix}, \quad \begin{pmatrix} g^{pq}\partial_q g_{cb} \\ g^{pq}\partial_f\partial_q g_{cd} + (1/2)(\partial_f g^{pq})\partial_q g_{cd} \end{pmatrix},$$

$$\begin{pmatrix} \partial_p g_{cb} \\ \partial_f\partial_p g_{cd} + (1/2) g_{pq}(\partial_f g^{qr})\partial_r g_{cd} \end{pmatrix}.$$

The last matrix shows that, unlike $m_1$, the dimensionality $m_2$ of the second normal complex of $V_m$ in $V_n$ depends not only on the nature of $g_{cb}$ but also on that of $g_{qp}$.

If $g_{qp}=g_{qp}(x^r)$, that is, if the complementary $V_{n-m}$ are totally geodesic in $V_n$ (cf. Theorem 2.1 for $m_1=0$), the matrix (4.2) reduces to

$$\begin{pmatrix} \partial_p g_{cb} \\ \partial_p\partial_f g_{cd} \end{pmatrix}.$$

This matrix is of rank $m_1+m_2$, and therefore $g_{cb}$, $\partial_f g_{cd}$ can be expressed in terms of $x^a$ and $m_1+m_2$ (but not less) functions of $x^\kappa$ which are independent with regard to $x^p$. But

$$(4.3)\qquad \begin{aligned} g_{cb} &= g_{cb}(x^a, \rho_1, \cdots, \rho_{m_1}), \\ \partial_f g_{cd} &= \phi_0 + \phi_1\partial_f\rho_1 + \cdots + \phi_{m_1}\partial_f\rho_{m_1}, \end{aligned}$$

where the $\phi$'s are some functions of $x^a$ and the $\rho$'s. Therefore the first and second normal complexes of $V_m$ in $V_n$ are of dimension $m_1$ and $m_2$, if and only if (4.3) is true and $\partial_f\rho_1, \cdots, \partial_f\rho_{m_1}$ are expressible in terms of $x^a$, $\rho_1, \cdots, \rho_{m_1}$, and $m_2$ other functions $\rho_{m_1+1}, \cdots, \rho_{m_1+m_2}$, which, together with $\rho_1, \cdots, \rho_{m_1}$, form $m_1+m_2$ functions independent with regard to $x^p$.

This being the case, we have

$$\partial_p g_{cb} = \frac{\partial g_{cb}}{\partial \rho_1}\partial_p\rho_1 + \cdots + \frac{\partial g_{cb}}{\partial \rho_{m_1}}\partial_p\rho_{m_1},$$

$$\partial_p\partial_f g_{ed} = \frac{\partial\partial_f g_{ed}}{\partial \rho_1}\partial_p\rho_1 + \cdots + \frac{\partial\partial_f g_{ed}}{\partial \rho_{m_1+m_2}}\partial_p\rho_{m_1+m_2}.$$

But if $w^a$, $y^a$, $z^a$ are three arbitrary vectors in $V_m$, the vectors $y^c w^b H_{cb}^{\cdot\cdot p}$ and $z^f y^e w^d D_f H_{ed}^{\cdot\cdot p}$ span the first two normal complexes of $V_m$ in $V_n$. Therefore by an argument similar to that which led to Theorem 2.2, we conclude that the first two normal complexes of $V_m$ at points of a fixed $V_{n-m}$ are orthogonal to a family of $V_{n-m-m_1-m_2}$ in $V_{n-m}$.

The above result can easily be extended to cover the normal complexes of higher order of $V_m$ in $V_n$; indeed we have the following two theorems.

THEOREM 4.1. *In a $V_n$ with fundamental tensor*

$$g_{\lambda\kappa} = \begin{pmatrix} g_{cb}(x^a) & 0 \\ 0 & g_{qp}(x^r) \end{pmatrix},$$

*the normal complexes of the subspaces $x^p = \text{const.}$ are of dimension $m_1, m_2, \cdots$ if and only if the matrices*

$$\begin{pmatrix} \partial_p g_{cb} \end{pmatrix}, \begin{pmatrix} \partial_p g_{cb} \\ \partial_p\partial_f g_{ed} \end{pmatrix}, \cdots$$

*are of ranks $m_1, m_1+m_2, \cdots$, respectively.*

THEOREM 4.2. *If a $V_n$ admits two families of complementary $V_m$ and $V_{n-m}$ and if $V_{n-m}$ are totally geodesic in $V_n$, then the first $l$ $(l=1, 2, \cdots)$ normal complexes of $V_m$ at points of any fixed $V_{n-m}$ are orthogonal to a family of $V_{n-m-m_1-m_2\cdots-m_l}$ in $V_{n-m}$.*

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# ON THE JOIN OF TWO COMPLEXES

C. E. CLARK

1. **Introduction.** In this note we point out an isomorphism between the $(r+1)$-dimensional Betti group of the join (defined below) of two complexes and a subgroup of the $r$-dimensional Betti group of the product of the two complexes. Using this isomorphism the Betti groups of the join are derived from those of the product in case the complexes are finite.[1]

2. **Definition of the join $(K_1, K_2)$ of $K_1$ and $K_2$.** To define the join of two complexes we first define the join $(\sigma, \tau)$ of a $p$-dimensional simplex $\sigma$ and a $q$-dimensional simplex $\tau$, $p, q=0, 1, \cdots$. This join is a $(p+q+1)$-dimensional simplex with a $p$-dimensional side associated with $\sigma$ and the opposite side, which is $q$-dimensional, associated with $\tau$. These sides will not be distinguished from $\sigma$ and $\tau$, respectively. Now consider the complexes $K_1$ and $K_2$. Consider the set consisting of the simplexes $\sigma_\alpha$ of $K_1$, the simplexes $\tau_\beta$ of $K_2$, and the simplexes $(\sigma_\alpha, \tau_\beta)$. In a natural way this set forms a complex. We define the *join* $(K_1, K_2)$ of $K_1$ and $K_2$ to be the first barycentric subdivision of this complex.

3. **The rays.** By the rays of $(\sigma, \tau)$ we mean the straight line segments each of which joins a point of $\sigma$ and a point of $\tau$. These rays cover $(\sigma, \tau)$. Also no two rays intersect except possibly at an end point. The rays of all $(\sigma_\alpha, \tau_\beta)$ of $(K_1, K_2)$ are called the *rays*.

Let $N_i$, $i=1, 2$, be the subcomplex made up of the simplexes of $(K_1, K_2)$ that have at least one vertex in $K_i$ together with the faces of all such simplexes. It is known that each ray meets the intersection $N_1 \cap N_2$ in exactly one point.[2] Furthermore $N_i$ and $N_1 \cap N_2$ can be homotopically deformed in $N_i$ along the rays into $K_i$, $i=1, 2$.[2] It follows that $N_1 \cap N_2$ and the product $K_1 \times K_2$ are homeomorphic (the complexes being considered as point sets).

4. **The theorem.** We prove this theorem.

THEOREM 1. *There is an isomorphism between the* $(r+1)$*-dimensional*

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[1] The Betti groups of the join of two finite complexes are known. They were computed by H. Freudenthal in his paper *Die Bettischen Gruppen der Verbindung Zweier Polytope*, Fund. Math. vol. 29 (1937) pp. 145–150.

[2] For a proof see our paper *Simultaneous invariants of a complex and subcomplex*, Duke Math. J. vol. 5 (1939) pp. 62–71.

*Betti group of* $(K_1, K_2)$ *and the subgroup of those homology classes of the* $r$*-dimensional Betti group of* $N_1 \cap N_2$ *which contain cycles that bound both in* $N_1$ *and* $N_2$, $r \geqq 0$; *all chains considered in this paper are finite and have integral coefficients.*

In this theorem $K_1$ and $K_2$ may be infinite.

PROOF. We note that $(K_1, K_2) = N_1 + N_2$. Hence we know that there is a homomorphism from the $(r+1)$-dimensional Betti group of $N_1 + N_2$ onto those homology classes of the $r$-dimensional Betti group of $N_1 \cap N_2$ which contain cycles that bound both in $N_1$ and $N_2$, $r \geqq 0$; furthermore, the kernel of this homomorphism consists of the homology classes that contain Summenzyklen, that is, cycles that are equal to the sum of two cycles, one in $N_1$ and the other[3] in $N_2$. To prove Theorem 1 we shall show that a Summenzyklus bounds in $(K_1, K_2)$. To prove this, consider a cycle $Z$ of $N_1$. This cycle can be homotopically deformed in $N_1$ along the rays into a singular cycle of $K_1$. We know that $(K_1, K_2)$ contains the join of $K_1$ and an arbitrary point of $K_2$. Hence any cycle of $K_1$ can be homotopically deformed in $(K_1, K_2)$ into a cycle of a vertex of $K_2$. Since the dimension of $Z$ is greater than zero, it follows that $Z$ is homologous to zero in $(K_1, K_2)$. Similarly, a cycle of $N_2$ with dimension greater than zero bounds in $(K_1, K_2)$.

5. **Some properties of $N_1 \cap N_2$.** From now on $K_1$ and $K_2$ are finite. For any chain $A$ of a complex let $|A|$ denote the closure of the set of those simplexes at which[4] $A \neq 0$. We know that with each pair of chains $C_1 \subset K_1$ and $C_2 \subset K_2$ there is associated a chain $C_1 \times C_2 \subset N_1 \cap N_2$; the dimension of $C_1 \times C_2$ is the sum of the dimensions of $C_1$ and $C_2$; and $|C_1 \times C_2| = |C_1| \times |C_2|$.[5] It follows that the projection of $|C_1 \times C_2|$ into $K_1$ or $K_2$ along the rays is a subset of $|C_1|$ or $|C_2|$, respectively. Also if $C_2$ is a zero-dimensional cycle, and $C_2 = 0$ at all but one vertex of $K_2$ at which $C_2 = 1$, then $C_1 \times C_2$ and $C_1$ are isomorphic, and homotopic deformation of $C_1 \times C_2$ along the rays shows that $C_1 \times C_2 \sim C_1$ in $N_1$ (all the rays used in the deformation meet at a point of $K_2$).

Let

$$Z_i^p \subset K_1 \tag{1}$$

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[3] See Alexandroff-Hopf, *Topologie*. I, p. 293, Theorem V. This theorem and its applicability were pointed out by the referee.

[4] An $r$-dimensional chain is a function defined over all $r$-dimensional simplexes of a complex.

[5] These and the following properties of $N_1 \cap N_2 = K_1 \times K_2$ are proved in Alexandroff-Hopf, *Topologie*. I, pp. 299–310.

and

$$Z^q_j \subset K_2 \tag{2}$$

be homology bases[6] for the $p$-dimensional cycles of $K_1$ and the $q$-dimensional cycles of $K_2$. We know that there is a set of $s$-dimensional chains $u^s_k \subset K_1$, $s = 2, 3, \cdots$, in $(1=1)$ correspondence with the $(s-1)$-dimensional torsion coefficients of $K_1$, and there is a set of $t$-dimensional chains $v^t_l \subset K_2$, $t = 2, 3, \cdots$, in $(1=1)$ correspondence with the $(t-1)$-dimensional torsion coefficients of $K_2$ such that a homology basis for the $r$-dimensional cycles of $N_1 \cap N_2$ is given by

$$Z^r_{ij} = Z^p_i \times Z^q_j, \quad p + q = r, \text{ order of } Z^r_{ij} \neq 1, \tag{3}$$

and

$$C^r_{kl} = c^r_{kl}(u^s_k \times v^t_l)^{\cdot}, \quad s + t = r + 1, \; c^r_{kl} \neq 1, \tag{4}$$

where $c^r_{kl}$ is the reciprocal of the greatest common divisor of the torsion coefficients associated with $u^s_k$ and $v^t_l$. Furthermore the order of $Z^r_{ij}$ is the greatest common divisor of the orders[7] of $Z^p_i$ and $Z^q_j$, and the order of $C^r_{kl}$ is the reciprocal of $c^r_{kl}$.

6. **The cycles of $N_1 \cap N_2$ that bound both in $N_1$ and $N_2$.** We choose a cycle from (1) with $p=0$ and a cycle from (2) with $q=0$. Each of these cycles will be denoted by the same symbol $Z^0_1$.

THEOREM 2. *A homology basis for the subgroup mentioned in Theorem* 1 *consists of the following subset of* (3) *and* (4): *if* $r>0$, *the basis consists of* $Z^r_{ij}$ *with both* $p>0$ *and* $q>0$, $Y^r_{ij} = Z^r_i \times (Z^0_1 - Z^0_j)$, $j \neq 1$, $X^r_{ij} = (Z^0_1 - Z^0_i) \times Z^r_j$, $i \neq 1$, *and all* $C^r_{ij}$; *if* $r=0$, *the basis consists of* $W^0_{ij} = (Z^0_1 - Z^0_i) \times (Z^0_1 - Z^0_j)$, $i \neq 1$, $j \neq 1$.

PROOF. In the first place consider $Z^r_{ij}$ with both $p \neq 0$ and $q \neq 0$. We know that $Z^r_{ij}$ can be homotopically deformed in $N_1$ along the rays into $|Z^p_i|$, a $p$-dimensional subcomplex of $K_1$. Since $p<r$, this implies that $Z^r_{ij}$ bounds in $N_1$. From symmetry, if $p \neq 0$ and $q \neq 0$, the cycle $Z^r_{ij}$ bounds both in $N_1$ and $N_2$.

Consider next $C^r_{kl}$. Since $s+t=r+1$, $s>1$, and $t>1$, it follows that

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[6] By a homology basis for the $r$-dimensional cycles of a complex we mean a set of cycles obtained by expressing the $r$-dimensional Betti group as the usual direct sum of free cyclic groups and finite cyclic groups whose orders are the torsion coefficients and by choosing a cycle from a generator of each summand.

[7] By the order of a cycle we mean the order of its homology class. Also it is understood that a free group has order zero and that the greatest common divisor of zero and a positive integer is that integer.

$s<r$ and $t<r$. Hence $C^r_{kl}$ bounds both in $N_1$ and $N_2$ because $C^r_{kl}$ is homotopic in $N_1$ to a cycle of $|u^s_k|$ and is homotopic in $N_2$ to a cycle of $|v^t_l|$.

Next consider $Z^r_{ij}$ with $p=r\neq 0$. We can assume that $Z^0_j=0$ at all but one vertex of $K_2$ and that $Z^0_j=1$ at the exceptional vertex. Then as shown above $Z^r_{ij}\sim Z^r_i$ in $N_1$. From this we shall deduce that $Z^r_{ij}$ does not bound in $N_1$. Suppose $Z^r_{ij}$ does bound in $N_1$. Then $Z^r_i=\dot{F}$, $F\subset N_1$. We deform $F$ in $N_1$ along the rays into the singular chain $F'\subset K_1$. The singular cycle $Z^r_i=\dot{F}'$ which contradicts the definition of $Z^r_i$.

From symmetry we see that $Z^r_{ij}$ with $q=r\neq 0$ does not bound in $N_2$.

Now consider $Y^r_{ij}$. Since $Z^r_i\times Z^0_1$ and $Z^r_i\times Z^0_j$ are each homologous in $N_1$ to $Z^r_i$, it follows that $Y^r_{ij}$ bounds in $N_1$. That $Y^r_{ij}$ also bounds in $N_2$ follows from the fact that $Y^r_{ij}$ can be homotopically deformed into a complex consisting of two vertices. From symmetry we know that $X^r_{ij}$ bounds both in $N_1$ and $N_2$.

Finally if $r=0$, we see that $W^0_{ij}$ bounds both in $N_1$ and $N_2$.

The theorem follows easily from these facts.

**7. The Betti groups of $(K_1, K_2)$.** Theorems 1 and 2 imply this theorem.

FREUDENTHAL'S THEOREM.[1] *From* (1) *delete one cycle with* $p=0$ *and from* (2) *delete one cycle with* $q=0$; *each association of one of the remaining* $Z^p_i$ *with one of the remaining* $Z^q_j$, $p+q=r$, *represents a generator of the* $(r+1)$*-dimensional Betti group of* $(K_1, K_2)$; *the order of this generator is the greatest common divisor of the orders of* $Z^p_i$ *and* $Z^q_j$; *furthermore each* $C^r_{kl}$ *in* (4) *represents a generator of order* $1/c^r_{kl}$; *all such generators with their orders define the* $(r+1)$*-dimensional Betti group of* $(K_1, K_2)$.

**8. The case $r=-1$.** The join $(K_1, K_2)$ is connected because two points of $K_i$ can be joined to an arbitrary point of $K_j$, $j\neq i$.

PURDUE UNIVERSITY

# A MIXED BOUNDARY VALUE PROBLEM
## SOME REMARKS ON A PROBLEM OF A. WEINSTEIN

ALBERT E. HEINS

At present the unilateral Laplace transform has had many interesting applications. To cite several types we have (a) initial value problems in ordinary differential equations, (b) initial value and boundary value problems in partial differential equations with one space variable, (c) "initial value problems" in ordinary difference equations, and (d) "initial value" and "boundary value problems" in partial difference equations.

Titchmarsh[1] and his collaborators, Cooper[2] and Busbridge,[3] have indicated that much can be done with the finite Laplace transform, that is,

$$\int_a^b e^{-sx} f(x)dx = g(s). \tag{1}$$

The transform (1) contains as a special case a finite Fourier transform, first used by Stokes in 1850 for the solution of certain boundary value problems in mathematical physics. The finite Fourier transform has been recently revived by Doetsch[4] and Kniess.[5] It cannot be applied as widely as (1), since it assumes, a priori, that the boundary value problem naturally has a Fourier series as an expansion. On the other hand (1) slips quite neatly into an expansion which is natural to the boundary value problem.

By applying a transformation of the type (1) to a linear differential equation (ordinary or partial) with constant coefficients under given boundary conditions, boundary functions which are superfluous are introduced. Picone and others have demonstrated that by solving the reduced equation and noting the fact that $g(s)$ is an entire function of the parameter $s$, these superfluous boundary elements may be eliminated. This procedure may become exceedingly difficult to carry through. The method we employ here makes use of a regularity condition which is introduced by rendering the boundary conditions symmetric.

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[1] Titchmarsh, J. London Math. Soc. vol. 14 (1939) p. 118.

[2] Cooper, J. London Math. Soc. vol. 14 (1939) p. 124.

[3] Busbridge, J. London Math. Soc. vol. 14 (1939) p. 128.

[4] Doetsch, Math. Ann. vol. 112 (1935) p. 52.

[5] Kniess, Math. Zeit. vol. 44 (1939) p. 266.

We propose to treat here a mixed boundary value problem in potential theory with the aid of (1). This problem was first investigated by A. Weinstein in 1927 and later by Cooper,[6] Bochner and Poritsky. The author wishes to thank Dr. Alexandre Weinstein for having brought this problem to his attention and for having discussed it with him.

We are concerned here with the following problem. Find the conditions under which the solution of the harmonic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

under the mixed boundary conditions

$$u(x, 0) = 0, \qquad u(x, 1) = k u_y(x, 1)$$

may be bounded or of finite exponential order for $-\infty < x < \infty$. We modify this problem by considering the solution of (2) which is odd and continuous in $y$ and which satisfies the mixed boundary conditions

$$u(x, 1) = k u_y(x, 1), \qquad -u(x, -1) = k u_y(x, -1).$$

We write

$$g(x, s) = \int_{-1}^{1} e^{-sy} u(x, y)dy.$$

Since $u(x, y)$ is a harmonic function we may differentiate twice under the integral sign with respect to $x$, and we have

$$\frac{\partial^2 g}{\partial x^2} = -\int_{-1}^{1} e^{-sy} \frac{\partial^2 u}{\partial y^2} dy.$$

Upon integrating by parts and noting the boundary conditions of the modified problem we have

$$\begin{aligned}\frac{\partial^2 g}{\partial x^2} + s^2 g &= -e^{-s}u_y(x, 1) + e^{s}u_y(x, -1) - s[e^{-s}u(x, 1) - e^{-s}u(x, -1)]\\ &= u_y(x, 1)[(e^{s} - e^{-s}) - sk(e^{s} + e^{-s})].\end{aligned}$$

But

$$u_y(x, 1) = \frac{(\partial^2 g/\partial x^2) + s^2 g}{e^{s} - e^{-s} - sk(e^{s} + e^{-s})} \tag{3}$$

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[6] Loc. cit.

and (3) is finite for all $s$. Hence

$$g_{xx}(s_j) + s_j^2 g(s_j) = 0$$

for all $s_j$ which are roots of

$$e^s - e^{-s} - sk(e^s + e^{-s}) = 0. \tag{4}$$

Hence

$$g(x, s_j) = A(s_j)e^{is_jx} + B(s_j)e^{-is_jx} = \int_{-1}^{1} e^{-s_jy}u(x, y)dy.$$

Now if $k>1$,[7] (4) has a real root at $s=0$ and an infinite sequence of conjugate imaginary roots. If $k<1$, (4) has three real roots and an infinite sequence of conjugate imaginary roots. As a matter of notation let $s_j = i\alpha_j$ ($i=(-1)^{1/2}$). If $u(x, y)$ is to be bounded for all $x$, $s_j$ is real and hence all $A(s_j)$ and $B(s_j)$ are zero save for the coefficients $A$ and $B$ corresponding to the real $s_j$. If $u(x, y)$ is to be of finite exponential order, say $O(e^{\lambda x})$, then $|s_j| < \lambda$ and all $A(s_j)$ and $B(s_j)$ are zero for any $|s_j|$ greater than $|\lambda|$. Thus without going to the expense of solving a nonhomogeneous differential equation and evaluating complex integrals as Cooper did, we can get the same results he did by elementary methods.

We can of course obtain Cooper's expansion by noting that the set of functions $\{\sin \alpha_j y\}$ ($is_j = \alpha_j$) is a complete set of orthogonal functions over $-1 \leqq y \leqq 1$. If we then assume that $u(x, y)$ is of bounded variation with respect to $y$ in $-1 \leqq y \leqq 1$ (or any other such condition which will insure an adequate representation) we have

$$\sum_{j=-\infty}^{\infty} C_j e^{i\alpha_j y} \int_{-1}^{1} e^{-i\alpha_j y'} u(x, y')dy' = u(x, y) \tag{5}$$

where the $C_j$'s are the normalizing factors of the set $\{\sin \alpha_j y\}$. Then if $u(x, y)$ is bounded with respect to $x$, (5) appears as a sum of two terms (that is, the case $k>1$). If $u(x, y)$ is of finite exponential order with respect to $x$, (5) appears as a finite sum. It is to be noted that since no conditions other than order conditions were put on $x$, for large $x$, the $A$'s and $B$'s remain undetermined.

The technic we have employed can be applied to the solution of Laplace's equation in two or three variables, when the boundary of the domain over which we solve this equation is a rectangle. The same technic may be applied to the solution of characteristic value problems associated with the differential equation

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[7] The critical case $k=1$, follows as the discussion in the text. In this case a first degree polynomial in $x$ arises instead of the exponential functions.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0$$

for various types of boundary conditions when the boundary is rectangular.

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# ON THE CONVERGENCE OF A CONTINUED FRACTION

T. F. GLASS AND WALTER LEIGHTON

It is known [1] that sufficient conditions for the convergence of the continued fraction

$$b_0 + \frac{a_1}{1} + \frac{a_2}{1} + \cdots, \tag{1}$$

where the elements are complex numbers, are

$$|a_2| \geqq 5, \quad |a_{2n}| \geqq 25/4, \quad |a_{2n-1}| \leqq 1/4, \quad n = 2, 3, 4, \cdots. \tag{2}$$

The purpose of this note is to extend this result.

THEOREM. *If* $|a_{2n+1}| \leqq r \leqq 1/4$ $(n = 1, 2, 3, \cdots)$ *and if the numbers* $a_{2n} = \rho_{2n} e^{i\theta_{2n}}$ $(n = 1, 2, 3, \cdots)$ *satisfy the conditions*

$$\rho_{2n} \geqq 2(1 + r)^2[1 - \cos(\theta_{2n} + \theta_0)], \qquad 0 \leqq \theta_{2n} < \pi - \theta_0, \tag{3}$$

$$\rho_{2n} \geqq 4(1 + r)^2, \qquad \pi - \theta_0 \leqq \theta_{2n} \leqq \pi + \theta_0, \tag{4}$$

$$\rho_{2n} \geqq 2(1 + r)^2[1 - \cos(\theta_{2n} - \theta_0)], \qquad \pi + \theta_0 < \theta_{2n} \leqq 2\pi, \tag{5}$$

*where* $\theta_0 = 2 \arcsin r$, *the continued fraction* (1) *converges.*

To prove the theorem we employ the continued fraction

$$1 + \frac{x_1}{1} + \frac{x_2}{1} + \cdots \tag{6}$$

where

$$x_{2n} = \frac{(1 + a_{2n-1})(1 + a_{2n+1})}{a_{2n}}, \qquad n = 2, 3, \cdots, \tag{7.1}$$

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$$x_{2n+1} = a_{2n+1}, \qquad n = 1, 2, \cdots, \tag{7.2}$$

$$x_1 = -a_1, \qquad x_2 = \frac{1 + a_2}{a_2}. \tag{7.3}$$

Since the convergence of (1) is independent of the choice of $b_0$, we set $b_0 = 1 - a_1$, and it follows [1] that the 2$n$th convergent of (1) formally equals the $(2n+1)$st convergent of (6), while the $(2n+1)$st convergent of (1) is equal to the 2$n$th convergent of (6). Thus (1) and (6) will converge or diverge together. By the first hypothesis of the theorem and (7.2) the numbers $x_{2n+1}$ are bounded and lie in $\bar{R}$ the closed parabola $|z| - \Re(z) = 1/2$. It is thus sufficient to prove that the numbers $x_{2n}$ defined in (7.1) and (7.3) subject to conditions (3), (4), and (5) are bounded and lie in the parabolic region described above [2].

To this end let

$$\begin{aligned} s_{2n+1} &= (1 + a_{2n-1})(1 + a_{2n+1}) = a_{2n}x_{2n} \\ &= r_{2n+1}e^{i\phi_{2n+1}}, \qquad n = 2, 3, \cdots, \end{aligned}$$

and set $x_{2n} = t_{2n}e^{i\omega_{2n}}$. It is clear that $t_{2n} \leq \max r_{2n+1}/\min \rho_{2n} = (1+r)^2/\min \rho_{2n}$.

First suppose that $a_{2n}$ lies in the region defined by (4). Then $\min \rho_{2n} = 4(1+r)^2$, $t_{2n} \leq 1/4$ and $x_{2n}$ will lie in $\bar{R}$. Next suppose that (3) holds, that is, that $a_{2n}$ lies outside the cardioid $\rho = 2(1+r)^2[1 - \cos(\theta+\theta_0)]$ and in the angle $0 \leq \theta < \pi - \theta_0$. Hence $\min \rho_{2n} = 2(1+r)^2[1 - \cos(\theta+\theta_0)]$ for each $\theta$ and thus

$$t_{2n} \leq \frac{1}{2[1 - \cos(\theta_{2n} + \theta_0)]}, \qquad 0 \leq \theta_{2n} < \pi - \theta_0. \tag{8}$$

Further $\omega_{2n} = \phi_{2n+1} - \theta_{2n}$ from which it follows that $-\theta_0 - \theta_{2n} \leq \omega_{2n} \leq \theta_0 - \theta_{2n}$, and hence that $-\pi \leq \omega_{2n} \leq \theta_0$. It is clear that the right-hand member of (8) decreases steadily from $1/2(1 - \cos\theta_0)$ to $1/4$ as $\theta_{2n}$ increases from 0 to $\pi - \theta_0$ and at the same time $\omega_{2n}$ decreases steadily from $\phi_{2n+1}$ to $\phi_{2n+1} + \theta_0 - \pi$. The proof for the case when (3) holds may now be completed by proving that the point $(1/2\ (1 - \cos\theta_0), \theta_0)$ and the points

$$(1/2[1 - \cos(\theta + \theta_0)], -\theta - \theta_0), \qquad 0 \leq \theta < \pi - \theta_0, \tag{9}$$

lie in $\bar{R}$, as an examination of a simple figure will show. The first point evidently lies on the parabola since the equation of the parabola in polar coordinates is

$$r = \frac{1}{2(1 - \cos\theta)}. \tag{10}$$

The substitution $\lambda = -\theta - \theta_0$ proves immediately that the points (9) lie on (10).

By symmetry it is clear that when $a_{2n}$ lies in the region defined by (5) the corresponding complex number $x_{2n}$ lies in $\bar{R}$. This completes the proof of the theorem.

COROLLARY 1. *If* $|a_{2n+1}| \leqq r \leqq 1/4$ *and* $|a_{2n}| \geqq 4(1+r)^2$, *the continued fraction* (1) *converges.*

It is clear that an analogous theorem to the above may be proved with the roles of the even and the odd elements interchanged.

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2. W. T. Scott and H. S. Wall, *A convergence theorem for continued fractions*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 155–172.

THE MARION INSTITUTE AND
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# EXPANSIONS OF QUADRATIC FORMS

RUFUS OLDENBURGER

**1. The problem.** A quadratic form $Q$ with coefficients in a field $K$, whose characteristic is different from 2, is usually given as a linear combination

$$\sum_{i=1}^{n} a_{ij}x_ix_j \tag{1}$$

of products $\{x_i x_j\}$, where $(a_{ij})$ is symmetric. The sum (1) is one of the type

$$\sum_{i=1}^{\tau} L_iM_i, \tag{2}$$

where the $L$'s and $M$'s are linear forms. In general the decomposition (1) is not the most economical way of writing $Q$ as a sum of the type (2) in the sense that $\tau$ is a minimum for $Q$. In treating algebras associated with quadratic forms E. Witt[1] showed that the form $Q$ is equivalent under a nonsingular linear transformation to a decomposition

$$\sum_{i=1}^{\sigma} y_iz_i + \sum_{i=1}^{r-2\sigma} \nu_iu_i^2, \tag{3}$$

where the last sum is a nonzero form, and $r$ is the rank of $Q$. In the present paper we shall show that the minimum $\tau$ for $Q$ is $r-\sigma$. Thus this minimum $\tau$ is determined by the rank $r$ and the "characteristic" $\sigma$ of $Q$. This characteristic[2] is the maximum number $\sigma$ of linearly independent linear forms $L_1, \cdots, L_\sigma$ such that the rank of $Q+\lambda_1L_1^2+\cdots+\lambda_\sigma L_\sigma^2$ is the same as the rank of $Q$ for all values of the $\lambda$'s. The form $Q$ has characteristic $\sigma$ *if and only if* $Q$ has the *canonical splitting* $G+H$, where $G$ has characteristic $\sigma$ and rank $2\sigma$, while $H$ has characteristic 0 and rank $r-2\sigma$. The form $G$ has a decomposition (2) with $\tau=\sigma$. The decomposition (3) is one such that the first sum is a form $G$ of the type described and the other a form $H$. Thus it will be proved

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[1] E. Witt, *Theorie der Quadratischen Formen in beliebigen Körpern*, J. Reine Angew. Math. vol. 176 (1937) p. 35.

[2] Rufus Oldenburger, *The index of a quadratic form for an arbitrary field*, Bull. Amer. Math. Soc. abstract 48-5-162.

that the decomposition (3) corresponding to a canonical splitting $G+H$ of $Q$ is one with a minimum number of terms.

Like the rank of $Q$ the characteristic of $Q$ has the property that this characteristic changes at most by 1 under addition of a term $\lambda L^2$, $L$ linear, to $Q$. We shall prove here that actually the minimum $\tau$, the characteristic $\sigma$ defined above, and the index[3] (if $K$ is real) possess this property of changing at most by 1 under additions of the type $LM$ to $Q$, where $L$ and $M$ are arbitrary linear forms.

We recall that the rank $r$ of $Q$ is the minimum $\tau$ for which $Q$ can be written as a sum (2), where for each $i$ the forms $L_i$ and $M_i$ are linearly dependent. Thus both the rank alone, and the rank and characteristic of $Q$, yield minimum properties of expansions of $Q$ invariant under nonsingular linear transformations on the variables in $Q$.

It will be understood throughout the present paper that the coefficients are in a field $K$ of the type specified above. The field $K$ is otherwise unrestricted, except where $K$ is taken to be the real or complex fields.

## 2. Solution of the minimum problem.

The following lemma needs no proof.

LEMMA 1. *The characteristic of a quadratic form $Q$ is invariant under nonsingular linear transformations on $Q$.*

The lemma to follow was proved elsewhere.[4]

LEMMA 2. *The characteristic of a quadratic form $Q$ changes at most by 1 under addition to $Q$ of a term $\lambda L^2$, $L$ linear and $\lambda$ in the given field.*

LEMMA 3. *The characteristic of a quadratic form $Q$ is at least as great as the characteristic of each form $Q^*$ obtained from $Q$ by imposing homogeneous linear relations on the variables in $Q$.*

We write $Q$ as in (1), and suppose that $Q^*$ is obtained from $Q$ by equating $x_1, \cdots, x_{e-1}$ to zero for some $e$. The matrix $A=(a_{ij})$ of $Q$ can be written as

$$\begin{Vmatrix} * & * \\ * & B \end{Vmatrix},$$

where $B$ is the matrix $(a_{ij})\,[i, j=e, \cdots, n]$ of $Q^*$, and the asterisks in $A$ indicate minors of $A$. We let $\alpha$ designate the characteristic of the form $Q^*$. Since $Q^*$ has a canonical splitting, as described in §1, the form $Q^*$ is equivalent under a nonsingular transformation to the form

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[3] The *index* of a real quadratic form $Q$ is the number $k$ of $+$ signs in a canonical form $x_1^2+\cdots+x_k^2-x_{k+1}^2-\cdots-x_r^2$ to which $Q$ is equivalent.

[4] See the above reference to a paper by R. Oldenburger.

$$\sum_{i=1}^{\alpha} u_i v_i + F, \tag{4}$$

where $F$ is a form of characteristic 0 whose variables are linearly independent of the $u$'s and $v$'s. With the form (4) we may associate the symmetric matrix

$$\begin{Vmatrix} 0 & 0 \\ 0 & C \end{Vmatrix}$$

of order $n-e+1$, where $C$ is a nonsingular minor of the type

$$\begin{Vmatrix} 0 & 0 & I \\ 0 & D & 0 \\ I & 0 & 0 \end{Vmatrix},$$

the minor $I$ being an identity matrix of order $\alpha$ [arising from the summation in (4)]. It follows from elementary matrix considerations that there is a nonsingular matrix $N$ such that

$$N'AN = \begin{Vmatrix} E & 0 \\ 0 & 0 \end{Vmatrix},$$

where $E$ is a nonsingular minor with the shape

$$E = \begin{Vmatrix} * & * & * & I \\ * & * & * & 0 \\ * & * & D & 0 \\ I & 0 & 0 & 0 \end{Vmatrix},$$

and $N'$ designates the transpose of $N$. We let $T = T(y_1, \cdots, y_n)$ be a quadratic form in $y_1, \cdots, y_n$ associated in the usual manner with $N'AN$. The rank of $T + \lambda_1 y_1^2 + \cdots + \lambda_\alpha y_\alpha^2$ is the same as the rank of $T$ for all values of the $\lambda$'s. Since $T$ is equivalent to $Q$ the characteristic of $Q$ is at least $\alpha$.

Suppose now that we impose homogeneous linear relations $Z_1 = 0, \cdots, Z_s = 0$ on the variables $x_1, \cdots, x_n$ in $Q$. It is no restriction to take these $Z$'s to be linearly independent forms. We may therefore use these $Z$'s and enough of the $x$'s to obtain a set of linearly independent forms, which we may employ as the $n$ variables in terms of which $Q$ is expressed. By Lemma 1 this change of variables leaves the characteristic invariant. Thus the problem which arises when the $Z$'s are set equal to zero reverts to the above case where $x_1 = \cdots = x_{s-1} = 0$.

THEOREM 1. *The minimum $\tau$ for which a quadratic form $Q$ with rank $r$ and characteristic $\sigma$ has the expansion* (2), *where the $L$'s and $M$'s are linear forms, is $r-\sigma$.*

We suppose that $Q$ is written as (2), where $\tau$ is a minimum. If for some $i$ and element $k_i$ we have $M_i = k_i L_i$, we write $k_i L_i^2$ in place of $L_i M_i$ ($i$ not summed). Thus we can split the sum (2) into $R+S$, where

$$R = \sum_{i=1}^{s} L_i M_i, \qquad S = \sum_{i=1}^{t} \nu_i N_i^2, \tag{5}$$

$L_i$ being linearly independent of $M_i$ for each $i$, and the $N$'s being linear forms. The $L$'s form a set of linearly independent linear forms, since otherwise we can write $R$ as a sum of products of linear forms with less terms. As in §1 we write a canonical splitting of $Q$ as $G+H$. Since $Q$ has rank $r$, we may take $Q$ to be a form in $r$ independent variables. Since the rank of $S$ is $t$, we have $t \geqq r-2s$. If $s<\sigma$, we have $s+t>r-\sigma$, whence the decomposition corresponding to the canonical splitting $G+H$ has less terms than (2). Thus $s \geqq \sigma$, and we can write $s=\sigma+\rho$ for a $\rho \geqq 0$.

We relabel the subscripts on the $L$'s, $M$'s, and $N$'s if necessary so that the forms in the set $\Sigma$, where

$$\Sigma = (L_1, \cdots, L_{\sigma+\rho}, M_1, \cdots, M_\zeta, N_1, \cdots, N_\xi),$$

yield a minimal basis for the $L$'s, $M$'s and $N$'s. Here $\xi \geqq r-\sigma-\rho-\zeta$. If (2) is a more economical decomposition than that which arises from $G+H$, we have $t \leqq r-\rho-2\sigma-1$. Now $t \geqq r-\sigma-\rho-\zeta$. These inequalities yield $\zeta \geqq \sigma+1$. We suppose that $\zeta$ satisfies this inequality. We let $Q'$ designate the form

$$Q - \sum_{i=\xi+1}^{t} \nu_i N_i^2.$$

Since by Lemma 2 the characteristic changes at most by 1 under each subtraction with $\nu_i N_i^2$ ($i$ not summed), the index $\alpha$ of $Q'$ is such that

$$\alpha \leqq \sigma + t - \xi.$$

Eliminating $t$ with the aid of an inequality relation above, we have $\alpha \leqq \zeta - 1$.

We take the linear forms in the set $\Sigma$ to be the variables in terms of which the form $Q'$ above is expressed. Setting $L_{\zeta+1}, \cdots, L_{s+t} = 0$, we obtain from $Q'$ a form $Q''$ with index $\zeta$. By Lemma 3, we have $\alpha \geqq \zeta$, giving us a contradiction. It follows that $\tau = r - \sigma$.

For the complex field the characteristic $\sigma$ of $Q$ is $[r/2]$, whereas for the real field $\sigma$ is the minimum of the indices of $Q$ and $-Q$. These results yield Corollary 1.

COROLLARY 1. *For the complex field the minimum number $\tau$ of Theorem* 1 *is $r - [r/2]$. For the real field $\tau$ is the maximum of the indices of $Q$ and $-Q$.*

Witt proved[8] that a form is a zero form if and only if the characteristic $\sigma$ of this form is greater than 0.

COROLLARY 2. *The form $Q$ of Theorem* 1 *is a zero form if and only if $\tau \neq r$.*

By Theorem 1 the sum (2), where $\tau$ is a minimum, is a sum with the $R$ and $S$ of (5) satisfying $R = G$, $S = H$, the sum $G + H$ being a canonical splitting of $Q$.

Although addition of a term $LM$, $L$ and $M$ linear, may change the rank $r$ of $Q$ by 2, this is not true of the index $\sigma$ and $\tau = r - \sigma$ as we shall now prove.

THEOREM 2. *Under addition of a term $LM$, $L$ and $M$ linear, to a quadratic form $Q$, the characteristic $\sigma$, and the minimum number $\tau$ for decompositions of type* (2), *change at most by* 1.

We write $Q$ as a sum (2) where $\tau$ takes on the minimum value $r - \sigma$, the rank of $Q$ being $r$. We let $\tau'$, $r'$, $\sigma'$ designate the analogues for $Q' = Q + LM$ of $\tau$, $r$, $\sigma$ for $Q$. Since

$$Q' = \sum_{i=1}^{\tau} L_i M_i + LM,$$

we clearly have $\tau' \leqq \tau + 1$. Thus $\tau$ changes at most by 1 under the addition of $LM$ to $Q$.

We suppose that $L$ and $M$ are linearly independent of each other and of the variables in $Q$ so that $r' = r + 2$. We write $Q$ as $R + S$, where $R$ and $S$ are given by (5) with $s = \sigma$, $t = r - \sigma$, whence

$$Q = \sum_{i=1}^{\sigma} L_i M_i + \sum_{i=1}^{r-\sigma} \nu_i N_i^2. \tag{6}$$

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[8] See the above reference to a paper by E. Witt

The form $R'$, where $R'=R+LM$, has index $\sigma+1$ and rank $2(\sigma+1)$, from which it follows that $Q'$ has the canonical splitting $R'+S'$ with $S'=S$. Thus $Q'$ has index $\sigma+1$.

If $L$ and $M$ are taken linearly dependent, or one or both of the forms $L$, $M$ are restricted to be linear forms in the variables of $Q$, by Lemma 3 we obtain from the form $Q'$ of the preceding paragraph a form $Q^*$ whose characteristic does not exceed $\sigma+1$. Thus in any case $\sigma'\leqq\sigma+1$, whence also $\sigma\leqq\sigma'+1$.

If the rank of $Q+LM$ is less than the rank of $Q$, Theorem 2 implies that the characteristic of $Q+LM$ does not exceed that of $Q$, whereas if the addition of $LM$ to $Q$ decreases the rank of $Q$ by 2, this addition also decreases the characteristic of $Q$.

We have the following analogue of Theorem 2.

THEOREM 3. *Under addition of a term $LM$, $L$ and $M$ linear, to a real quadratic form $Q$, the index of $Q$ changes at most by* 1.

We write $Q$ as the sum $P+N$ of a positive definite form $N$ and a negative definite form $N$, the rank of $Q$ being the sum of the ranks of $P$ and $N$. We suppose that $Q$ is written in any way as a sum $P'+N'$, where $P'$ and $N'$ are positive definite and negative definite forms respectively. The index $\alpha$ of $Q$ is the rank of $P$. We let $\beta$ designate the rank of $P'$, whence

$$P' = \sum_{i=1}^{\beta} p_i P_i^2$$

for linear forms $P_1, \cdots, P_\beta$. We suppose that $\alpha>\beta$. Setting $P_1=\cdots=P_\beta=0$, we have $P\neq 0$, while $P+N$ is negative definite, a contradiction. Thus $\alpha\leqq\beta$.

The form $LM$ can be written as the difference $R^2-S^2$, where $R$ and $S$ are linear forms, or one of the terms $R$, $S$ is zero. The form $Q+LM$ is a sum of the positive definite form $P+R^2$, and negative definite part $N-S^2$. Since the rank of $P+R^2$ differs at most by 1 from the rank of $P$, the index of $Q+LM$ does not exceed $\alpha+1$. It follows that the indices of $Q$ and $Q+LM$ differ at most by 1.

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# SECTIONS OF CONTINUOUS COLLECTIONS

J. H. ROBERTS AND PAUL CIVIN

In the present note we establish the following

THEOREM. *Suppose $G$ is a continuous collection*[1] *of closed and compact sets filling a separable metric space $X$. Suppose further that the space $G$, considered as a decomposition space, has dimension at most $n$. Then there is a closed subset $K$ of $X$, such that for each $g \in G$, the set $g \cdot K$ is nonvacuous and consists of at most $(n+1)$ points.*

We call such a point set $K$ an $(n+1)$-section of the collection $G$. Thus a 1-section of $G$ is a true section. G. T. Whyburn[2] has shown that if the elements of $G$ are 0-dimensional and $G$ is a dendrite, then $G$ admits a true section. The present result gives only a 2-section, but there is no hypothesis on the dimension of the elements of $G$. For $n=1$, it is known that in general $G$ does not admit a true section. For $n>1$ it is not known whether the present result gives the best possible constant.

We first establish the theorem in the 0-dimensional case.

LEMMA. *Suppose $G$ is 0-dimensional, and $\epsilon$ is a given positive number. Suppose $W$ is an open set in $X$ such that $W \cdot g \neq 0$ for each $g \in G$. Then there is an open set $E$ in $X$ such that $\overline{E} \subset W$, $E \cdot g \neq 0$ for every $g \in G$, and the diameter of $E \cdot g < \epsilon$ for each $g \in G$.*

Let $f(x)$ be a homeomorphism of $M$, a subset of the Cantor set, into $G$.[3] In the product space $M \times X$, consider the set $A$ of points $(x, y)$ with $x \in M$ and $y \in f(x)$. For $x \in X$ there is a unique $y = y(x)$ in $M$ such that $x \in f(y)$. The function $t(x) = (y(x), x)$ is a homeomorphism of $X$ into $A$.

In the space $A$, the open set $t(x)$ and the continuous collection $H$ of elements $t(g)$ for $g \in G$ satisfy the properties of $W$ and $G$ stated in the hypothesis of the lemma. Furthermore, the diameter of a set $Z$ in $A$ is not smaller than the diameter of $t^{-1}(Z)$. Hence all we need show is that there exists an open set $E$ satisfying the theorem relative to the open set $t(W) = U$ and the continuous collection $H$.

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[1] *A continuous collection filling a space $X$*, is a collection $G$ of sets $g$ such that: (1) If $x \in X$, then $x \in g$ for exactly one $g$. (2) If $x \in g$, $x_n \in g_n$ and $x_n \to x$, then $\lim g_n = g$.

[2] *A theorem on interior transformations*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 414–416.

[3] P. Urysohn, *Sur les multiplicités Cantoriennes*, Fund. Math. vol. 7 (1926) p. 77.

For each $p\in U$ there is an open set $U_p$ such that (1) $U_p\supset p$, (2) $\overline{U}_p\subset U$, (3) the diameter of $U_p<\epsilon$, and (4) the projection[4] $V_p$ of $U_p$ upon $M$ is both open and closed. The collection $\{V_p\}$ is an open covering of $M$ and therefore there is a countable subcollection $\{V_{p_i}\}$ covering $M$. The collection of sets $\{W_i\}$ where $W_i$ is defined by the relations

$$W_1 = V_{p_1}, \qquad W_i = V_{p_i} - \sum_{j=1}^{i-1} V_{p_j}$$

is a covering of $M$ by mutually exclusive open sets. Let $Y_i$ denote the open subset of $U_{p_i}$ whose projection is $W_i$ and let $E=\sum_{i=1}^{\infty} Y_i$. The open set $E$ has the required properties relative to the space $A$, the open set $U$, and the continuous collection $H$.

In order to prove that $\overline{E}\subset U$ it is sufficient to show that $\overline{E}=\sum_{i=1}^{\infty}\overline{Y}_i$. Suppose $p\in\overline{E}$ and $p\notin\sum_{i=1}^{\infty}\overline{Y}_i$. Then there is a sequence $p_n\to p$ and $p_n\in Y_{i_n}$. Suppose $p\in g$, and $\pi(g)\subset W_j$, where $\pi$ denotes the projection of $A$ on $M$. Since $p_n\to p$, $\pi(p_n)\to\pi(p)$. But $\pi(p_n)\not\subset W_j$ for more than a finite number of $n$. This contradicts the fact that $W_j$ is open.

Now $E$ intersects each $g$ since the sequence $\{W_i\}$ is a covering of $M$. Also, since the sets $W_i$ are mutually exclusive, if $Y_i\cdot g\neq 0$ then $Y_j\cdot g=0$ for $i\neq j$. Then, as $Y_i\subset U_{p_i}$ and the diameter of $U_{p_i}<\epsilon$, the diameter of $E\cdot g<\epsilon$ for every $g\in H$. This proves the lemma.

The theorem for the 0-dimensional case follows by considering a sequence of positive numbers $\{\epsilon_n\}$ with $\epsilon_n\to 0$ as $n\to\infty$ and a sequence of open sets $\{E_n\}$ such that $\overline{E}_{n+1}\subset E_n$, $E_n\cdot g\neq 0$ for $g\in G$, and the diameter of $E_n\cdot g<\epsilon_n$, for $g\in G$. The common part $K$ of the sets $E_n$ is closed. For $g\in G$, the set $K\cdot g$ consists of exactly one point, since $g$ is compact and $\epsilon_n\to 0$.

The theorem for the $n$-dimensional case follows by considering an at most $(n+1)$-to-one closed mapping $f(x)$ of a subset $M$ of the Cantor set[5] into $G$. In the product space $M\times X$ consider the set $A$ of points $(y, x)$ with $y\in M$ and $x\in f(y)$. The sets $(y, x)$ for $y$ fixed and $x\in f(y)$ form a 0-dimensional continuous collection $H$ which fills $A$. The mapping $t(y,x)=x$ is a closed, at most $(n+1)$-to-one mapping of $A$ into $X$. By the theorem for the 0-dimensional case, there is a true section $K$ of the collection $H$ in the space $A$. The set $t(K)$ gives the required $(n+1)$-section of the continuous collection $G$.

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[4] That is, $V_p$ is the set of $x\in M$ such that $(x, y)\in U_p$.

[5] See J. H. Roberts, *A theorem on dimension*, Duke Math. J. vol. 8 (1941) p. 572, Theorem 9.1. The mapping $\phi_n$ as actually defined is a closed mapping, although this result is not specifically stated in the theorem.

# FUNCTIONAL TOPOLOGY

MARSTON MORSE

**Introduction.** The present paper gives the fundamental existence proof for a homotopic critical point under hypotheses which are less stringent than those employed in the author's recent fascicule on functional topology and abstract variational theory [2]. This relaxation of hypotheses seems to be necessary if one is to extend the variational theory in the large to non-regular problems. Non-regular integrals include some of the most important and interesting integrals such as the Jacobi least action integral[1] in the three body problem of celestial mechanics. The author believes that a topological basis for the planetary orbits will be disclosed by studying the contour manifolds of this Jacobi integral.

The results of this paper will be used to extend the theory in the large to non-regular problems in two papers by Morse and Ewing.

1. **The metric spaces $M$, $L$, $J$.** We are concerned with a compact metric space $M$ of elements $p, q, r, \cdots$ and distance $pq, pr, \cdots$. We shall deal with two functions $J(p)$ and $L(p)$, bounded, single-valued, and lower semi-continuous on $M$. In the applications $p$ will be a curve joining two fixed points in some space, the distance $pq$ will be the Fréchet distance between curves, while $J(p)$ and $L(p)$ will be integrals along $p$, with $L$ the length of $p$.

Beside the metric $M$ we shall use two other metrics, an $L$-metric with a distance

$$| pq | = pq + | L(p) - L(q) |,$$

and a $J$-metric with a distance,

$$pq + | J(p) - J(q) |.$$

We shall refer to the corresponding spaces as the spaces $L$ and $J$. We shall make the following hypothesis.

HYPOTHESIS. *Convergence on $L$ to a point $p$ shall imply convergence on $J$ to $p$.*

We shall not assume that convergence on $J$ to $p$ implies convergence on $L$ to $p$. Convergence on $L$ or $J$ clearly implies convergence on $M$. Terms such as neighborhood, compact, and so on

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[1] For other examples see paper [1] by McShane.

will be preceded by the letters $M$, $L$, or $J$ according to which metric is used to define them. Every $M$-neighborhood of a point $p$ contains an $L$-neighborhood of $p$, but not conversely. Every $J$-neighborhood of $p$ contains an $L$-neighborhood of $p$, but not conversely.

The subset of points on which $J \leqq c$ will be denoted by $J^c$. The set $J^c$ is $M$-compact since $J$ is lower semi-continuous on $M$. But $J^c$ will not in general be $L$ or $J$-compact. In fact an $M$-compact subset $A$ would be $L$-compact if and only if $L(p)$ were $M$-continuous on $A$. Similarly with $J$-compactness. In seeking to cover sets such as $J^c$ with a finite number of neighborhoods we accordingly use $M$-neighborhoods, although it would be simpler if we could use $L$-neighborhoods.

On the other hand the deformations which we shall use in later papers are adequate only if their continuity is $L$-continuity. We shall thus be using $L$-continuous deformations defined over $M$-neighborhoods. But $J$-neighborhoods enter also, since there are important properties which can be established for $J$-neighborhoods but not for $L$ or $M$-neighborhoods.

Our chains and cycles shall be defined on $L$ using $L$-continuity. They shall be finite singular chains and cycles, taken mod 2, see [3, p. 146]. The point set bearing a singular chain is $L$-compact and hence $J$-compact. We shall term the least upper bound of $J$ on an arbitrary set $E$, the *$J$-height* of $E$. In general we shall say that a point $p$ is *above* or *below* $q$ according as $J(p)$ is greater or less than $J(q)$. We shall admit *relative* $k$-cycles $u$ in which the modulus is always a set $J^c$ in which $c$ is less than the $J$-height of $u$.

2. **$J$-deformations.** Let $E$ be a subset of $L$. Let $I$ be an interval $0 \leqq \tau \leqq a$ $(a>0)$. By $E \times I$ we shall mean the product of $E$ and $I$, assigning the usual metric to the space $E \times I$. We shall admit deformations $D$ of $E$ which replace a point $p$ found on $E$ at the time $\tau = 0$ by a point $q(p, \tau)$ on $L$ at the time $\tau$ $(0 \leqq \tau \leqq a)$. If $D$ is to be admissible we require:

(a) *That $q(p, \tau)$ map $E \times I$ continuously into $L$.*

(b) *That for $p$ fixed $q(p, \tau)$ map $I$ continuously into $M$ uniformly with respect to $(p, \tau)$ on $E \times I$.*

We shall say that $D$ is a *weak $J$-deformation*, if for $p$ on $E$ and for $q = q(p, \tau)$, $J(p) - J(q) \geqq 0$ for each $\tau$ on $I$. A weak $J$-formation $q(p, \tau)$ will be said to be *proper* on $E$, if when $\tau$ is bounded from 0 on $I$,

$$J(p) - J(q) > \epsilon > 0, \qquad q = q(p, \tau),$$

where $\epsilon$ is a constant independent of $p$ on $E$.

A point $p$ will be said to be ***homotopically J-ordinary*** if some $J$-neighborhood of $p$ admits a proper $J$-deformation. A point which is not homotopically $J$-ordinary will be termed ***homotopically J-critical***.

We shall say that $J$ is ***upper-reducible*** at $p$, if for each constant $a > J(p)$ there exists a weak $J$-deformation of some $M$-neighborhood of $p$ which is proper for points initially above $a$.

The reader may wonder why a $J$-neighborhood is used in the above definition of a $J$-critical point while an $M$-neighborhood is used in the definition of upper-reducibility. A $J$-neighborhood is used in this connection because otherwise we are unable to prove that a rectifiable curve which is not an extremal is homotopically $J$-ordinary.[2] This reflects the fact that being an extremal is a consequence of properties possessed by weak neighborhoods such as $J$-neighborhoods. On the other hand an $M$-neighborhood is used in the definition of upper-reducibility since otherwise we could not prove the fundamental Theorem 3.1 of this paper. For, as one sees from its definition, upper-reducibility is not concerned merely with points $q$ at which $J(q)$ is near $J(p)$, so that a $J$-neighborhood would be too restrictive.

We come to products of deformations. Let $D$ be a deformation of a set $A$. The set of final images of points of $A$ under $D$ will be denoted by $D(A)$. Let $B_1, \cdots, B_n$ be a set of weak $J$-deformations such that $B_1$ is applicable to $A$, $B_2$ to $B_1(A)$, and more generally $B_{i+1}$ is applicable to $B_i \cdots B_1(A)$. In such circumstances the deformations $B_n \cdots B_1$ will be said to define a product deformation $\Delta$. The deformation $\Delta$ is applicable to $A$. One sees that $\Delta$ is a weak $J$-deformation of $A$.

The following lemma shows how a weak $J$-deformation can be extended as a weak $J$-deformation beyond a local domain of definition.

LEMMA 2.1. *Let $B$ be a subset of $L$ and $s$ a point of $B$. Let $S_r$ be the intersection with $B$ of a spherical $M$-neighborhood of $s$ of radius $r$. Suppose that $S_{2e}$ admits a weak $J$-deformation $D$. Then $D$ can be replaced by a weak $J$-deformation $\theta$ of $B$ such that $\theta \equiv D$ for points initially on $S_e$ and $\theta$ is the null deformation for points initially exterior to $S_{2e}$.*

Suppose the time $\tau$ in $D$ varies on the interval $(0, a)$. Under $\theta$, $\tau$ shall likewise vary on $(0, a)$. Points of $B$ initially on $S_e$ shall be deformed under $\theta$ as under $D$ while points of $B$ initially exterior to $S_{2e}$ shall be held fast. Points $p$ of $B$ at a distance $ps$ from $s$ such that

$$(2.1) \qquad e \leqq ps \leqq 2e,$$

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[2] This theorem is the basic generalization of the Euler theorem that a curve (of class $C'$) which is not an extremal is not minimizing.

shall be deformed under $\theta$ as follows. Let $t(p)$ divide the interval $(0, a)$ in the ratio inverse to that in which $ps$ divides the interval (2.1). That is, let

$$\frac{t(p)}{a} = \frac{2e - ps}{e}, \qquad e \leq ps \leq 2e.$$

Under $\theta$ points $p$ of $B$ which satisfy (2.1) initially shall be deformed as under $D$ until $\tau$ reaches $t(p)$ and shall be fast thereafter.

Let $q(p, \tau)$ be the image of $p$ under $\theta$. Recall that $ps$, and hence $t(p)$, vary continuously as $p$ varies $L$-continuously on $\overline{S_{2e}-S_e}$. It will be convenient to set $t(p) = a$ for $p$ on $S_e$ and $t(p) = 0$ for $p$ on $B - S_{2e}$. Then $t(p)$ is defined and continuous as $p$ varies on $B$. To establish the continuity of $q(p, \tau)$ one breaks the domain of the pairs $(p, \tau)$ into the two domains

$$(2.3) \qquad [0 \leq \tau \leq t(p)] \qquad [p \text{ on } B],$$

$$(2.4) \qquad [t(p) \leq \tau \leq a] \qquad [p \text{ on } B],$$

with the set on which $\tau = t(p)$ in common. On the second domain $q(p, \tau)$ is constant. On the first domain $q(p, \tau)$ equals the point function defining $D$. The functions $q(p, \tau)$ defined over these separate domains obviously combine to define a function $q(p, \tau)$ with the properties of a weak $J$-deformation of $B$.

3. **The fundamental theorem.** Two different sets of neighborhoods $U$ and $V$, respectively, covering a given space, will be said to be *equivalent* if each neighborhood $U$ of each point $p$ contains a neighborhood $V$ of $p$, and conversely. In this sense the set of $J$-neighborhoods is equivalent to the set of neighborhoods $V(p)$ defined by conditions of the form

$$(3.1) \qquad pq < \delta, \qquad J(q) < J(p) + e,$$

where $p$ is fixed and $\delta$ and $e$ are arbitrary positive constants. This is a consequence of the lower semi-continuity of $J(p)$ for $p$ on $M$. For the relation $pq < \delta$ implies the relation

$$(3.2) \qquad J(p) - e < J(q)$$

if $\delta$ is sufficiently small. Conditions (3.1) and (3.2) taken jointly define a set of neighborhoods equivalent to the set of $J$-neighborhoods, so that the set of neighborhoods (3.1) is equivalent to the set of $J$-neighborhoods. We shall therefore feel free to replace $J$-neighborhoods by neighborhoods of the type (3.1).

We continue with a fundamental theorem.

THEOREM 3.1. *Let $C$ be an $M$-compact set with $J$-height $c$, containing no homotopic $J$-critical points at the level $c$. If $J$ is upper-reducible at each point of $C$ there exists a weak $J$-deformation of a $J$-neighborhood of $C$ into a set with $J$-height less than $c$.*

I. If $p$ is a point of $C$ at the level $c$, $p$ is homotopically $J$-ordinary. There accordingly exists a spherical $M$-neighborhood $V(p)$ of $p$ and a positive constant $c(p) > c$ such that the intersection $J^{c(p)} \cdot V(p)$ admits a proper $J$-deformation $D(p)$.

II. If $p$ is a point of $C$ below $c$ the upper-reducibility of $J$ at $p$ implies the following: there exists a spherical $M$-neighborhood $V(p)$ of $p$ and a constant $a(p) < c$ such that $V(p)$ admits a weak $J$-deformation $D(p)$ which is a proper deformation of the subset of $V(p)$ above $a(p)$.

Let $U(p)$ and $R(p)$ be spherical $M$-neighborhoods of $p$ with radii one-third and one-sixth that of $V(p)$, respectively. Since $C$ is $M$-compact there exists a finite set of neighborhoods $R(p)$, say neighborhoods $R_1, \cdots, R_n$ of points $p_1, \cdots, p_n$, respectively, covering $C$. For points $p_i$ at the level $c$, set $\mu = \min c(p_i)$. For points $p_i$ below $c$ set $\nu = \max a(p_i)$. We shall apply Lemma 2.1 setting $s = p_i$, $B = J^\mu$ and $e$ is equal to the radius of $U_i = U(p_i)$. With this choice of $s$, $B$ and $e$ we infer the existence of a weak $J$-deformation $H_i$ of $J^\mu$ which deforms $U_i \cdot J^\mu$ as does $D(p_i)$.

Let $r$ equal the minimum of the radii $r_i$ of the spheres $R_i$. Let $\theta_i$ be a weak $J$-deformation defined by taking the time interval in $H_i$ so short that the maximum $M$-displacement under $\theta_i$ is less than $r/n$. The product deformation $\Delta = \theta_n \cdots \theta_1$ is a weak $J$-deformation of $J^\mu$ and displaces no point an $M$-distance in excess of $r \leqq r_i$. In particular we can conclude that

$$\theta_{i-1} \cdots \theta_1(R_i J^\mu) \subset U_i \cdot J^\mu. \tag{3.3}$$

Let $S_i$ be the set defined by the left number of (3.3) and let $C_i$ be the subset of $S_i$ above $\nu$. The deformation $\theta_i$ is proper on $C_i$ since $\theta_i = D(p_i)$ on $C_i$. If $C_i \neq 0$, the $J$-height of $C_i$ is accordingly diminished under $\theta_i$ by a positive constant $\eta_i$. If $C_i = 0$, set $\eta_i = 0$. Let $2\eta$ be a positive constant less than each $\eta_i \neq 0$ and such that $c + \eta < \mu$ (recall that $c < \mu$). Then $\Delta$ deforms the set

$$(R_1 + \cdots + R_n)\ J^{c+\eta} \tag{3.4}$$

into a set with a $J$-height at most $\max(\nu, c - \eta) < c$.

The set (3.4) contains a $J$-neighborhood of $C$ and the proof of the theorem is complete.

A class $K$ of $k$-cycles on $L$, either relative or absolute, will be said to form a *$J$-class* if any cycle into which a cycle $u$ of $K$ can be carried by a weak $J$-deformation is also in $K$. With this understood we have the following corollary of the theorem.

COROLLARY 3.1. *Suppose $J$ is upper-reducible at each point of an $M$-compact set $C$. If there exists a $J$-class $K$ of $k$-cycles such that $K$ includes a cycle in each $J$-neighborhood of $C$ but no cycle with a $J$-height less than that of $C$ there exists at least one homotopic $J$-critical point with the $J$-height of $C$.*

In applying this corollary one is led to the two following specializations.

COROLLARY 3.2. *Let $K$ be a homology class of absolute $k$-cycles, non-bounding on $L$, and let $c$ be the greatest lower bound of $J$-heights of cycles of $K$. If $J$ is upper-reducible at each point of $J^a$ there exists a homotopic $J$-critical point of $J$ at the level $c$.*

Corollary 3.2 follows from Corollary 3.1 upon taking $C$ as $J^a$.

COROLLARY 3.3. *Let $K$ be a homology class composed of $k$-cycles* mod $J^a$, *non-bounding* mod $J^a$ *on $L$, and let $c$ be the greatest lower bound of $J$-heights of cycles of $K$. If $c>a$ and if $J$ is upper-reducible at each point of $J^a$ there exists a homotopic $J$-critical point at the level $c$.*

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THE INSTITUTE FOR ADVANCED STUDY

# RECURRENCE FORMULAS FOR CERTAIN DIVISOR FUNCTIONS

D. H. LEHMER

Numerical functions giving the excess of the number of divisors of $n$ of one sort over the number of divisors of a second sort were introduced over a century ago from the theory of binary quadratic forms and from the theory of elliptic functions. Later a systematic discussion of several of these functions was made by J. W. L. Glaisher, who, by means of theta function identities, found recurrences with gaps for these functions. The purpose of this note is to point out that these functions are sums over divisors of certain periodic Lucas functions and that sums of Lucas functions in general satisfy triangular number recurrence relations as given by (6) below. This formula is specialized in several ways, first to periodic Lucas functions, then to degenerate functions, and then to the more typical cases related to the Pell equation and Fibonacci's series.

Formula (6) may be obtained in the following elementary manner from the famous triple product identity of Gauss and Jacobi.[1]

$$(1)\qquad \prod_{r=1}^{\infty}\left\{(1-x^{2r})(1+ax^{2r-1})(1+a^{-1}x^{2r-1})\right\}=\sum_{n=-\infty}^{\infty}a^{n}x^{n^2}.$$

If in (1) we set $x=t^{1/2}$ and $a=-\alpha t^{1/2}$ we obtain at once

$$(2)\quad (1-\alpha)\prod_{r=1}^{\infty}\left\{(1-t^{r})(1-\alpha t^{r})(1-\alpha^{-1}t^{r})\right\}=\sum_{n=-\infty}^{\infty}(-\alpha)^{n+1}t^{n(n+1)/2}.$$

If now we introduce $\theta$ by $\alpha=e^{2\theta i}$ and write

$$(3)\qquad P(\theta,t)=\prod_{r=1}^{\infty}\left\{(1-t^{r})(1-2t^{r}\cos 2\theta+t^{2r})\right\},$$

then, on combining the terms on the right of (2) for $n=r-1$ and $-r$,

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[1] K. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Königsberg, 1829, §64; *Gesammelte Werke*, Berlin, 1881, vol. 1 pp. 232–234. Other elementary proofs of (1) (not depending on elliptic functions) have been given by A. Cauchy, C. R. Acad. Sci. Paris vol. 17 (1843) pp 523–531; *Oeuvres*, vol. 8, pp. 42–50; A. Enneper, *Elliptische Functionen*, Halle, 1876, pp. 74–77; J. Tannery and J. Molk, *Éléments de la theorie des fonctions elliptiques*, Paris, 1896, vol. 2 pp. 10–13; G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford, 1938, pp. 280–281.

we can write (2) in the form

$$P(\theta, t) = \sum_{r=1}^{\infty} t^{r(r-1)/2}(-1)^{r+1} \frac{\sin (2r-1)\theta}{\sin \theta}. \tag{4}$$

Differentiating (3) and (4) with respect to $\theta$ we find, on substituting for $P(\theta, t)$ from (4):

$$\begin{aligned} &\left\{ \sum_{k=1}^{\infty} (-1)^k t^{k(k-1)/2} \frac{\sin (2k-1)\theta}{\sin \theta} \right\} \left\{ \sum_{r=1}^{\infty} \frac{t^r}{1 - 2t^r \cos 2\theta + t^{2r}} \right\} \\ &= (1/4) \csc^2 \theta \sum_{k=1}^{\infty} (-1)^k t^{k(k-1)/2} \left\{ \frac{k \cos (2k-1)\theta}{\cos \theta} - \frac{\sin 2k\theta}{\sin 2\theta} \right\}. \end{aligned} \tag{5}$$

Let us consider the general Lucas function

$$U_n = \frac{a^n - b^n}{a - b}$$

where $a$, $b$ are the roots of $x^2 - Px + Q = 0$. Supposing that $Q \neq 0$ we define $q$ by

$$q = Q^{-1/2}$$

where a definite square root of $Q$ is taken. Finally we introduce the divisor sum $D_n$ by

$$D_n = \sum_{\delta | n} U_\delta q^\delta$$

where $\delta$ ranges over the positive divisors of $n$. Let the parameter $\theta$ in (5) be taken so that $2 \cos 2\theta = Pq$, and hence

$$\frac{\sin 2n\theta}{\sin 2\theta} = q^{n-1} U_n.$$

Then, since

$$U_n = PU_{n-1} - QU_{n-2},$$

we have

$$\begin{aligned} U_n q^n &= Pq U_{n-1} q^{n-1} - q^2 Q U_{n-2} q^{n-2} \\ &= 2 \cos 2\theta U_{n-1} q^{n-1} - U_{n-2} q^{n-2}. \end{aligned}$$

Therefore

$$(1 - 2x \cos 2\theta + x^2) \sum_{m=1}^{\infty} x^m q^m U_m = xq.$$

Setting $x = t^\nu$ and summing over $\nu = 1, 2, 3, \cdots$, we get

$$q\sum_{\nu=1}^{\infty} \frac{t^{\nu}}{1-2t^{\nu}\cos 2\theta + t^{2\nu}} = \sum_{m=1}^{\infty}\sum_{\nu=1}^{\infty} t^{\nu m} q^{m} U_{m} = \sum_{n=1}^{\infty} D_{n} t^{n}.$$

Substituting this into (5) and comparing coefficients of $t^n$ on both sides of the resulting equation we find

$$\tag{6} \begin{aligned} &D_n - D_{n-1}\frac{\sin 3\theta}{\sin\theta} + D_{n-3}\frac{\sin 5\theta}{\sin\theta} - D_{n-6}\frac{\sin 7\theta}{\sin\theta} + - \cdots \\ &\quad = \begin{cases} (q/4)\csc^2\theta(-1)^{k+1}\left\{\dfrac{k\cos(2k-1)\theta}{\cos\theta} - \dfrac{\sin 2k\theta}{\sin 2\theta}\right\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad n = k(k-1)/2 & \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ \text{otherwise.} & \end{cases} \end{aligned}$$

Here the left side involves only such $D$'s as have subscripts differing from $n$ by triangular numbers.

Of the special cases of (6) perhaps the most interesting are those in which $q=1$ and $\theta$ is a simple rational multiple of $\pi$. In these cases the function $D_n$ belongs to a general class which is composed of functions of which the typical one may be denoted by

$$E_m(n) = E_m(n \mid \alpha_1, \alpha_2, \cdots ; \beta_1, \beta_2, \cdots)$$

and is defined as the excess of the number of those divisors of $n$ of one of the forms $mx+\alpha_i$ over the number of those divisors of $n$ of one of the forms $mx+\beta_i$.

Thus for $q=1$, $\theta=\pi/4$ we have

$$U_n = \frac{\sin n\pi/2}{\sin \pi/2} = \begin{cases} +1 & \text{if} \quad n = 4x+1 \\ -1 & \text{if} \quad n = 4x+3 \\ \ \ 0 & \text{otherwise.} \end{cases}$$

Hence

$$D_n = \sum_{\delta\mid n} U_\delta = E_4(n) = E_4(n \mid 1;\, 3).$$

In this case (6) gives us

$$\begin{aligned} &E_4(n) - E_4(n-1) - E_4(n-3) + E_4(n-6) + E_4(n-10) - \cdots \\ &\qquad\qquad = \begin{cases} (-1)^{[(k-1)/2]}[k/2] & \text{if} \quad n = k(k-1)/2 \\ \qquad 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This formula is due to Glaisher.[2]

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[2] J. W. L. Glaisher, *On the function which denotes the difference between the number of* $4m+1$*-divisors and the number of* $4m+3$*-divisors of a number*, Proc. London Math. Soc. (1) vol. 15 (1884) pp. 104–122 (especially pp. 107, 110).

If $\theta=\pi/3$ we find that $U_n=0$, 1 or $-1$ according as $n$ is congruent to 0, 1, or 2 modulo 3. Hence

$$D_n = E_3(n) = E_3(n \mid 1; 2).$$

In this case (6) gives

$$E_3(n) - E_3(n-3) - E_3(n-6) + E_3(n-15) + E_3(n-21) - \cdots$$
$$= \begin{cases} (2k-1)/3 & \text{if } n = k(k-1)/2 \text{ and } k \equiv 2 \pmod 3 \\ -[(k+1)/3] & \text{if } n = k(k-1)/2 \text{ and } k \not\equiv 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Here the numbers 0, 3, 6, 15, 21, 36, 45, $\cdots$ are those triangular numbers which are multiples of 3. This formula, for the case $n\equiv 1 \pmod 3$, was given by Glaisher.[3] If $\theta=\pi/6$ then

$$D_n = E_6(n) = E_6(n \mid 1, 2; 4, 5).$$

The recurrence (6) gives, in this case,

$$E_6(n) - 2E_6(n-1) + E_6(n-3) + E_6(n-6) - 2E_6(n-10)$$
$$+ E_6(n-15) + E_6(n-21) - \cdots$$
$$= \begin{cases} -k & \text{if } n = k(k-1)/2 \text{ and } k \equiv 0 \pmod 3 \\ k-1 & \text{if } n = k(k-1)/2 \text{ and } k \equiv 1 \pmod 3 \\ 1 & \text{if } n = k(k-1)/2 \text{ and } k \equiv 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Here the coefficient of $E_6(n-\Delta)$ is $+1$ or $-2$ according as $\Delta$ is or is not a multiple of 3.

In case $\theta$ is taken as $\pi/8$, the function $D_n$ is no longer an integer, for all $n$, but has the value

$$D_n = E_8(n \mid 1, 3; 5, 7) + 2^{1/2}E_8(n \mid 2; 6).$$

The equation (6) now yields two formulas obtained by equating separately the rational and irrational parts of both sides. Combining these two results for simplicity we obtain the following, where

$$E_8(n) = E_8(n \mid 1, 3; 5, 7).$$

If $n$ is even

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[3] J. W. L. Glaisher, *On the function wick denotes the excess of the number of divisors of a number which* $\equiv 1$, mod 3, *over the number which* $\equiv 2$, mod 3, Proc. London Math. Soc. (1) vol. 21 (1890) pp. 395–402.

$$E_8(n) - E_8(n-1) + E_8(n-3) - E_8(n-6) - E_8(n-10) + \cdots$$
$$= \begin{cases} 4[k/4](-1)^{k+1+[k/4]} & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

If $n$ is odd

$$E_8(n) + E_8(n-1) - E_8(n-3) - E_8(n-6) - E_8(n-10) - \cdots$$
$$= \begin{cases} (-1)^{[k/4]}\{6[(k+1)/4] - [k/2] + (-1)^k 2\} & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

In the first of these the periodic sign pattern is

$$+ - + - - + - + + - + - - + - + \cdots$$

while in the second formula it is

$$+ + - - - - + + + + - - - - + + \cdots$$

As a by-product we have for $n$ odd:

$$E_8'(n-1) - E_8'(n-3) - E_8'(n-15) + E_8'(n-21)$$
$$+ E_8'(n-45) - E_8'(n-55) - \cdots$$
$$= \begin{cases} (-1)^{[k/4]}2[(k+1)/4] & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

Here we have

$$E_8'(n) = E_8(n \mid 1, 2, 3; 5, 6, 7).$$

The numbers 1, 3, 15, 21, 45, $\cdots$ are the odd triangular numbers and the sign pattern is

$$+ - - + + - - + + - - + + - - + \cdots .$$

The values 0 and $\pi/2$ of $\theta$ correspond to degenerate values of Lucas $U_n$ which becomes $n$ and $(-1)^{n+1}n$, respectively. If we take the limiting forms of (6) in these cases we get the following two formulas

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + 9\sigma(n-10) - 11\sigma(n-15) + \cdots$$
$$= \begin{cases} (-1)^k k(k-1)(2k-1)/6 & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise;} \end{cases}$$

$$\lambda(n) + \lambda(n-1) + \lambda(n-3) + \lambda(n-6) + \lambda(n-10) + \cdots$$
$$= \begin{cases} n & \text{if } n \text{ is triangular} \\ 0 & \text{otherwise.} \end{cases}$$

In the first formula $\sigma(n)$ denotes the sum of the divisors of $n$ and in the second $\lambda(n)$ denotes the excess of the sum of the odd divisors of $n$ over the sum of its even divisors. The first relation, apparently, was first published by Sardi[4] and later given by Halphen[5] and Glaisher.[6,7] The second is due to Glaisher[1] and Lipschitz.[8]

Two final examples of (6) in which $\theta$ is complex will be given. The first of these corresponds to $q=1$ and $\theta = \arcsin i$. Let $(x_m, y_m)$ be the $m$th multiple solution of the Pell equation

$$x^2 - 2y^2 = 1,$$

and $(t_m, u_m)$ be the $m$th solution of the modified Pell equation

$$t^2 - 2u^2 = -1,$$

so that

$$\begin{aligned} y_1 &= 2, & y_2 &= 12, & y_3 &= 70, \cdots, & y_{i+1} &= 6y_i - y_{i-1}, \\ t_1 &= 1, & t_2 &= 7, & t_3 &= 41, \cdots, & t_{i+1} &= 6t_i - t_{i-1}, \\ u_1 &= 1, & u_2 &= 5, & u_3 &= 29, \cdots, & u_{i+1} &= 6u_i - u_{i-1}. \end{aligned}$$

Finally let

$$S(n) = \sum_{d|n} y_d.$$

Then (6) becomes

$$t_1S(n) - t_2S(n-1) + t_3S(n-3) - t_4S(n-6) + t_5S(n-10) - \cdots$$
$$= \begin{cases} (-1)^k(2ku_k - y_k)/4 & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

The case of $\theta = \arcsin(i/2)$ has to do with the Fibonacci series:

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, \cdots, F_{i+1} = F_i + F_{i-1}.$$

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[4] C. Sardi, *Sulle somme dei divisori dei numeri*, Giornale di Mathematica vol. 7 (1869) pp. 112–115.

[5] G. Halphen, *Sur une formule récurrente concernant les sommes des diviseurs des nombres entiers*, Bull. Soc. Math. France vol. 5 (1877) pp. 158–160.

[6] J. W. L. Glaisher, *Theorems relating to the sum of the uneven divisors of a number*, Quart. J. Math. vol. 19 (1883) pp. 216–223 (especially p. 220).

[7] J. W. L. Glaisher, *On the sum of the divisors of a number*, Proc. Cambridge Philos. Soc. vol. 5 (1884), pp. 108–120.

[8] R. Lipschitz, *Sur les sommes des diviseurs des nombres*, C. R. Acad. Sci. Paris vol. 100 (1885) pp. 845–847. (The recurrence (7), which is misquoted in Dickson's *History of the theory of numbers* vol. 1, p. 302, was given without proof six years earlier by Glaisher, Nouvelle Correspondance Mathématique vol. 5 (1879) p. 176.)

In fact if we let

$$T(n) = \sum_{s|n} F_{2s}$$

then

$$T(n) - 4T(n-1) + 11T(n-3) - 29T(n-6) + \cdots$$
$$= \begin{cases} (-1)^k k F_{2k-1} - F_{2k} & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

Here the $m$th term of the sequence

$$1, 4, 11, 29, 76, 199, \cdots$$

is

$$F_{2m} + F_{2m-2}.$$

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# ON PARTICULAR SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

K. L. NIELSEN AND B. P. RAMSAY

1. **Introduction.** The boundary value and characteristic value problems are classical questions in the theory of partial differential equations of elliptic type. A method for actual solution of these problems consisting of approximations by expressions $W_n = \sum_{\nu=1}^{n} \alpha_\nu^{(n)} \phi_\nu(x, y)$, where $\phi_\nu(x, y)$ are *particular solutions of the considered differential equation*, has been given by Bergman (see [1]).[1] Here the $\alpha_\nu^{(n)}$ are constants which are to be determined by the requirement that the values of $W_n$ on the boundary approximate the given data (for details see [1]).[2]

In applying this method it is important for practical purposes to obtain a simple procedure for the construction of the particular solu-

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[1] The numbers in the brackets refer to the bibliography.

[2] This method is in a certain sense the reverse of the Rayleigh-Ritz method in which the approximating expressions satisfy the boundary conditions but do not satisfy the given equation.

tions. In this connection Bergman [2] has proved that to every equation

$$L(U) = U_{z\bar{z}} + aU_z + bU_{\bar{z}} + cU = 0, \tag{1}$$

where $a$, $b$, $c$ are functions of $z=x+iy$ and $\bar{z}=x-iy$ and $U_z=\partial U/\partial z$, $U_{\bar{z}}=\partial U/\partial \bar{z}$, $U_{z\bar{z}}=\partial^2 U/\partial z\partial \bar{z}$, there exist functions $E(z, \bar{z}, t)$ such that

$$P(f) = \int_{-1}^{+1} E(z, \bar{z}, t) f(z(1 - t^2)/2)dt/(1 - t^2)^{1/2}, \tag{2}$$

where $f$ is an arbitrary analytic function of one complex variable, will be a particular solution of $L(U)=0$.[3]

To expedite the numerical computation for practical problems we further desire that $E$ has a simple structure so that we may easily evaluate the integral (2) for arbitrary values of $z$, $\bar{z}$. In this note we shall determine certain types of equations, $L(U)=0$, for which $E$ has the simple form

$$E(z, \bar{z}, t) = \exp(Nt^n + Mt^m), \tag{3}$$

where $N=N(z, \bar{z})$ and $M=M(z, \bar{z})$. We shall further show how $M$ and $N$ may be determined from the coefficients $a$, $b$, $c$.

We note further that our results can also be successfully used for the study of the singularities of the functions $U$ satisfying (1). In our case it is possible to show that the functions $U$ possess certain singularities which may be characterized independently of the representation (2) by the property that these $U$ satisfy certain ordinary differential equations in addition to $L(U)=0$.

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[3] The equation (1) is equivalent to the system of two equations $\Delta U^{(1)}/4+AU^{(1)}_x/2+BU^{(1)}_y/2+CU^{(2)}_x/2+DU^{(2)}_y/2+c_1U^{(1)}-c_2U^{(2)}=0$, $\Delta U^{(2)}/4-CU^{(1)}_x/2-DU^{(1)}_y/2+AU^{(2)}_x/2+BU^{(2)}_y/2+c_2U^{(1)}+c_1U^{(2)}=0$ where $U=U^{(1)}+iU^{(2)}$, $c=c_1+ic_2$, $A=[(a+\bar{a})+(b+\bar{b})]/2$, $B=[(a-\bar{a})-(b-\bar{b})]/2i$, $C=-[(a-\bar{a})+(b-\bar{b})]/2i$, $D=[(a+\bar{a})-(b+\bar{b})]/2$. If $a=\bar{b}$ and $c$ is real then these equations are completely independent of each other and it is only necessary to take the real part of (2) (see [1, p. 540]).

If $a=b=c=0$; that is, if $L(U)=\Delta U=0$, then we may take $E=1$. That is to say that every analytic function of one complex variable is a complex harmonic function. Thus analogous to the case of complex harmonic functions the totality of solutions of the equation $L(U)=0$ is the sum of two subclasses obtained by the operator $P$; one from analytic functions and the other from anti-analytic functions of one complex variable. Using these results and the theorem of Runge, Bergman has further shown that the representation (2) yields a method for constructing a denumerable set, $S$, of particular solutions (which is independent of the domain) which is complete for the totality of solutions in every star-domain with the center at the origin (see [1]).

2. **The equation $L(U)=0$ for which (2) with $E$ of the form (3) is a class of particular solutions.** As is easily seen the equation (1) can always be transformed into the equation

$$L'(V) = V_{z\bar{z}} + BV_z + CV = 0 \tag{4}$$

(see also [2, p. 1172]). Thus it suffices to consider this equation. To simplify the notation in the following theorem we shall use $p^{(\nu)}$ to denote an arbitrary function of $z$ *only* and $q^{(\nu)}$ to denote an arbitrary function of $\bar{z}$ *only*; we shall thus omit the arguments. $k_\nu$ shall denote constants and $p_z^{(\nu)} = dp^{(\nu)}/dz$, $q_{\bar{z}}^{(\nu)} = dq^{(\nu)}/d\bar{z}$. $f$ shall denote an arbitrary analytic function of one complex variable.

THEOREM I. *In the equation $L'(V)=0$ let the coefficients $B$ and $C$ have the following forms*

$$\begin{array}{llll}
\mathrm{I}_1: & B = 0 & \textit{and} & C = -q^{(1)}q_{\bar{z}}^{(1)}/2,\\
\mathrm{I}_2: & B = -p_z^{(1)} & \textit{and} & C = -q^{(2)}q_{\bar{z}}^{(2)}/2,\\
\mathrm{II}: & B = 0 & \textit{and} & C = -3k_2\bar{z}q_{\bar{z}}^{(3)}/4 - q^{(3)}q_{\bar{z}}^{(3)}/2,\\
\mathrm{III}: & B = -k_1 & \textit{and} & C = -q_{\bar{z}}^{(4)}q^{(4)}/2,\\
\mathrm{IV}_1: & B = -q^{(5)} & \textit{and} & C = (dB/d\bar{z})/2,\\
\mathrm{IV}_2: & B = -q^{(6)} - p_z^{(2)} & \textit{and} & C = -q_{\bar{z}}^{(6)}/2,\\
\mathrm{V}: & B = 2k_2\bar{z} + q^{(7)} & \textit{and} & C = q_{\bar{z}}^{(7)}/2.
\end{array} \tag{5}$$

*Then every function*

$$V = \int_{-1}^{+1} \exp\,(Nt^n + Mt^m) f(z(1-t^2)/2)\,dt/(1-t^2)^{1/2} \tag{6}$$

*is a particular solution of* (4) *and moreover, in the respective cases*

(7)

| | $N$ | $M$ | $n$ | $m$ |
|---|---|---|---|---|
| $\mathrm{I}_1$ | $z^{1/2}q^{(1)}$ | $0$ | 1 | 0 |
| $\mathrm{I}_2$ | $z^{1/2}q^{(2)}$ | $p^{(1)}$ | 1 | 0 |
| $\mathrm{II}$ | $k_2z^{3/2}$ | $-3k_2z^{3/2}/2 - z^{1/2}q^{(3)}$ | 3 | 1 |
| $\mathrm{III}$ | $k_1\bar{z}$ | $z^{1/2}q^{(4)}$ | 2 | 1 |
| $\mathrm{IV}_1$ | $zq^{(5)}$ | $0$ | 2 | 0 |
| $\mathrm{IV}_2$ | $\bar{z}q^{(6)}$ | $p^{(2)}$ | 2 | 0 |
| $\mathrm{V}$ | $k_2z^2$ | $-2k_2\bar{z}^2 - \bar{z}q^{(7)}$ | 4 | 2 |

*On the other hand if* (6) *with $n$, $m$, and $M$ given by* (7) *are solutions of $L'(V)=0$ for an arbitrary $f$ then the coefficients $B$ and $C$ can be represented by the forms* (5).[4]

PROOF. The proof of this theorem is obtained by considering special cases of the general form of $E$, (3). A given form of $E$ will determine a solution, $V$, of the partial differential equation, $L'(V)=0$, provided the coefficients $B$ and $C$, satisfy the equation

$$G(E) = (1 - t^2)(E_{tz}) - E_z/t + 2zt(E_{zz^*} + BE_{z^*} + CE) = 0 \tag{8}$$

(see [2, p. 1171]).

With the given form (3) we then desire to determine $N$, $M$, $B$ and $C$ in such a way that this equation is satisfied. The required derivatives are

$$\tag{9} \left\{ \begin{aligned} E_z &= e^{Nt^n+Mt^m}[t^nN_z + t^mM_z],\\ E_t &= e^{Nt^n+Mt^m}[t^{m-1}mM + t^{n-1}nN],\\ E_{z^*} &= e^{Nt^n+Mt^m}[t^mM_{z^*} + t^nN_{z^*}],\\ E_{tz} &= e^{Nt^n+Mt^m}[t^{m-1}mM_z + t^{2m-1}mMM_z + t^{n-1}nN_z\\ &\qquad + t^{n+m-1}(mMN_z + nNM_z) + t^{2n-1}nNN_z],\\ E_{zz^*} &= e^{Nt^n+Mt^m}[t^mM_{zz^*} + t^{2m}M_zM_{z^*} + t^nN_{zz^*}\\ &\qquad + t^{n+m}(M_zN_{z^*} + M_{z^*}N_z) + t^{2n}N_zN_{z^*}]. \end{aligned} \right.$$

These values are to be substituted into (8). When the substitution is made and the equation is divided by the common factor $\exp(Nt^n + Mt^m)$ we have an equation which is zero for an arbitrary $t$; thus the coefficients of each power of $t$ must vanish. This gives us the following system of equations

$$\tag{10} \left\{ \begin{array}{ll} t^{m-1}: & (m-1)M_z = 0\\ t^{2m-1}: & mMM_z = 0,\\ t^{n-1}: & (n-1)N_z = 0,\\ t^{n+m-1}: & mMN_z + nNM_z = 0,\\ t^{2n-1}: & nNN_z = 0,\\ t^{m+1}: & (2Bz - m)M_z + 2zM_{zz^*} = 0,\\ t^{2m+1}: & 2zM_zM_{z^*} - mMM_z = 0,\\ t^{n+1}: & (2Bz - n)N_z + 2zN_{zz^*} = 0,\\ t^{n+m+1}: & -mMN_z - nNM_z + 2zM_zN_{z^*} + 2zM_{z^*}N_z = 0,\\ t^{2n+1}: & -nNN_z + 2zN_zN_{z^*} = 0,\\ t: & 2zC = 0. \end{array} \right.$$

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[4] This theorem includes the special cases discussed by Bergman in [2].

For various values of $m$ and $n$ this system will take different forms. For many values trivial[8] solutions arise but for others non-trivial solutions exist. We thus obtain the various cases of the theorem.

We shall develop the case for $m=2$ and $n=4$. The system (10) then is

$$(11)\quad \begin{cases} t^9: & N_z(2zN_z - 4N) = 0, \\ t^7: & M_z(2zN_z - 4N) + 4NN_z - 2MN_z + 2zN_zM_z = 0, \\ t^5: & 2MN_z + 4NM_z - 2MM_z - 4N_z + 2zM_zM_z \\ & \qquad\qquad + 2zN_{zz} + 2BzN_z = 0, \\ t^3: & 2MM_z + 3N_z - 2M_z + 2zM_{zz} + 2BzM_z = 0, \\ t: & M_z + 2Cz = 0. \end{cases}$$

From the first equation of (11) it appears that we must consider two cases: $N_z=0$ and $N_z \neq 0$. Let us consider the first case; then this equation is satisfied. The second equation becomes

$$(12)\qquad M_z(2zN_z - 4N) = 0.$$

We need not consider $M_z=0$ as with $N_z=0$ this would reduce to a trivial case; we thus take $M_z \neq 0$ and from (12) we get

$$2zN_z - 4N = 0.$$

Solving this equation we obtain

$$(13)\qquad N = kz^2.$$

The third equation of (11) then reads

$$4NM_z - 2MM_z + 2zM_zM_z = 0$$

and since $M_z \neq 0$ and by (13) it becomes

$$2kz^2 - M + zM_z = 0.$$

The solution of this equation is

$$(14)\qquad M = -(2kz^2 + zq).$$

We now consider the fourth equation of (11) which reads

$$(15)\qquad 2MM_z - 2M_z + 2zM_{zz} + 2BzM_z = 0.$$

From (14) we get $M_z = -zq_z$ and $M_{zz} = -q_z$; substituting these into the equation (15) we can solve for $B$, getting

$$(16)\qquad B = 2kz + q.$$

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[8] By a trivial solution we mean a solution which gives $B=C=M=N=0$.

The last equation of (11) reads

$$M_s + 2C_s = 0$$

so that we get

$$C = q_s/2. \tag{17}$$

We thus have case V of the theorem. The other cases can be proved in a similar manner.[6] It is easily seen that all the cases satisfy equation (8).

It remains to prove that if (2) is a solution of (4); that is, $L(P(f)) = 0$, for every $f$ then $G(E) = 0$. Let $f = (z(1-t^2)/2)^\mu$, then by taking the derivatives of $V = P(f)$ and substituting into (4) we see that it reduces to prove that from

$$\int_{-1}^{+1} \frac{(1-t^2)^\mu G(E)}{(1-t^2)^{1/2}}\, dt = 0, \qquad \mu = 0, 1, 2, \cdots, \tag{18}$$

it follows that $G(E) = 0$. This follows directly from the fact that in the integral $G(E)$ is an even function of $t$ and the completeness of the system $t^\mu$.

3. **A discussion of the form of** $E$, (3). The question of whether or not Theorem I gives all the partial differential equations (4) which have solutions (2) with $E$ in the form (3) has not been completely answered. However, the number of non-trivial solutions for $B$ and $C$ is definitely limited and it is possible to give certain relations between $m$ and $n$ in this case by studying the relations between the exponents of $t$ in $E$. For certain values of $m$ and $n$, $C = 0$ and for others only trivial solutions can result.

We proceed to analyze the system (10) by comparing the exponents of $t$. The last equation of this system shows that in general $C$ is zero unless $m$ and $n$ have values which will make at least one of the other exponents equal to 1. In order to avoid duplication of possible cases we may, without any loss of generality, make the following assumptions on the values of $m$ and $n$:

(19) we shall take $n < m$ if $n$ and $m$ have different signs and $|n| > |m|$ in all other cases.

We thus formulate the conditions for $m$ and $n$ for which $C = 0$ and in Theorem III, conditions for which $B$ and $C$ do not vanish simultaneously.

THEOREM II. (a) *When* $n > 0$ *and* $m > 0$, $C = 0$ *unless* $m = 1$ *or* $m = 2$.

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[6] A typewritten copy of the enlarged proof with a discussion of all the cases is available at the Brown University Library.

(b) *When* $n<0$ *and* $m<0$, $C=0$. (c) *When* $n<0$ *and* $m>0$, $C=0$ *unless* $m=1$, $m=2$, *or* $n=-m$.

The proof of this theorem follows directly by comparing the exponents of $t$ in (10).

THEOREM III. *The non-trivial solutions for $B$ and $C$ are possible*
(i) *for all* $m$ *only if* $n=2$, $n=m+2$, $n=m-2$, $n=-m$, $n=2-m$, $n=m+1$, $n=1$, $n=m/2$, $n=(m+2)/2$;
(ii) *for all* $n$ *only if* $m=0$ *or* $m=2$.

PROOF. Consider the coefficient of $t^{2n-1}$. If no other exponent is equal to $2n-1$ then we have $nNN_1=0$. By (19) we have $n\neq 0$ and thus for this equation to be satisfied $N_1=0$. Consider the coefficient of $t^{n+m-1}$. If no other exponent is equal to $n+m-1$ then using the fact that $N_1=0$ this coefficient is $nNM_1=0$ and since $n\neq 0$ we must have $M_1=0$ for this equation to be satisfied. But with $N_1=0$ and $M_1=0$ we see by substitution into (10) that $B=C=0$. Hence the non-trivial solutions for $B$ and $C$ for general values of $m$ and $n$ are possible only if either of the exponents $n+m-1$ or $2n-1$ is equal to some other exponent. Checking all these possibilities and using (19) we obtain the conditions of the theorem.

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# RESIDUE THEOREMS OF HARMONIC FUNCTIONS OF THREE VARIABLES

STEFAN BERGMAN

1. **Introduction.** The connection between analytic functions of one complex variable and harmonic functions of two real variables is used in the study of the latter functions. The analogy suggests generalizations of these methods of attack to the case of three variables.

Suppose that $O$ is some operation which transforms the class of analytic functions of the complex variable $u$ into the class of harmonic functions of three variables $x_1, x_2, x_3$. If $O$ possesses the property that to every harmonic polynomial of the $n$th degree there corresponds an expression of the form $A\ [u(x_1, x_2, x_3)]^n$, where $u$ is a linear expression in $x_1, x_2, x_3$, then such an operation yields a representation of spherical harmonics.

The existing representations of the spherical harmonics lead us to several operations $O$. The operation P described below (a generalization of the known Whittaker's formula) seems to be the most suitable for the questions considered in this note.

Let $F$ be the class of analytic functions of complex variables $u$ and $\zeta$, where $u = x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta$. By the operation

$$(1.1)\quad \mathrm{P}(f, \mathfrak{L}^1, \mathfrak{T}) = (1/2\pi i)\int_{\mathfrak{L}^1} f(u, \zeta)d\zeta, \qquad \mathfrak{X} = (x_1, x_2, x_3) \in \mathcal{U}^3(\mathfrak{T})$$

$F$ is transformed into the class of (complex) harmonic[1] functions (see [1, 2][2]). $\mathcal{U}^3(\mathfrak{T})$ denotes a sufficiently small neighborhood of the point $\mathfrak{T} = (t_1, t_2, t_3)$, and $\mathfrak{L}^1$ is a closed curve in the $\zeta$-plane.

$\mathrm{P}(f, \mathfrak{L}^1, \mathfrak{T})$ is a functional. If we carry out the integration for $\mathfrak{X} \in \mathcal{U}^3(\mathfrak{T})$ and we continue analytically the obtained harmonic function $H(\mathfrak{X})$ to a point $\mathfrak{Y} \in \mathcal{U}^3(\mathfrak{B})$, then the continued function is not necessarily equal to $\mathrm{P}(f, \mathfrak{L}^1, \mathfrak{B})$. This is the reason why it is necessary to indicate the point $\mathfrak{T}$ in the neighborhood of which the integration is carried out.

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[1] Here and hereafter harmonic functions mean harmonic functions of three *real* variables $x_1, x_2, x_3$, except when we specifically state that we mean harmonic functions of two variables. By taking the real or imaginary part in the formulas for complex harmonic functions we obtain analogous results for real harmonic functions.

[2] The numbers in brackets refer to the bibliography.

E $[\cdots]$ means the set of points satisfying the relations indicated in the brackets. Superscripts over a manifold indicate its dimensionality.

The analogy suggests further consideration of harmonic vectors $\mathfrak{H}(\mathfrak{X})=H^{(1)}(\mathfrak{X})\mathfrak{i}_1+H^{(2)}(\mathfrak{X})\mathfrak{i}_2+H^{(3)}(\mathfrak{X})\mathfrak{i}_3$, in addition to a single function, where $H^{(k)}$ are harmonic functions connected by certain differential relations.[3] $\mathfrak{i}_k$ is the unit vector in the $x_k$ direction.

In §§2 and 3 we shall consider harmonic vectors

$$\begin{aligned}\mathfrak{H}(\mathfrak{X}) = \mathfrak{P}(f, \mathfrak{Q}^1, \mathfrak{T}) \equiv \mathrm{P}[f, \mathfrak{Q}^1, \mathfrak{T}]\mathfrak{i}_1 + \mathrm{P}[(i/2)(\zeta + \zeta^{-1})f, \mathfrak{Q}^1, \mathfrak{T}]\mathfrak{i}_2 \\ + \mathrm{P}[(1/2)(\zeta - \zeta^{-1})f, \mathfrak{Q}^1, \mathfrak{T}]\mathfrak{i}_3.\end{aligned} \tag{1.2}$$

Clearly, curl $\mathfrak{H}=0$, div $\mathfrak{H}=0$ at every point at which all components of $\mathfrak{H}$ are regular.

Consider vectors $\mathfrak{H}(\mathfrak{Y})=H^{(1)}(\mathfrak{Y})\mathfrak{i}_1+H^{(2)}(\mathfrak{Y})\mathfrak{i}_2$, $\mathfrak{Y}=(y_1, y_2)$ of two real variables $y_1$, $y_2$ for which we have curl $\mathfrak{H}=0$, div $\mathfrak{H}=0$. $\int_{\mathfrak{J}^1}\mathfrak{H}\cdot d\mathfrak{Y}=0$ if $\mathfrak{J}^1$ is a closed curve, and if $\mathfrak{J}^1$ can be shrunk to a point in a regularity domain of $\mathfrak{H}$. ($\cdot$ means the scalar product.) If singularities of $\mathfrak{H}$ lie inside of $\mathfrak{J}^1$ or $\mathfrak{H}$ is a many valued function the residue theorem yields well known relations.

Let $\mathfrak{R}^2$ be some simply connected domain in which $\mathfrak{J}^1$ lies, and let $\mathfrak{H}=\mathfrak{H}_1+\sum_{k=1}^{n}\mathfrak{S}_k$, where $\mathfrak{H}_1$ is a vector which is regular in $\mathfrak{R}^2$. Then

$$\int_{\mathfrak{J}^1}\mathfrak{H}\cdot d\mathfrak{Y} = \int_{\mathfrak{J}^1}\mathfrak{H}_1\cdot d\mathfrak{Y} + \sum_{k=1}^{n}\int_{\mathfrak{J}^1}\mathfrak{S}_k\cdot d\mathfrak{Y}.$$

In this connection we make the trivial simplification that we limit ourselves to the case of suitable standard vectors $\mathfrak{S}$ which possess one singular point (or one connected singular line in the case of three variables).

If $\mathfrak{S}(\mathfrak{Y})=\mathrm{Im}\ [(z-\alpha)^{-1}]\mathfrak{i}_1+\mathrm{Re}\ [(z-\alpha)^{-1}]\mathfrak{i}_2$ then $\int_{\mathfrak{J}^1}\mathfrak{S}(\mathfrak{Y})\cdot d\mathfrak{Y}=2\pi$ if $\alpha$ lies inside of $\mathfrak{J}^1$. Since $\mathfrak{S}$ can be considered as a harmonic vector in three variables the above relation may be formulated in the following form. Let $\mathfrak{S}(\mathfrak{X})$ be a harmonic vector which in appropriate coordinates $y_1$, $y_2$, $y_3$ may be written

$$\mathfrak{S} = \mathrm{Im}\,(y_1 + iy_2 - \alpha)^{-1}\mathfrak{i}_1 + \mathrm{Re}\,(y_1 + iy_2 - \alpha)^{-1}\mathfrak{i}_2 + 0\cdot\mathfrak{i}_3.$$

Then $\int_{\mathfrak{J}^1}\mathfrak{S}(\mathfrak{X})\ d\mathfrak{X}=2\pi$ if $\mathfrak{J}^1$ cannot be reduced to a point without cutting E $[y_3=0,\ y_1+iy_2=\alpha]$. As it is known (see [1, 2]) there exist algebraic harmonic functions with singularity lines or points [see

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[3] Harmonic vectors are met in physical applications especially in the theory of a perfect incompressible fluid.

The above mentioned differential relations can vary in nature depending upon the question which has to be considered. We limit ourselves in this paper to special relations, namely, curl $\mathfrak{H}=0$, div $\mathfrak{H}=0$ in §§2 and 3 and (4.3) in §4. However, in many instances with certain changes we may extend our results to much more general relations.

(2.5) p. 166] which cannot be represented as harmonic functions of two variables. In §§2 and 3 we introduce harmonic vectors $\mathfrak{S}(\mathfrak{X})$ with such singularities and consider

$$\sum_k \int_{\mathfrak{X}_k^{(1)}}^{\mathfrak{X}_k^{(2)}} \mathfrak{S}_k(\mathfrak{X}) \cdot d\mathfrak{X}.$$

Here $\mathfrak{S}_k$ are branches of $\mathfrak{S}$ and $\mathfrak{X}_k^{(\mathrm{H})}$ points which satisfy certain algebraic relations. We show that the above expression is equal to a quantity connected with the singularity, which may be interpreted as the residue of $\mathfrak{S}$.[4]

A theorem of related character concerning surface integrals is given in §4.

As pointed out before, the operation P transforms a polynomial in $u$ and $\zeta$ into a harmonic polynomial of the same degree.

As we shall see below a rational function $f$ is in general transformed into an algebraic harmonic function of $x_1, x_2, x_3$. The question arises whether every algebraic harmonic function $A(x_1, x_2, x_3)$ can be represented in the form $\mathrm{P}(f, \mathfrak{Q}^1, \mathfrak{T})$ where $f$ is a rational function in $u$ and $\zeta$. This question has not been answered. However, from the result of [2, p. 637] follows that $A(x_1, x_2, x_3)$ can be represented in the above form with $f = \int_0^1 a(u, \zeta, T)dT$, $a$ being an algebraic function in $u, \zeta, T$.

2. **Singularities of harmonic functions.** The introduction of the operator (1.1) gives a simple procedure for the construction of standard functions with singularities. Suppose that $f(u, \zeta)$ has poles, say for instance $f = p/q$, where $p$ and $q$ are polynomials in $u$ and $\zeta$. For simplicity's sake we shall suppose that $q = [u - \phi(\zeta)]\zeta^s$ where $\phi(\zeta)$ is a polynomial in $\zeta$ and $s$ a non-negative integer.

In order to study the functional

$$(2.1)\quad \mathrm{P}[p/q, \mathfrak{Q}^1, \mathfrak{T}] = \frac{1}{2\pi i}\int_{\mathfrak{Q}^1} \frac{p(u, \zeta)}{q(u, \zeta)}\, d\zeta, \quad \mathfrak{X} \in \mathcal{V}^3(\mathfrak{T}),\ \mathfrak{X} = (x_1, x_2, x_3),$$

as a function of $\mathfrak{T}$ it-is useful to consider (2.1) in the (five-dimensional) space $x_1, x_2, x_3, \xi, \eta$; where $\zeta = \xi + i\eta$.

$$(2.2)\qquad Z^3 = \mathrm{E}\{q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta] = 0\}$$

is the singularity manifold of $p/q$. $Z^3$ can also be written in the form

$$(2.3)\qquad Z^3 = \mathrm{E}[\zeta = \zeta_\nu(\mathfrak{X}), \nu = 1, 2, \cdots, n]$$

where $\zeta_\nu$ are algebraic functions of $x_1, x_2, x_3$ and $n$ the degree of $q$ in $\zeta$.

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[4] This result has a certain similarity with Abel's theorem in the theory of integrals of algebraic functions.

At every point $\mathfrak{X}$, except those which lie in

$$\mathcal{V}^1 = \mathrm{E}\left\{ \prod_{k\neq\nu} [\zeta_\nu(\mathfrak{X}) - \zeta_k(\mathfrak{X})]^2 = 0 \right\}$$

we have $n$ distinct branches $Z_\nu^3 = \mathrm{E}\ [\zeta = \zeta_\nu(\mathfrak{X})], \nu = 1, 2, \cdots, n$ of $Z^3$.

Suppose $\mathfrak{D} \notin \mathcal{V}^1$, and $\mathcal{C}_\nu^1$, $\nu = 1, 2, \cdots, n$, are chosen in such a way that one and only one $\zeta_\nu(\mathfrak{D})$ lies inside of each $\mathcal{C}_\nu^1$. Then by the residue theorem $\mathrm{P}[p/q, \mathcal{C}_\nu^1, \mathfrak{D}] = S_\nu(\mathfrak{X})$, where $S_\nu$ is the corresponding branch of

$$S(\mathfrak{X}) = \frac{p[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta]}{\partial\{q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta]\}/\partial\zeta} \tag{2.4}$$

$$q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta] = 0.$$

$S(\mathfrak{X})$ is an algebraic function which possesses $n$ branches. We denote the (multiply covered) $x_1, x_2, x_3$-space in which $S(\mathfrak{X})$ is univalent by $\mathfrak{R}^3$; its sheets (which are supposed to be obtained by appropriate cuts), by[5] $\mathfrak{R}_\nu^3$. The singularity line of $S(\mathfrak{X})$ is

$$\begin{gathered} q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta] = 0, \\ \frac{\partial q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta]}{\partial\zeta} = 0. \end{gathered} \tag{2.5}$$

EXAMPLE. $\mathrm{P}[(u - ai)^{-1}\zeta^{-1}, \mathfrak{Q}^1, \mathfrak{T}]$, $\mathfrak{Q}^1 = \mathrm{E}[\zeta = e^{it}, 0 \leq t \leq 2\pi]$, $a \geqq 0$, $t_1 \geqq 0$ is a two valued function. (See [1, p. 655].) If $a > 0$ then the singularity line $\mathrm{E}[x_1 = 0, x_2^2 + x_3^2 = a^2]$ is the branch line. For $a = 0$ this branch line shrinks to a point, both branches $S_k(\mathfrak{X}) = (-1)^{k+1}/r$ for $k = 1, 2$, $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, are connected only at the origin.

We further note that $\mathrm{P}[(1/2)(\zeta + \zeta^{-1})u^{-1}\zeta^{-1}, \mathfrak{Q}^1, \mathfrak{T}] = -ir^{-1}\cos\phi \cdot \tan(\theta/2)$, $\mathrm{P}[(1/2i)(\zeta - \zeta^{-1})u^{-1}\zeta^{-1}, \mathfrak{Q}^1, \mathfrak{T}] = -ir^{-1}\sin\phi\tan(\theta/2)$ for $t_1 > 0$. Here $r, \theta, \phi$ are polar coordinates.

In study of the representation (2.1) the points $\mathfrak{X}$ which satisfy the equation $q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta] = 0$ (for an appropriate $\zeta$) are of importance. In this connection it is useful to introduce certain curves, $\mathcal{L}^1(\zeta)$.

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[5] We suppose here that $\zeta = 0$ lies outside of $\mathcal{C}_\nu^1$. Otherwise, and also for the evaluation of (2.1) if the degree of $q$ (in $u$) is higher than 1, see [2, pp. 644–650], especially the formulas (7) and (10).

We note that the $\mathfrak{R}_\nu^3$ are not necessarily connected with each other in the *real* $x_1, x_2, x_3$-space.

By the correspondence

$$\mathfrak{X} \to u = x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta \tag{2.6}$$

we associate to every point $\mathfrak{X}$ of the $x_1, x_2, x_3$-space a point $u$ of the complex plane. On the other hand to the point $A+iB$ of the $u$-plane there corresponds a line $\mathfrak{P}^1(\zeta, A+iB)$ in the $x_1, x_2, x_3$-space. If we write $x_2+ix_3=\rho e^{i\varphi}$, $\zeta=\tau e^{i\alpha}$ then from $u=A+iB$ it follows that $ie^{i(\varphi-\alpha)}\tau^{-1}+2x_1\rho^{-1}+ie^{-i(\varphi-\alpha)}\tau=2\rho^{-1}(A+iB)$ or

$$\begin{aligned}\mathfrak{P}^1(\tau e^{i\alpha}, A + iB) = \mathrm{E}\Bigg[\rho^2 = 4\left(\frac{B}{\tau + \tau^{-1}}\right)^2 + 4\left(\frac{A - x_1}{\tau - \tau^{-1}}\right)^2,& \\ \varphi = \alpha + \operatorname{arc\,cos}\frac{2B\rho^{-1}}{\tau + \tau^{-1}}\Bigg].&\end{aligned} \tag{2.7}$$

We put

$$\mathfrak{L}^1(\zeta) = \mathfrak{P}^1[\zeta, \phi(\zeta)] \tag{2.8}$$

where $\phi(\zeta)$ is the polynomial introduced on p. 165. For $\mathfrak{X}\in\mathfrak{L}^1(\zeta_0)$ we have

$$\begin{aligned}q(u, \zeta_0) &= q[(x_1 + (i/2)(x_2 + ix_3)\zeta_0^{-1} + (i/2)(x_2 - ix_3)\zeta_0), \zeta_0] \\ &= \zeta_0^{s}[u_0 - \phi(\zeta_0)] = 0;\end{aligned}$$

therefore

$$\mathfrak{L}^1(\zeta_0) = \mathfrak{Z}^3 \cap \mathrm{E}(\zeta = \zeta_0). \tag{2.9}$$

Since in[6] $\mathfrak{R}_1^3$, $\mathfrak{Z}^3=\mathfrak{S}_{\nu=1}^{n}\mathfrak{Z}_\nu^3$, we can write $\mathfrak{L}^1(\zeta)=\mathfrak{S}_{\nu=1}^{n}\mathfrak{L}_\nu^1(\zeta)$ where $\mathfrak{L}_\nu^1(\zeta_0)=\mathfrak{Z}_\nu^3\cap\mathrm{E}[\zeta=\zeta_0]$.

Let $\mathfrak{Q}^1$ be a closed curve in the $\zeta$-plane. We suppose (for simplicity's sake) that one and only one point $\zeta_k(\mathfrak{D})$, say $\zeta_1(\mathfrak{D})$, lies inside of $\mathfrak{Q}^1$. Then $\mathrm{P}[p/q, \mathfrak{Q}^1, \mathfrak{T}]=S_1(\mathfrak{X})$ for $\mathfrak{T}=\mathfrak{D}$. However, if $\mathfrak{T}$ varies in $\mathfrak{R}_1^3$ new branches $\mathfrak{Z}_\nu^3$ can slip in or out from $\mathfrak{Q}^1$. Since $\mathfrak{Z}^3$ is a continuous manifold, $\mathfrak{Z}_\nu^3$ can enter $\mathfrak{Q}^1$ only for such values of $\mathfrak{X}$ for which $\mathfrak{Z}_\nu^3$ has intersection with $\mathrm{E}[\zeta\in\mathfrak{Q}^1]$, that is for $\mathfrak{X}\in\mathfrak{Z}_\nu^3 \cap \mathrm{E}[\zeta\in\mathfrak{Q}^1]=\mathfrak{S}_{\zeta\in\mathfrak{Q}^1}\mathfrak{L}_\nu^1(\zeta)$ $=\mathfrak{M}_\nu^2(\mathfrak{Q}^1)$. Thus, if the point $\mathfrak{T}$ in the operator (2.1) moves and passes through $\mathfrak{M}^2(\mathfrak{Q}^1)$, the integral $\mathrm{P}[p/q, \mathfrak{Q}^1, \mathfrak{T}]$ has a jump,[7] it increases or decreases by $S_\nu(\mathfrak{X})$.

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[6] $\mathfrak{R}_1^3$ means the sheet of $\mathfrak{R}^3$ in which $S_1(\mathfrak{X})$ is defined. (See p. 166). $\mathfrak{S}$ denotes the topological sum. $\cap$ means the intersection.

We suppose again that the point $\zeta=0$ lies neither inside $\mathfrak{l}_\nu^1$ nor inside of $\mathfrak{Q}^1$. Otherwise we must take into account the residue at $\zeta=0$ since the integrand of (2.1) can have a pole at this point.

[7] The surfaces $\mathfrak{M}^2(\mathfrak{Q}^1)$ are denoted as surfaces of separation of the representation (2.1).

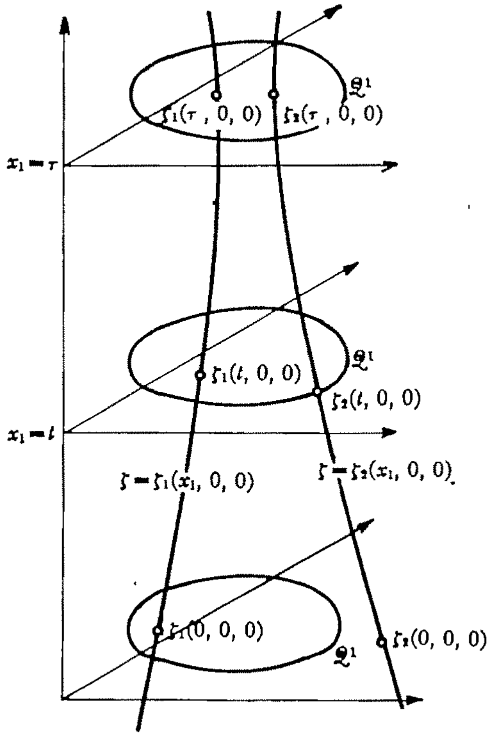

FIG. 1. The intersection of $Z^3$ with $E[x_2=0, x_3=0]$

The curves $\mathfrak{L}^1(\zeta)$ play also a role in another connection. Consider a closed curve $\mathfrak{J}^1$ in the schlicht $x_1, x_2, x_3$-space. To $\mathfrak{J}^1$ there corresponds by (2.6) a closed curve $\mathcal{F}^1(\zeta)$ in the $u$-plane. In the following we consider the case where the point $u=\phi(\zeta)$ lies inside of $\mathcal{F}^1(\zeta)$. If we construct the surface $\mathfrak{K}^2=\mathfrak{S}_{Z\in\mathfrak{J}^1(\zeta)}\mathfrak{P}^1(\zeta, Z)$, then $\phi(\zeta)$ lies inside of $\mathcal{F}^1(\zeta)$ if and only if $\mathfrak{K}^2$ cannot be reduced to a line without cutting $\mathfrak{L}^1(\zeta)=\mathfrak{P}^1(\zeta, \phi(\zeta))$. In general, this will happen if and only if the curve $\mathfrak{J}^1$ cannot be reduced to a point without cutting $\mathfrak{L}^1(\zeta)$.

The statement that $\mathfrak{J}^1$ cannot be reduced to a point without cutting $\mathfrak{L}^1(\zeta)$ we shall understand in the above sense.

The essential idea of the method of attack which will be used in the next section is the following:

Let $\mathfrak{J}^1$ be a closed curve in the $x_1, x_2, x_3$-space. If we divide it in an appropriate way, $\mathfrak{J}^1=\mathfrak{S}_\nu\mathfrak{J}^1_\nu$, then, by the operation P, the differential $f(u, \zeta)du$ will be transformed into $\mathfrak{S}_\nu(\mathfrak{X})\cdot d\mathfrak{X}$, for $\mathfrak{X}\in\mathfrak{J}^1_\nu$, that is, we have $[\int_{\mathfrak{L}^1} f(u, \zeta)d\zeta]du=\mathfrak{S}_\nu(\mathfrak{X})\cdot d\mathfrak{X}$. $\mathfrak{J}^1_\nu$ means that part of $\mathfrak{J}^1$ within whose points $f(u, \zeta)$ will be transformed into the branch $\mathfrak{S}_\nu(\mathfrak{X})$.

Here $\mathfrak{S}_\nu(\mathfrak{X})$ is an appropriate branch of the algebraic harmonic vector $\mathfrak{S}(\mathfrak{X})$. (See (3.1).) Thus

$$(2.10)\qquad (1/2\pi i)\int_{\mathfrak{J}^1}\left\{\int_{\mathfrak{L}^1}[f(u,\zeta)d\zeta]\right\}du = \sum_{\nu}\int_{\mathfrak{J}^1_\nu}\mathfrak{S}_\nu(\mathfrak{X})\cdot d\mathfrak{X}.$$

Changing the order of the integration in the left side member and applying the residue theorem for functions of one complex variable, we evaluate (2.10).

**3. Residue formula for line integrals.** Let $\mathfrak{J}^1$ be a closed differentiable curve in the schlicht $x_1, x_2, x_3$-space. We place it in $\mathfrak{R}^3_1$ and suppose that $\mathfrak{J}^1$ intersects the cuts which bound $\mathfrak{R}^3_1$ at one (and only one) point $\mathfrak{E}$. Thus $\mathfrak{J}^1$ is an open connected curve in $\mathfrak{R}^3_1$. Its end points (which coincide in the schlicht space with $\mathfrak{E}$) are denoted by $\mathfrak{E}_1$ and $\mathfrak{E}_2$.

Let $\mathfrak{L}^1$ be a closed curve in the $\zeta$-plane. We suppose (for simplicity's sake) that $\zeta_1(\mathfrak{E}_1)$ lies inside and $\zeta_\nu(\mathfrak{E}_1)$, $\nu=2, \cdots, n$, lie outside of $\mathfrak{L}^1$. (See Figure 1.)

THEOREM. *Let*

$$(3.1)\qquad \begin{gathered}\mathfrak{S}(\mathfrak{X}) = \left[\frac{p(u,\zeta)}{\partial q(u,\zeta)/\partial\zeta}\right]\left[\mathfrak{t}_1 + \frac{i(\zeta+\zeta^{-1})}{2}\mathfrak{t}_2 + \frac{(\zeta-\zeta^{-1})}{2}\mathfrak{t}_3\right],\\ u \equiv [x_1 + (i/2)(x_2+ix_3)\zeta^{-1} + (i/2)(x_2-ix_3)\zeta]\end{gathered}$$

*where* $\zeta=\zeta(x_1, x_2, x_3)$ *is a root of the equation*

$$(3.2)\qquad q(u,\zeta) \equiv [u-\phi(\zeta)]\zeta^s = 0,$$

*be a harmonic vector, the branches of which we denote by* $\mathfrak{S}_\nu(\mathfrak{X})$, $\nu=1, 2, \cdots, n$. *The corresponding sheets of the multiply covered space of* $\mathfrak{S}(\mathfrak{X})$ *may be denoted by* $\mathfrak{R}^3_\nu$.

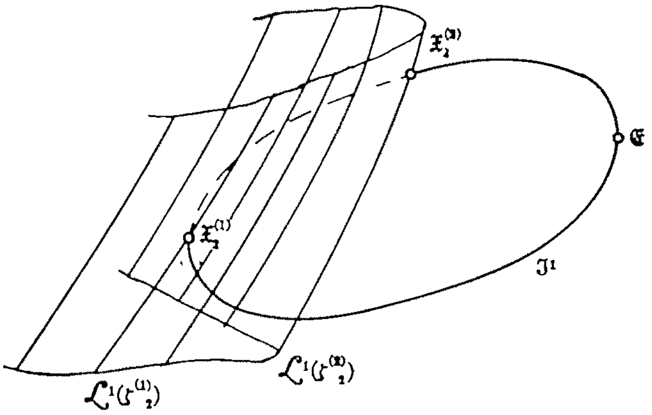

FIG. 2. The curve $\mathfrak{J}^1$ and the surface $\mathfrak{M}^2_2(\mathfrak{L}^1)$

*Let* $\mathfrak{J}^1$ *and* $\mathfrak{Q}^1$ *be curves in the* $x_1$, $x_2$, $x_3$*-space and in the* $\zeta$*-plane, respectively, with the above described properties.*

*Let finally*[8] $\{\mathfrak{X}_{\nu_\mu}^{(k)}\}$, $k=1, 2$; $\mu=1, 2, \cdots, p$ *be the points of* $\mathfrak{J}^1$ *which satisfy the equation* (3.2) *for appropriate values*[9] $\zeta=\zeta_\nu^{(k)}$, $\zeta_\nu^{(k)}\in\mathfrak{Q}^1$.

*Then*

$$(3.3)\quad \int_{\mathfrak{E}_1}^{\mathfrak{E}_2}\mathfrak{S}_1(\mathfrak{X})\cdot d\mathfrak{X}+\sum_{\mu=1}^{p}\int_{\mathfrak{X}_{\nu_\mu}^{(1)}}^{\mathfrak{X}_{\nu_\mu}^{(2)}}\mathfrak{S}_{\nu_\mu}(\mathfrak{X})\cdot d\mathfrak{X}=2\pi i\int_{\mathfrak{S}\mathfrak{N}_k^1}p[\phi(\zeta),\zeta]d\zeta$$

*where* $\mathfrak{N}_k^1$ *are the intervals of* $\mathfrak{Q}^1$, *for which the filaments* $\mathcal{L}^1(\zeta)$, *of* $\mathcal{M}^2(\mathfrak{Q}^1)$ *lie inside of* $\mathfrak{J}^1$; $\zeta_{\nu_\mu}^{(k)}$ *are their end points.*

REMARK. We suppose here that the point $\zeta=0$ does not lie inside of $\mathfrak{Q}^1$. Otherwise the components of the vector assume a form slightly different from that indicated in (3.1). These components in this case can be easily computed using the formulas of [2, §2].

PROOF. We consider the double integral

$$I=\int_{\mathfrak{J}^1}\int_{\mathfrak{L}^1}\frac{p(u,\zeta)d\zeta du}{q(u,\zeta)}.$$

The integrand is an absolutely integrable function; we can replace the double integral by the iterated one and change the order of the integration. Thus,

$$I=\int_{\mathfrak{J}^1}\left\{dx_1\int_{\mathfrak{L}^1}pq^{-1}d\zeta+dx_2\int_{\mathfrak{L}^1}(i/2)(\zeta+\zeta^{-1})pq^{-1}d\zeta\right.$$
$$\left.+\,dx_3\int_{\mathfrak{L}^1}(1/2)(\zeta-\zeta^{-1})pq^{-1}d\zeta\right\}.$$

By (3.1), (2.4) and (1.2) we have $I=\int_{\mathfrak{J}^1}\mathfrak{P}(pq^{-1},\mathfrak{Q}^1,\mathfrak{X})\cdot d\mathfrak{X}$. According to the results on p. 167, $\mathfrak{P}(pq^{-1}, \mathfrak{Q}^1, \mathfrak{X}) = \mathfrak{P}(pq^{-1}, \mathfrak{Q}^1, \mathfrak{E}_1) + \sum_{\mu=1}^{s}\mathfrak{S}_{\nu_\mu}(\mathfrak{X})$ for all points of $\mathfrak{J}^1$ which lie between $\mathfrak{X}_{\nu_\mu}^{(1)}$ and $\mathfrak{X}_{\nu_\mu}^{(2)}$, $\mu = 1, 2, \cdots$, p. Whence

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[8] For simplicity's sake we suppose that $\mathfrak{J}^1$ intersects every $\mathcal{M}^2(\mathfrak{Q}^1)$ in two points (if at all), and that it has no intersections with $\mathcal{M}_1^2(\mathfrak{Q}^1)$.

[9] The points $\mathfrak{X}$ of $\mathfrak{R}_1^3$ which satisfy the equation (3.2) for a fixed $\zeta$ form the part of $\mathcal{L}^1(\zeta)$ which lies in $\mathfrak{R}_1^3$. It consists of $n$ branches $\mathcal{L}_p^1(\zeta)$, $p=1, 2, \cdots, n$. The points $\mathfrak{X}$ of $\mathfrak{R}_1^3$ which satisfy (3.2) with $\zeta\in\mathfrak{Q}^1$ form $n$ surfaces $\mathcal{M}_p^2(\mathfrak{Q}^1)=\mathfrak{S}_{\zeta\in\mathfrak{L}^1}\mathcal{L}_p^1(\zeta)$. (See p. 167.) The points $\{\mathfrak{X}_{\nu_\mu}^{(k)}\}$ are the intersections of these surfaces with $\mathfrak{J}^1$.

The branch $\mathcal{M}_\nu^2(\mathfrak{Q}^1)$ which corresponds to a given branch of $\mathfrak{S}(\mathfrak{X})$ can be determined as follows. For $\mathfrak{X}=\mathfrak{E}_1$ we have $n$ roots of the equation (3.2), $\zeta=\zeta_\nu(\mathfrak{E}_1)$, $\nu=1, 2, \cdots, n$. To every $\zeta_\nu(\mathfrak{E}_1)$ corresponds a uniquely determined branch $\mathfrak{S}_\nu(\mathfrak{X})$ of $\mathfrak{S}(\mathfrak{X})$. (See p. 166.) If the argument $\mathfrak{X}$ varies continuously in $\mathfrak{R}_1^3$, $\zeta_\nu(\mathfrak{X})$ will move in the $\zeta$-plane. Let $\mathfrak{X}=\mathfrak{X}_\tau$ be a value such that $\zeta_\nu^*=\zeta_\nu(\mathfrak{X}_\tau)\in\mathfrak{Q}^1$. Since each $\mathcal{L}^1(\zeta)$ consists of $n$ disconnected branches the point $\mathfrak{X}_\tau$ lies in one of the lines $\mathcal{L}_p^1(\zeta_\nu^*)$. This branch of $\mathcal{L}^1(\zeta_\nu^*)$ corresponds to the branch $\mathfrak{S}_\nu(\mathfrak{X})$, and therefore $p=\nu$. The branch of $\mathcal{M}^2(\mathfrak{Q}^1)$ to which $\mathcal{L}_\nu^1(\zeta_\nu^*)$ belongs becomes $\mathcal{M}_\nu^2(\mathfrak{Q}^1)$.

$$I = \int_{\mathfrak{E}_1}^{\mathfrak{E}_2}\mathfrak{S}_1(\mathfrak{X})\cdot d\mathfrak{X} + \sum_{\mu=1}^{p}\int_{\mathfrak{X}_{\nu_\mu}^{(1)}}^{\mathfrak{X}_{\nu_\mu}^{(2)}}\mathfrak{S}_{\nu_\mu}(\mathfrak{X})\cdot d\mathfrak{X}.$$

Changing the order of integration we obtain

$$I = \int_{\mathfrak{Q}^1} d\zeta\left[\int_{\mathcal{F}^1(\zeta)} [p(\tau, \zeta)/q(\tau, \zeta)]d\tau\right] = \int_{\mathfrak{Q}^1} J(\zeta)d\zeta.$$

$\mathcal{F}^1(\zeta)$ is a closed curve in the complex $\tau$-plane and $q = [\tau - \phi(\zeta)]\zeta^{n}$. (See p. 167.) $J(\zeta) = 2\pi i p[\phi(\zeta), \zeta]$ or 0 if $\tau = \phi(\zeta)$ lies inside or outside of $\mathcal{F}^1(\zeta)$, respectively.

According to the considerations on p. 168, $\phi(\zeta)$ lies inside of $\mathcal{F}^1(\zeta)$ if and only if $\mathfrak{J}^1$ cannot be reduced to a point without cutting $\mathcal{L}^1(\zeta)$. $I$ therefore is equal to the right-hand member of (3.3), which completes our proof.

EXAMPLE. We take $p = 1$, $q = \zeta u$, $\mathfrak{Q}^1 = \mathrm{E}[\zeta = e^{it}, 0 \leqq t \leqq 2\pi]$. The vector $\mathfrak{S}(\mathfrak{X})$ (see (3.1)) which we obtain in this case has two branches $\mathfrak{S}_1(\mathfrak{X}) = r^{-1}[\mathfrak{t}_1 + x_2(x_1 + r)^{-1}\mathfrak{t}_2 + x_3(x_1 + r)^{-1}\mathfrak{t}_3]$ and $\mathfrak{S}_2(\mathfrak{X}) = -r^{-1}[\mathfrak{t}_1 + x_2(x_1 - r)^{-1}\mathfrak{t}_2 + x_3(x_1 - r)^{-1}\mathfrak{t}_3]$, $r^2 = x_1^2 + x_2^2 + x_3^2$. The branches $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are not connected in the real $x_1$, $x_2$, $x_3$-space. They are of a different form from (3.1) since $\zeta = 0$ lies inside of $\mathfrak{Q}^1$. We obtain $\mathfrak{S}_1(\mathfrak{X})$ and $\mathfrak{S}_2(\mathfrak{X})$ choosing the $x_1$ component of $\mathfrak{X}$ to be positive or negative, respectively. Further we have in this case:

$\mathcal{L}^1(\zeta) = \mathrm{E}[x_1 = 0, x_2\cos t + x_3\sin t = 0]$ where $t = -i \log \zeta$; $\mathcal{M}^2(\mathfrak{Q}^1) = \mathfrak{S}\ \mathcal{L}^1(\zeta) = \mathrm{E}[x_1 = 0]$.

Let $\mathfrak{J}^1 = \mathfrak{J}_1^1 + \mathfrak{J}_2^1$ be a closed curve in the $x_1$, $x_2$, $x_3$-space, where $\mathfrak{J}_1^1$ denotes the part which lies in $\mathrm{E}[x_1 > 0]$, $\mathfrak{J}_2^1$ the part in $\mathrm{E}[x_1 < 0]$. By $\mathfrak{X}_1 = (r_1 \cos\phi_1\mathfrak{t}_2 + r_1 \sin\phi_1\mathfrak{t}_3)$ and $\mathfrak{X}_2 = (r_2 \cos\phi_2\mathfrak{t}_2 + r_2 \sin\phi_2\mathfrak{t}_3)$, $0 < \phi_1 < \pi/2$, $\pi/2 < \phi_2 < \pi$, we denote the intersections of $\mathfrak{J}^1$ with $\mathcal{M}^2(\mathfrak{Q}^1)$ $(= \mathrm{E}[x_1 = 0])$. (We suppose that this intersection consists of two points.) Then

$$\sum_{\nu=1}^{2}\int_{\mathfrak{J}_\nu^1}\mathfrak{S}_\nu(\mathfrak{X})\cdot d\mathfrak{X} = (1/2\pi i)\int_{\mathfrak{Q}^1}\zeta^{-1}d\zeta\left[\int_{\mathcal{F}^1(\zeta)} u^{-1}du\right]$$

$$= \int_{\mathcal{N}_1^1}\zeta^{-1}d\zeta - \int_{\mathcal{N}_2^1}\zeta^{-1}d\zeta,$$

where $\mathcal{N}_1^1 = \mathrm{E}[\zeta = e^{it}, \phi_1 + \pi/2 \leqq t \leqq \phi_2 + \pi/2]$, $\mathcal{N}_2^1 = \mathrm{E}[\zeta = e^{it}, \phi_1 + 3\pi/2 \leqq t \leqq \phi_2 + 3\pi/2]$.

REMARK 1. The generalization of the above considerations if $q$ is a polynomial of a higher degree than 1 presents no difficulties. (See [2].)

We note further that our considerations can be generalized without essential difficulties to the case where $p$ and $q$ are entire functions.

REMARK 2. We supposed that $\mathfrak{Q}^1$ is a closed curve. However, our method can be applied also if $\mathfrak{Q}^1$ is an open curve. An operator $P(f, \mathfrak{Q}^1, \mathfrak{T})$ with $\mathfrak{Q}^1$ being an open curve transforms rational functions $f$ into harmonic functions with logarithmic singularities. The fact that the latter harmonic functions were not studied in detail prevented us from discussing this case here.

### 4. Residue formulas for surface integrals. Generalized Cauchy's formula.

There exists another possibility of generalization of the residue formula, namely, in considering surface integrals.

Various relations for surface integrals can be obtained using Green's formula. In particular a well known formula for the value of the function in the domain in terms of boundary values of the function and its derivative can be established in this way.

In the case of two variables for a harmonic vector $\mathfrak{H}$ with curl $\mathfrak{H}=0$, div $\mathfrak{H}=0$, $\mathfrak{H}(\mathfrak{X})$ may be expressed in terms of the boundary values of its components. (In Cauchy's integral formula for analytic functions which may be obtained in this way, the derivatives disappeared.)

It is clear that this procedure can be applied to harmonic vectors of several variables, provided that corresponding differential relations exist between the derivative of the components.[10] As an illustration we shall derive a formula which can be considered as a generalization of Cauchy's formula.

*Let $H(\mathfrak{X})$ be a harmonic function which is regular in a (schlicht) domain $\mathfrak{D}^3$ with twice continuously differentiable boundary $\mathfrak{B}^2$. Let $G_\nu$ be twice continuously differentiable functions, $\nu=1, 2, \cdots, n$, such that*

$$\nabla H = \sum_{\nu=1}^{n} \nabla G_\nu \times \mathfrak{a}_\nu, \tag{4.1}$$

*($\times$ means vector product), where $\mathfrak{a}_\nu = a_{\nu 1}\mathfrak{i}_1 + a_{\nu 2}\mathfrak{i}_2 + a_{\nu 3}\mathfrak{i}_3$ are vectors with constant coefficients. Then*

$$H(\mathfrak{X}) = (1/4\pi) \iint_{\mathfrak{B}^2} \left[ H\nabla r^{-1} + \nabla r^{-1} \times \sum_{\nu=1}^{n} G_\nu \mathfrak{a}_\nu \right] d\mathfrak{o} \tag{4.2}$$

*where $d\mathfrak{o} = dx_2 dx_3 \mathfrak{i}_1 + dx_1 dx_3 \mathfrak{i}_2 + dx_1 dx_2 \mathfrak{i}_3$.*

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[10] In particular this procedure has been used in the theory of functions of several complex variables to obtain generalizations of Cauchy's residue formula. See Bergman, *Partial differential equations*, Publication of Brown University, 1941 Supplementary Note III pp. SIII 8–SIII 9.

See also the further remarks concerning the other possibilities of generalizations of Cauchy's formulas mentioned there and Bergman, Revista de Ciencias, Lima, Peru, vol. 44 (1942) pp. 377–394 especially p. 391.

PROOF. By Green's formula,

$$H(\mathfrak{X}) = (1/4\pi)\left[\int\!\!\int_{\mathfrak{B}^2} H\nabla r^{-1}\cdot d\mathfrak{o} - \int\!\!\int_{\mathfrak{B}^2} r^{-1}\nabla H\cdot d\mathfrak{o}\right]$$
$$= (1/4\pi)(I_1 - I_2).$$

By (4.1)

$$I_2 = \int\!\!\int_{\mathfrak{B}^2}\left[\frac{1}{r}\left(\frac{\partial H}{\partial x_1}\,dx_2dx_3 + \frac{\partial H}{\partial x_2}\,dx_1dx_3 + \frac{\partial H}{\partial x_3}\,dx_1dx_2\right)\right]$$
$$= \int\!\!\int_{\mathfrak{B}^2} r^{-1}\left\{\sum_{\nu=1}^{n}\left(a_{3\nu}\frac{\partial G_\nu}{\partial x_2} - a_{2\nu}\frac{\partial G_\nu}{\partial x_3}\right)dx_2dx_3\right.$$
$$+\sum_{\nu=1}^{n}\left(-a_{3\nu}\frac{\partial G_\nu}{\partial x_1} + a_{1\nu}\frac{\partial G_\nu}{\partial x_3}\right)dx_1dx_3$$
$$\left.+\sum_{\nu=1}^{n}\left(a_{2\nu}\frac{\partial G_\nu}{\partial x_1} - a_{1\nu}\frac{\partial G_\nu}{\partial x_2}\right)dx_1dx_2\right\}.$$

Integrating by parts we obtain

$$I_2 = -\int\!\!\int_{\mathfrak{B}^2}\left(\nabla r^{-1}\times\sum_{\nu=1}^{n} G_\nu\mathfrak{a}_\nu\right)\cdot d\mathfrak{o}$$

since the line integrals cancel each other. Suppose now that we have a harmonic vector $\mathfrak{H}(\mathfrak{X}) = \sum_{k=1}^{3} H^{(k)}(\mathfrak{X})\mathfrak{i}_k$, such that

$$\nabla H^{(k)} = \sum_{\nu=1}^{3}\nabla H^{(\nu)}\times\mathfrak{a}_\nu^{(k)},\qquad k = 1, 2, 3. \tag{4.3}$$

(4.2) yields

$$H^{(k)}(\mathfrak{X}) = (1/4\pi)\int\!\!\int_{\mathfrak{B}^2}\left[H^{(k)}\nabla r^{-1} + \nabla r^{-1}\times\sum_{\nu=1}^{3} H^{(\nu)}\mathfrak{a}_\nu^{(k)}\right]\cdot d\mathfrak{o}. \tag{4.4}$$

Thus:

$$\mathfrak{H}(\mathfrak{X}) = (1/4\pi)\int\!\!\int_{\mathfrak{B}^2}\left[\mathfrak{H}(\nabla r^{-1}\cdot d\mathfrak{o})\right.$$
$$\left.+\sum_{k=1}^{3}\left[\left(\nabla r^{-1}\times\sum_{\nu=1}^{3} H^{(\nu)}\mathfrak{a}_\nu^{(k)}\right)\cdot d\mathfrak{o}\right]\mathfrak{i}_k\right]$$
$$= (1/4\pi)\int\!\!\int_{\mathfrak{B}^2}\left[\mathfrak{H}(\nabla r^{-1}\cdot d\mathfrak{o})\right.$$
$$\left.+\sum_{\nu=1}^{3}\sum_{k=1}^{3}\left[H^{(\nu)}(d\mathfrak{o}\times\nabla r^{-1})\cdot\mathfrak{a}_\nu^{(k)}\right]\mathfrak{i}_k\right].$$

In particular if div $\mathfrak{H}=0$ and curl $\mathfrak{H}=0$ then the relation (4.3) is satisfied, and we can represent $\mathfrak{H}(\mathfrak{X})$ in the above form (4.4).[11]

REMARK. $d\mathfrak{o}=\mathfrak{n}_1 do$, where $\mathfrak{n}_1$ is the unit vector in the direction of the interior normal, and $do$ the area of the surface element. (We note that $dx_m dx_n$ means $+|dx_m dx_n|$ or $-|dx_m dx_n|$, the sign being chosen in such a manner that the above relation for $d\mathfrak{o}$ is valid.)

(4.2) can also be obtained using the Stokes' formula. In fact

$$I_2 = \sum_\nu \iint_{\mathfrak{B}^2} \mathfrak{n}_1 r^{-1}(\nabla G_\nu \times \mathfrak{a}_\nu)do = \sum_\nu \iint_{\mathfrak{B}^2} \mathfrak{n}_1 r^{-1}(\nabla \times G_\nu \mathfrak{a}_\nu)do$$

since the $a_{k\nu}$ are constants. By Stokes' formula

$$\iint_{\mathfrak{B}^2} \mathfrak{n}_1(\nabla \times r^{-1}G_\nu \mathfrak{a}_\nu)do = \iint_{\mathfrak{B}^2} \mathfrak{n}_1 r^{-1}(\nabla \times G_\nu \mathfrak{a}_\nu)do$$
$$+ \iint_{\mathfrak{B}^2} \mathfrak{n}_1 G_\nu(\nabla \times r^{-1}\mathfrak{a}_\nu)do = 0,$$

since $\mathfrak{B}^2$ is a closed surface. Therefore

$$I_2 = -\sum_\nu \iint_{\mathfrak{B}^2} \mathfrak{n}_1 G_\nu(\nabla \times r^{-1}\mathfrak{a}_\nu)do.$$

We also note that formulas expressing the pressure on an immersed body (analogous to those of the two-dimensional theory) can be obtained using the fact that $\iint_{\mathfrak{B}^2}[\mathfrak{n}_1|\mathfrak{H}|^2-2\mathfrak{H}(\mathfrak{n}_1\mathfrak{H})]do$ does not change when $\mathfrak{B}^2$ is deformed, if curl $\mathfrak{H}=$ div $\mathfrak{H}=0$.

REMARK. The method presented here can obviously be applied if the components $H^{(n)}$ of a vector $\mathfrak{H}(\mathfrak{X})$ are connected by the relations

$$\sum_{n=1}^{3} \mathfrak{a}_{kn}\cdot\nabla H^{(n)} + \mathfrak{b}_k\cdot\mathfrak{H} + C_k = 0, \qquad k = 1, 2, 3,$$

where $\mathfrak{a}_{kn}=(a_{kn}^{(1)}\mathfrak{i}_1+a_{kn}^{(2)}\mathfrak{i}_2+a_{kn}^{(3)}\mathfrak{i}_3)$, $\mathfrak{b}_k=(b_k^{(1)}\mathfrak{i}_1+b_k^{(2)}\mathfrak{i}_2+b_k^{(3)}\mathfrak{i}_3)$, and the determinants $a_{kn}^{(s)}\neq 0$, for $n=1, 2, 3$.

Analogous formulas are also valid if $H^{(k)}$ satisfy certain elliptic differential equations and if we introduce instead of $r^{-1}$ a fundamental solution of this equation.

## BIBLIOGRAPHY

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2. ———, Math. Ann. vol. 99 (1928) pp. 629–658.

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[11] In the case where curl $\mathfrak{H}=$ div $\mathfrak{H}=0$ similar formulas were given by von Mises (an address presented at the New York meeting of the Society on April 4, 1942, by invitation of the Program Committee) and by Fulton and Rainich (Amer. J. Math. vol. 54 (1932) pp. 235–241).

# THE PROBLEM OF SECURING TEACHERS OF COLLEGIATE MATHEMATICS FOR WARTIME NEEDS

The eighteen-year-old draft law and the inauguration of the proposed Army and Navy programs for colleges and universities are creating serious personnel problems for departments of mathematics. In an attempt to alleviate this situation, partially at least, the American Mathematical Society and the Mathematical Association of America are establishing in the Philadelphia Office of the Society a bureau to provide information regarding available teachers of collegiate mathematics. The cooperation of all mathematicians is needed in order to make this plan a success.

The government has indicated that approximately 250,000 trainees will be sent to a selected group of about 300 colleges and universities. It is estimated that the full-time services of at least 2500 teachers of mathematics will be required for this program. In addition, institutions will necessarily provide a normal program for men under 18, for women, who are electing mathematics in increasing numbers, and for students rejected, for any reason, by Selective Service. Institutions will also continue special contracts already in existence such as meteorology and aviation pre-flight schools.

To provide for all this instruction there are available about 3,000 persons now functioning as teachers of mathematics in four-year colleges. To ensure the proper handling of this instruction, readjustments of and additions to teaching staffs of many institutions will be necessary. For some institutions, additional personnel might be secured from the following sources:

(1) Members of departments of mathematics in institutions which do not assume government programs;

(2) Persons now teaching in non-critical fields who have had sufficient training in mathematics to make them valuable in this field. These persons should be recruited for the teaching of mathematics rather than being allowed to assume non-academic employment. This should probably be the main source of supply.

It is in order to promote an orderly solution of these problems that the American Mathematical Society and the Mathematical Association of America have established the bureau of information regarding teachers of collegiate mathematics. The purpose is two-fold:

(1) To collect information concerning persons who are trained in mathematics and who are or will be available for teaching during the emergency at institutions other than their own. It is requested that

the names and present addresses of persons who are available for employment as teachers of collegiate mathematics be reported to the address given below. Additional information concerning qualifications will be secured directly from the mathematicians so reported.

(2) To furnish to chairmen of departments of mathematics who need additional personnel the information obtained through (1). As the employment offered will, in most cases, be temporary, an attempt will be made to furnish the chairman with an adequate list, mainly of persons living in his vicinity. *The purpose of the bureau is to furnish only information and not recommendations.* The department chairman will, of course, investigate on his own responsibility the qualifications of the available teachers.

The country's awakened consciousness of the need for mathematics as a preparation for service in the armed forces imposes on the mathematical profession the grave responsibility to meet this need by providing the best possible teaching. The Society and Association aim to make a definite contribution to the war effort by supplying information regarding available teachers of mathematics. To do this, the full cooperation of all mathematicians is needed. It is impossible to predict what portion of the demand will be met in this manner. It is likely that teachers will become available less rapidly than the demands for additional personnel are made, but the cooperation of mathematicians will enable the bureau to function effectively.

All correspondence in this connection should be addressed to:

Committee on Available Teachers
of Collegiate Mathematics
110 Bennett Hall
University of Pennsylvania
Philadelphia, Pa.

*Committee:*
W. D. Cairns
Arnold Dresden
J. R. Kline

March, 1943

# OCCUPATIONAL CLASSIFICATION OF MATHEMATICIANS

## MEMORANDUM F

Occupational Bulletins No. 10 and No. 23 have recently been amended by Selective Service Headquarters. These are the directives upon which chairmen of departments of mathematics have been basing requests for the occupational classification of members of their staffs and of students majoring in mathematics.

It is to be noted particularly that, according to both Bulletins, a graduate student in mathematics may be considered for occupational classification only if, in addition to pursuing graduate studies, he is acting as a graduate assistant and is engaged in:

(a) classroom instruction for not less than twelve hours per week; or

(b) scientific research certified by a recognized federal agency as related to the war effort.

The amended Bulletins supersede the original directives. In making requests for occupational classification, chairmen of departments of mathematics are advised to attach copies of the amended Bulletins to Form 42A. Copies of the revised Bulletins will be furnished, upon request, by the office of the Secretary. In all other details, the procedure recommended in Memorandum E, distributed on October 28, 1942, and published in the November, 1942, Bulletin of the American Mathematical Society, remains unchanged.

J. R. KLINE,
*Secretary*

January 30, 1943

# ON THE ZEROS OF THE DERIVATIVES OF A FUNCTION AND ITS ANALYTIC CHARACTER

G. POLYA

## 1. Introduction.

How do the zeros of the $n$th derivative $f^{(n)}(x)$ behave when $n$ becomes very large? How does this behavior depend on the analytic nature of the function $f(x)$? At first sight this question appears to be a little out of the way. In fact it is intimately connected with essential parts of the theory of functions. In the Hadamard theory of the singularities of the Taylor series we consider the sequence of the derivatives $f(x), f'(x), f''(x), \cdots$ for a fixed complex value of $x$; in the theory of quasi-analytic functions we consider the same sequence for variable real $x$, and we consider especially the maximum absolute value of each of its terms in a given interval; now we consider again the same sequence $f(x), f'(x), f''(x), \cdots$ for variable $x$, that may be complex or real, and we consider especially the values of $x$ for which its terms vanish. In all three cases the main object of consideration is the connection of the chosen feature of the sequence $f(x), f'(x), f''(x), \cdots$ with the analytic nature of the function $f(x)$.

There is a special reason for giving a general survey of results about the zeros of successive derivatives just now. The subject received quite recently essential contributions from several mathematicians in this country, from R. P. Boas, Einar Hille, A. C. Schaeffer, I. J. Schoenberg, Gabor Szegö, J. D. Tamarkin, D. V. Widder, Norbert Wiener and the present speaker. All these contributions are linked together and seem to open the door to further results and to a well connected harmonious theory.

In this short talk I should like to emphasize the main outline and the essential divisions of the theory in so far as they are recognizable today. In order to do this as clearly and intuitively as I can, I have to discuss some of the older results, to leave out some of the newer ones and to sacrifice precise details except for a few central points.

Our subject has two main parts. In one we consider analytic functions of a complex variable, in the other real-valued functions of a real variable. I proceed to describe these two parts in this order.

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An address presented before the New York meeting of the Society on April 3, 1942, by invitation of the Program Committee; received by the editors May 8, 1942.

## THE COMPLEX DOMAIN

**2. Statement of the problem.** In the next sections (2 to 7) $f(z)$ denotes an analytic function of the complex variable $z$. Let us assume that $f(z)$ is single-valued and analytic in the whole plane, except maybe in certain isolated singular points.

Let us consider the zeros of $f(z)$; they form a more or less irregular distribution of points in the plane. Now let us pass to the derivative $f'(z)$ and consider its zeros; they are in general different from those of $f(z)$ and form a new distribution of points. If we pass, by differentiating once more, to $f''(z)$ and its zeros, the picture changes again, the points appear to have moved into a third position. Corresponding to the sequence of the derivatives $f(z), f'(z), f''(z), \cdots$ we have a sequence of distributions of points formed by the zeros of the successive derivatives. Can we discover in this sequence of point-distributions some definite trend?

It is not bad to start from a vague question that is intuitive and suggestive, provided that it leads us finally to some precise question. What precise question is suggested by the intuitive process we are considering, the migration of zeros in the course of successive differentiations?

In fact, not only one question, but several different precise questions can be put. But we need a few definitions.

(I) Let us consider any fixed regular point of $f(z)$; let us assume, to be concrete, that $z=0$ is a regular point. Let $r_n$ denote the absolute value of the zero of $f^{(n)}(z)$ which is nearest to the origin.

(II) Let us consider any fixed closed circle in which $f(z)$ is regular. Let us assume that $f(z)$ is regular in the closed unit circle $|z| \leqq 1$, and let $N_n$ denote the number of zeros of $f^{(n)}(z)$ (with multiplicity) in this circle.

(III) We define a set $D'$, attached to the function $f(z)$, in the following way. Consider a point $z$ of the plane. If there exists a circle of center $z$ in which only a finite number of derivatives vanishes, $z$ does not belong to $D'$. In the other case, when any (arbitrarily small!) fixed circle with center $z$ contains the zeros of an infinity of derivatives $f^{(n_1)}(z), f^{(n_2)}(z), \cdots$, $z$ belongs to $D'$. The set $D'$ is in some sense the final position of the zeros of $f^{(n)}(z)$; let us call $D'$ the *final* set of $f(z)$. If $f(z)$ is meromorphic or entire, the final set $D'$ can also be described as the derived set of the denumerable set $D$ of all zeros of all derivatives $f(z), f'(z), f''(z), \cdots$ (provided that points in which an infinity of derivatives vanishes are also counted as points of $D'$).

How do the zeros of the derivatives approach a regular point? What can we say about the sequence $r_1, r_2, r_3, \cdots$? Do the zeros of

$f^{(n)}(z)$ become denser as $n$ increases? What about the sequence $N_1, N_2, N_3, \cdots$? Does the configuration of the zeros of $f^{(n)}(z)$ tend to a final position? Can we find the final set $D'$?

3. **Meromorphic functions.** The answer to all these questions depends on the analytic character of the function $f(z)$. For functions of certain sorts we can discover a trend, and this trend is particularly easy to recognize when $f(z)$ is a meromorphic function. In this case the trend may be roughly described by saying that the poles of $f(z)$ act as repulsive centers on the zeros of $f^{(n)}(z)$.

We can give a precise description after a precise definition. Those points of the plane which are nearer to a pole $a$ of $f(z)$ than to any other pole of $f(z)$ constitute the *domain* of the pole $a$. If the domains of two poles, $a$ and $b$, have common boundary points, these are on the perpendicular bisector of the line that joins $a$ to $b$. Therefore the boundary of the domain of the pole $a$ is a polygonal line, and the domain itself is an open convex set. The zeros of $f^{(n)}(z)$ are pushed toward the polygonal boundary of the domain of the pole as $n$ increases. *The final set of a meromorphic function contains no point interior to the domain of any pole but contains all points which are on the common boundary of the domains of two or more poles* [20].

Observe that the final set depends only on the position of the poles of $f(z)$, but is independent of the multiplicity of these poles, or their residues, and so on.

4. **Functions with an essential singularity.** Let us consider next the case in which $f(z)$ has just one finite singular point, the point $z=0$, say. If this point is a pole the whole plane is its domain, the zeros of $f^{(n)}(z)$ are pushed towards $\infty$, they have no finite limiting point, the final set is empty. But if the point $z=0$ is an essential singular point, the situation is quite different; the trend of the movement of the zeros of $f^{(n)}(z)$ with increasing $n$ is much more difficult to recognize, and it is no longer true that this trend depends only on the position of the singular point. For example, if

$$f(z) = (1/z)e^{-1/z},$$

all zeros of $f^{(n)}(z)$ are positive and real and the final set is the positive half of the real axis. If we change $z$ into $ze^{i\alpha}$, where $\alpha$ is a fixed real number, $0<\alpha<2\pi$, the singularity of $f(z)$ remains at the point 0, but the zeros of $f^{(n)}(z)$ tend to condense along another ray issued from the singular point; the final set changes position.

5. **Entire functions.** Let us now consider a function with no finite

singular points, an entire function. Can we recognize the trend of the zeros of $f^{(n)}(z)$ as $n \to \infty$? We may say that this trend depends mainly on the rate at which $f(z)$ increases when $|z| \to \infty$, that is, on the order of $f(z)$. For certain entire functions of order 1, as, for example, for

$$f(z) = \sin z$$

the differentiation does not change essentially the density of the distribution of the zeros. If the order of $f(z)$ is less than 1 the differentiation tends to scatter the zeros; the zeros of $f^{(n)}(z)$ tend to move out to $\infty$ as $n$ increases; their distribution becomes thinner. If the order of $f(z)$ is greater than 1 the differentiation tends to concentrate the zeros; the zeros of $f(z)$ tend to move in from $\infty$ as $n$ increases; their distribution becomes denser. In short, the point at $\infty$ exerts on the zeros of $f^{(n)}(z)$ attraction or repulsion, according as the order of $f(z)$ is less than 1 or greater than 1.

Passing to precise statements, if $\lambda$ denotes the order of the entire function $f(z)$, we have

$$\limsup_{n \to \infty} (\log n)^{-1} \log r_n \geqq (1 - \lambda)/\lambda, \tag{1}$$

$$\liminf_{n \to \infty} (\log n)^{-1} \log N_n \leqq (\lambda - 1)/\lambda. \tag{2}$$

The inequality (1) was found, at least for $\lambda < 1$, by Ålander [1]. More relations and more precise relations of this kind were discovered by Gontcharoff and several authors following his line of research.[1] The inequality (2) seems to be new. Both inequalities are "exact," that is, the sign of equality may be attained for certain special functions. Furthermore we cannot exchange lim inf and lim sup or change either into lim without falling into false assertions.[2]

6. **Real entire functions.** An entire function $f(z)$ is called *real* if it takes real values for real $z$, or, what is the same, if the coefficients of its Maclaurin series are all real. The set of zeros of a real entire function is symmetrical with respect to the real axis. The real axis seems to exert an influence on the complex zeros of $f^{(n)}(z)$; it seems to at-

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[1] Gontcharoff [14]. See also Takeneka [30–34], Kakeya [16], Whittaker [35, 36], Schoenberg [28]. Boas [9] (see also Levinson [17]) introduced a quantity related to but different from $r_n$; let $s_n$ denote the radius of the largest circle of center $z=0$ in the interior of which $f(z)$ is regular and $f^{(n-1)}(z)$ is *univalent*. Visibly $r_n \geqq s_n$. It seems that $s_n$ has more simple properties than $r_n$; especially we have the following double inequality $\liminf_{n \to \infty} (\log n)^{-1} \log s_n \leqq (1-\lambda)/\lambda \leqq \limsup_{n \to \infty} (\log n)^{-1} \log s_n$.

[2] If $\lambda \leqq 1$ the final set may be empty and we may construct entire functions of any given order whose final set consists of just one point. *If $\lambda > 1$, must the final set necessarily contain a point?*

tract these zeros when the order is less than 2, and it seems to repel them when the order is greater than 2.

We may pass to exact statements, but these are, for the time being, hypothetical.

A. *If the order of the real entire function $f(z)$ is less than 2, and $f(z)$ has only a finite number of complex zeros, then its derivatives, from a certain one onwards, will have no complex zeros at all.*

B. *If the order of the real entire function $f(z)$ is greater than 2, and $f(z)$ has only a finite number of complex zeros, then the number of the complex zeros of $f^{(n)}(z)$ tends to infinity as $n \to \infty$.*

Both theorems are hypothetical, but A is better known than B. A weaker theorem than A, with 2/3 instead of 2, was first proved, then 2/3 was replaced by 1, and at last by 4/3.[3] So we know today that real entire functions of order less than 4/3 lose all their complex zeros after a finite number of differentiations provided they had only a finite number to start with. But the advance from 4/3 to 2, that is, to the full theorem A, is still to be made.

B is supported mostly by "experimental" evidence.[4] I quote two interesting examples

$$e^{-z^{2k}}, \qquad e^{-e^{z}}. \tag{3}$$

The final set of the first entire function (3) ($k$ is a positive integer) consists of $k$ straight lines passing through the origin and dividing the plane into $2k$ equal angles [20]. The final set of the second entire function (3) consists of an infinity of parallel lines, dividing the plane into congruent strips of width $2\pi$.[5] The final set of both functions contains the whole real axis, but also points outside the real axis (provided $k \geqq 2$).

**7. Another open question about real entire functions.** The consider-tion of the general trend of the zeros of $f^{(n)}(z)$ and some examples, principally (3), may lead us to the following hypothetical theorem.

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[3] The hypothetical theorem A (or rather a slightly more general theorem in which certain entire functions of order 2 are also considered) was found independently by two authors, A. Wiman and G. Polya The theorem with 2/3 was stated as proved by Polya [21, see p. 27, annotation 21]; proved, with reference to this first statement, almost simultaneously by Ålander [4] and Polya [22]. The theorem with 1 was proved by Wiman [39], that with 4/3 by Polya [23]. See also Wiman [38].

[4] A first remark in the direction of theorem B was made by Polya [18, 19]. (The role of the order 2 is indicated by the following theorem, due to Laguerre. *A real entire function that has only real zeros is or is not a limit of polynomials that have only real zeros according as its order is less than 2 or greater than 2*.) A more precise hypothetical theorem was formulated by A. Wiman and investigated by Ålander [1, 2] who used a geometrical method.

[5] Suggested by the present speaker, proved by G. Szegö (unpublished).

C. *If a real entire function of order greater than* 1 *remains bounded for real values of the variable, then its final set contains the whole real axis.*

I state this theorem without any pretension to the gift of prophecy. I state it because I think that the problem to prove or to disprove it, is neither too trivial nor completely inaccessible.

## THE REAL DOMAIN

8. **Notation and introductory results.** We change our notation. Henceforth $f(x)$ denotes a real-valued function of the real variable $x$, that is defined and possesses derivatives of all orders in a certain interval $I$, and $N_n$ denotes the number of changes of sign of $f^{(n)}(x)$ in $I$.

$f(x)$ need not be analytic in $I$. The analyticity of $f(x)$ is bound to the condition that to each point $a$ of $I$ corresponds a positive $\rho=\rho(a)$, such that, in the common part of the two intervals $I$ and $(a-\rho, a+\rho)$, $f(x)$ is represented by a convergent power series in $(x-a)$. If $f(x)$ is analytic in $I$, its analytic continuation is completely determined by its values at the points of $I$ and, if the continuation exists, it may turn out that $f(x)$ is meromorphic, or entire, or entire of finite order, and so on.

The number $N_n$ may be infinite. But if the interval $I$ is closed, and $f(x)$ is analytic in $I$, not only is $N_n$ a finite number for each $n$ but the increase of $N_n$ with $n$ is limited by the following theorem.

*If $f(x)$ is analytic in the closed interval $I$, then*

$$\liminf_{n\to\infty} n^{-1} N_n \text{ is finite.} \tag{4a}$$

*If $f(x)$ is an entire function*

$$\liminf_{n\to\infty} n^{-1} N_n = 0. \tag{4e}$$

*If $f(x)$ is an entire function of the finite order* $\lambda$

$$\liminf (\log n)^{-1} \log N_n \leq (\lambda - 1)/\lambda. \tag{4f}$$

*If $f(x)$ is an entire function of exponential type (that is, if its increase does not exceed the finite type of order* 1)

$$\liminf_{n\to\infty} N_n \text{ is finite.} \tag{4t}$$

The letters a, e, f, t will be used systematically in distinguishing formulas, a reminding us of analytic, e of entire, f of finite order, t of exponential type. Observe that the increase of $N_n$ is more and more restricted as we pass along the cases a, e, f, t.

The theorem we just stated starts from a given analytic character of the function (as analytic, entire and so on) and reaches conclusions concerning the zeros of the derivatives, especially conclusions concerning the behavior of the sequence $N_1, N_2, N_3, \cdots$; the theorem seems to be new, it is easy to prove by "complex variable" methods, and appropriate to introduce our subject. We pass now to results of a different character in which we start from the derivatives, especially from properties of the sequence $N_1, N_2, N_3, \cdots$, and reach conclusions about the analytic nature of the function, using mainly "real variable" methods.

9. **All derivatives are of constant sign.** The first relationships between the analytic character of a real function and the qualitative behavior of its derivatives were discovered by S. Bernstein. One of his simplest results is concerned with the condition

$$N_n = 0, \qquad n = 0, 1, 2, \cdots, \tag{5}$$

and may be stated as follows.

*If no derivative of $f(x)$ changes sign in an open interval $I$, then $f(x)$ is analytic in $I$.*

This result came as a great surprise at the time of its first publication [5], and it still occupies a central position. It is the common starting point of many questions and generalizations.

10. **An infinity of derivatives is of constant sign.** A first generalization of the condition (5) is due to S. Bernstein himself [5].

*If infinitely many derivatives $f^{(n_1)}, f^{(n_2)}, \cdots$ are of constant sign in the open interval $I$,*

$$N_{n_k} = 0, \qquad k = 1, 2, 3, \cdots, \tag{6}$$

*and if the sequence $n_1, n_2, \cdots$ does not increase more rapidly than a geometric progression, that is, if there is a fixed quantity $Q$ such that*

$$n_{k+1}/n_k < Q, \qquad k = 1, 2, 3, \cdots, \tag{7}$$

*then it still may be asserted that $f(x)$ is analytic in $I$.*

Recently R. P. Boas gave a new proof of this result and showed by interesting examples that the condition expressed by (6) and (7) cannot be essentially relaxed without endangering the truth of the theorem [10].

11. **All derivatives are of constant sign. Influence of the signs.**

Let us return to the central condition (5). If it holds, each derivative of $f(x)$ has a constant sign in the open interval $I$, and, therefore, each derivative is monotonic in $I$, and so is the absolute value of each derivative. In fact, since

$$(d/dx)[f^{(n)}(x)]^2 = 2f^{(n)}(x)f^{(n+1)}(x)$$

the absolute value of $f^{(n)}$ is steadily increasing or steadily decreasing, according as $f^{(n)}$ and $f^{(n+1)}$ are or are not of the same sign. We say that $f^{(m)}$ and $f^{(n)}$, where $m<n$, belong to the same *block* if and only if the absolute values of $f^{(m)}, f^{(m+1)}, \cdots, f^{(n-1)}, f^{(n)}$ all vary in the same sense, that is, all increase or all decrease. Thus $f^{(n)}$ and $f^{(n+1)}$ belong to different blocks if and only if

$$f^{(n)}(x)f^{(n+1)}(x) < 0$$

in $I$. Let $l_1, l_2, l_3, \cdots$ denote the lengths of the successive blocks into which the sequence $f, f', f'', \cdots$ is decomposed; we assume here that no block has infinite length, and that therefore there is an infinity of blocks. S. Bernstein discovered that the lengths of these blocks are connected with the analytic nature of $f$. He proved the following theorem.[6]

*If, in the open interval I,*

$$l_n = o(l_1 + l_2 + \cdots + l_n) \tag{7e}$$

*$f(x)$ is an entire function. If there exists a positive number* $\lambda$, $\lambda > 1$, *such that*

$$l_n = O\left([l_1 + l_2 + \cdots + l_n]^{(\lambda-1)/\lambda}\right) \tag{7f}$$

*$f(x)$ is an entire function of finite order not exceeding* $\lambda$. *If*

$$l_n = O(1) \tag{7t}$$

*$f(x)$ is an entire function of exponential type.*

12. **An infinity of derivatives is of constant sign. Influence of the signs.** D. V. Widder found recently [37] that $f(x)$ *is necessarily an entire function of exponential type if, in I,*

$$(-1)^m f^{(2m)}(x) \geqq 0, \qquad m = 0, 1, 2, \cdots. \tag{8}$$

This condition would imply, *if* we knew that no derivative of $f(x)$ vanishes in $I$, that $f^{(2m)}$ and $f^{(2m+1)}$ belong to different blocks, and that, therefore, no block has a length greater than two, $l_k \leq 2$. But, in fact, Widder's condition (8) does not say anything about the nonvanishing of the derivatives of odd order in $I$, and, therefore, Widder's theorem is not contained in the third part of Bernstein's theorem stated in the foregoing section; (8), although of much more special nature, does not imply (7t).

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[6] Bernstein's statement of his results [7, 8] is different in expression but not in substance from the statement given here.

R. P. Boas and the present speaker set out to find a common generalization of S. Bernstein's result, stated in the foregoing section, and of Widder's result, quoted in the present section. They obtained the following theorem [12, 13].

*Let* $\{n_k\}$ *and* $\{q_k\}$ *be sequences of positive integers,* $\{n_k\}$ *strictly increasing. Assume that* $f^{(n_k)}$ *and* $f^{(n_k+2q_k)}$ *do not change sign in* $I$,

$$N_{n_k} = N_{n_k+2q_k} = 0, \qquad k = 1, 2, 3, \cdots, \tag{9}$$

*that*

$$f^{(n_k)}(x) f^{(n_k+2q_k)}(x) \leq 0$$

*in* $I$, *and that*

$$q_1 + q_2 + \cdots + q_k = O(n_k).$$

*If*

$$n_k - n_{k-1} = o(n_k), \qquad q_k = o(n_k), \tag{10e}$$

*then* $f(x)$ *is an entire function. If there is a positive number* $\lambda$, $\lambda > 1$, *such that*

$$n_k - n_{k-1} = O(n_k^{(\lambda-1)/\lambda}), \qquad q_k = O(n_k^{(\lambda-1)/\lambda}), \tag{10f}$$

*then* $f(x)$ *is an entire function of finite order not exceeding* $\lambda$. *If*

$$n_k - n_{k-1} = O(1), \qquad q_k = O(1), \tag{10t}$$

*then* $f(x)$ *is an entire function of exponential type.*

This theorem is more general than that of Bernstein quoted in the foregoing section, since the condition (9) is less restrictive than (5). We derive Bernstein's theorem from the present one by putting $l_1 + l_2 + \cdots + l_k = n_k$ and $q_k = 1$, for $k = 1, 2, 3, \cdots$. We obtain Widder's result, quoted in this section, by putting $n_k = 2k$ and $q_k = 1$.

13. **No derivative changes sign more than a given number of times. The case of periodic functions.** Let us return again to the fundamental theorem considered in §9. S. Bernstein derived the analyticity of the function from the condition (5). Is it possible to derive analyticity from the less restrictive condition

$$N_n = O(1), \tag{11}$$

that is, does the condition that no derivative changes sign more than a fixed number of times imply the analyticity of the function? This question was first answered in the special case of periodic functions by Norbert Wiener and the present speaker [25], and led to the following precise result.

*If no derivative of the periodic function $f(x)$ of period $2\pi$ has more than $2m$ changes of sign in a period, $f(x)$ is a trigonometric polynomial of order not exceeding $m$,*

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_m \cos mx + b_m \sin mx$$

*with certain constants $a_0, a_1, b_1, \cdots, a_m, b_m$.*

Any derivative of a trigonometric polynomial of order $m$ is a trigonometric polynomial of the same order and has no more than $2m$ real zeros in a period. But if a periodic function is not a trigonometric polynomial the number of changes of sign of the $n$th derivative must tend to infinity with $n$; this is contained in the theorem. We can restate the theorem by saying that the number of changes of sign of the $n$th derivative of a periodic function does or does not tend to infinity, according as its Fourier series has or has not an infinity of coefficients different from zero. Thus the theorem discloses a new characteristic property of trigonometric polynomials, the uniform boundedness of the changes of sign of all derivatives.

**14. The number of changes of sign of the $n$th derivative has a prescribed bound depending on $n$. The case of periodic functions.** The theorem of the preceding section became a starting point of new questions and new generalizations. A first generalization may be formulated as the following tripartite theorem.

*Let $f(x)$ denote a periodic function and $N_n$ the number of changes of sign of $f^{(n)}(x)$ in a period. If*

$$N_n < 2n/\log n \tag{11e}$$

*for sufficiently large $n$, then $f(x)$ is an entire function. If there is a $\lambda$ $(\lambda > 1)$ such that*

$$\limsup\, (\log n)^{-1} \log N_n \leqq (\lambda - 1)/\lambda, \tag{11f}$$

*$f(x)$ is an entire function of finite order not exceeding $\lambda$. If*

$$N_n = O(1) \tag{11t}$$

*$f(x)$ is an entire function of exponential type.*

A similar but less precise tripartite statement was obtained by Norbert Wiener and the present speaker [25]. The exact form just stated is due to Gabor Szegö [29]. As it may be shown by suitable examples, the condition (11f) is completely sharp, and (11e) differs very little (if it differs at all) from a completely sharp condition.

The condition (11t) reproduces (11), and the third part of the theorem is identical with the theorem of the foregoing section; in fact, as is well known, a periodic entire function of exponential type is just a trigonometric polynomial. The conditions (11e), (11f), (11t) are parallel to the foregoing series of conditions (4e), (4f), (4t); (7e), (7f), (7t); and (10e), (10f), (10t). The hypotheses (11e), (11f), (11t) restrict more and more the increase of $N_n$; the last one does not allow any "increase" at all we may say. The corresponding conclusions (that $f(x)$ is entire, entire of finite order, entire of exponential type) may be expressed by a more and more rapid decrease of the coefficients of the Fourier series; in the last case the decrease is so abrupt that all coefficients are equal to 0 from a certain one onward.

15. **Other generalizations.** The theorem considered in §13 dealt with periodic functions, and therefore with Fourier series.

It is natural to pass from periodic functions to almost periodic functions and from Fourier series to Fourier integrals (trigonometric integrals); in both cases the number of changes of sign has to be replaced by (a suitable notion of) the density of changes of sign. The first extension was hinted by Norbert Wiener and the present speaker [25] the second by J. D. Tamarkin.

A different kind of generalization was effected by Einar Hille who subjected the role of the Fourier series in the proof of the theorem of §13 to a thorough analysis. The most important property of the Fourier series $\sum c_n e^{inx}$ from the point of view of that proof is the following. The operation of differentiation $d/dx$ changes the series in a simple way, it multiplies the coefficient $c_n$ by the simple factor $in$. But the differential operation $(1-x^2)(d^2/dx^2)-2xd/dx$ acts on the development into Legendre polynomials in a similar way. This observation leads to the following theorem, one of numerous analogous results discovered by Hille [15].

*Let $N_n^*$ denote the number of changes of sign of*

$$[(1-x^2)(d^2/dx^2)-2xd/dx]^n f(x)$$

*in the interval $-1 \leqq x \leqq 1$. If $N_n^* = O(1)$, the function $f(x)$ possesses a development into Legendre polynomials having only a finite number of non-vanishing coefficients, that is, $f(x)$ itself is a polynomial.*

16. **No derivative changes sign more than a given number of times. The general case.** The question raised at the beginning of §13 was

completely answered by A. C. Schaeffer who proved the following theorem.[7]

*If, in an open interval I, no derivative of $f(x)$ changes sign more than a fixed number of times, $f(x)$ is analytic in I.*

This theorem contains that of S. Bernstein, discussed in §9, since its condition (11) is less restrictive than condition (5). Schaeffer's proof is also very remarkable. It uses tools of the same general character as the proofs of the theorems discussed in §§9–12, namely inequalities between derivatives, but it adds to those known before an especially original inequality of this kind, another application of which yields a new proof of the theorem of §13, and seems to open a new vista on the hypothetical theorem C of §7.

Looking back at the new results and new analogies that have been added quite recently to our knowledge of the subject, we may obtain the impression of grasping the outlines of a well balanced theory leading to further results.

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39. ———, *Ueber die Realität der Nullstellen fast aller Ableitungen gewisser ganzer Funktionen*, ibid. vol. 114 (1937) pp. 617–621.

SMITH COLLEGE

# THE NOVEMBER MEETING IN SOUTH BEND

The three hundred ninety-first meeting of the American Mathematical Society was held at the University of Notre Dame, South Bend, Indiana, on Friday and Saturday, November 27–28, 1942 in connection with the Centennial Celebration of the University of Notre Dame. The attendance was about seventy including the following fifty-one members of the Society:

Leonidas Alaoglu, E. S. Allen, P. H. Anderson, W. L. Ayres, Henry Blumberg, L. M. Blumenthal, I. W. Burr, R. V. Churchill, C. E. Clark, A. H. Copeland, C. C. Craig, R. P. Eddy, Samuel Eilenberg, G. M. Ewing, N. J. Fine, C. H. Fischer, J. H. Giese, Michael Golomb, Cornelius Gouwens, L. M. Graves, G. E. Hay, M. H. Heins, T. H. Hildebrandt, A. S. Householder, M. H. Ingraham, J. H. Kenna, Joseph Landin, H. W. Linscheid, Eugene Lukacs, G. W. Mackey, Karl Menger, A. N. Milgram, C. J. Nesbitt, Jerzy Neyman, Ivan Niven, C. D. Olds, P. M. Pepper, Maxwell Reade, G. E. Schweigert, A. H. Smith, W. S. Snyder, C. E. Springer, A. H. Stone, D. M. Stone, R. M. Thrall, Bernard Vinograde, Abraham Wald, M. S. Webster, G. W. Whitehead, A. L. Whiteman, J. W. T. Youngs.

Sessions for the reading of contributed papers were held on Friday afternoon with Professor L. M. Blumenthal presiding and on Saturday morning with Professor R. V. Churchill presiding. Jointly with this meeting the University of Notre Dame held its annual Mathematical Symposium, the subject being *Modern statistics*. On Friday afternoon prior to the Society session Professor Jerzy Neyman of the University of California spoke on *Theory of confidence intervals* and at the close of the Society session Professor Abraham Wald of Columbia University spoke on *Outline of a general theory of statistical inference*. On Saturday morning before the session of the Society there were two more addresses of the Symposium: *Asymptotic properties of the likelihood ratio test* by Professor Wald and *On a class of tests equivalent in the limit to the likelihood ratio tests* by Professor Neyman. The presiding officers of the Symposium sessions were Professor C. C. Craig, Dean M. H. Ingraham, Professors Sewall Wright and A. H. Copeland. The Symposium addresses contributed greatly to the interest of the Society meetings. On Friday evening those attending the meetings were the guests of the University of Notre Dame at a buffet supper.

Titles and cross references to the abstracts of the papers read follow below. Papers 1 to 6 were read Friday afternoon, papers 7 to 12 on Saturday morning, and papers 13 to 20, whose abstract numbers are followed by the letter *t*, were read by title. Dr. G. S. Young was introduced by Professor W. L. Ayres. Paper 8 was read by Professor Nesbitt, 9 by Mr. Reed and 11 by Professor Smith.

1. Maxwell Reade: *Remarks on a paper of Beckenbach.* (Abstract 49-1-40.)

2. P. M. Pepper: *A new method for imbedding theorems.* (Abstract 49-1-76.)

3. A. N. Milgram: *A topologically invariant metric property of simple closed curves.* (Abstract 49-1-98.)

4. L. M. Graves: *Metrisation of weak convergence in Banach spaces.* (Abstract 49-1-96.)

5. G. S. Young: *Sets of axioms for the plane.* Preliminary report. (Abstract 49-1-102.)

6. Karl Menger: *Statistical generalizations of metric geometry.* (Abstract 49-1-75.)

7. Ivan Niven: *An unsolved case of the Waring problem.* (Abstract 49-1-10.)

8. C. J. Nesbitt and W. M. Scott: *Some remarks on algebras over an algebraically closed field.* (Abstract 49-1-9.)

9. Karl Menger and S. G. Reed: *On a surface not intersecting the set* $R_3^1$. (Abstract 49-1-97.)

10. G. M. Ewing: *Minimizing an integral on a class of continuous curves.* (Abstract 49-1-30.)

11. H. F. S. Jonah and A. H. Smith: *Zero order summability of the series conjugate to the derived Fourier series.* (Abstract 49-1-35.)

12. C. E. Springer: *Dual goedesics on a surface.* (Abstract 49-1-80.)

13. M. A. Basoco: *On the Fourier developments of a certain class of theta quotients.* (Abstract 48-11-310-*t*.)

14. Raphael Salem: *On some singular monotonic functions which are strictly increasing.* (Abstract 49-1-41-*t*.)

15. Ivan Niven: *The Pell equation in quadratic fields.* (Abstract 49-1-11-*t*.)

16. Maxwell Reade: *On a theorem of Fédoroff and Binney.* (Abstract 49-1-39-*t*.)

17. J. E. Wilkins: *A class of functions in the calculus of variations for multiple integrals in parametric form.* (Abstract 49-1-52-*t*.)

18. Brockway McMillan: *Networks of mechanisms.* Preliminary report. (Abstract 49-1-68-*t*.)

19. Max Coral: *Solution of quasi-linear partial differential equations through a characteristic initial curve.* (Abstract 49-1-29-*t*.)

20. R. L. Swain: *Approximate isometries in bounded spaces.* (Abstract 49-1-100-*t*.)

W. L. AYRES,
*Associate Secretary*

## THE NOVEMBER MEETING IN LOS ANGELES

The three hundred ninety-second meeting of the American Mathematical Society was held at the University of California at Los Angeles on Saturday, November 28, 1942. The attendance was about sixty, including the following thirty members of the Society:

O. W. Albert, Harry Bateman, Clifford Bell, L. E. Bell, Myrtie Collier, P. H. Daus, G. C. Evans, A. I. Forsythe, G. E. Forsythe, R. D. Gordon, P. G. Hoel, G. H. Hunt, D. H. Hyers, C. G. P. Kuschke, C. C. Lin, W. E. Mason, A. D. Michal, Knox Millsaps, W. T. Puckett, G. E. F. Sherwood, S. C. Snowdon, A. E. Taylor, B. P. Taylor, T. Y. Thomas, S. E. Urner, F. A. Valentine, L. F. Walton, W. M. Whyburn, E. R. Worthington, Max Zorn.

The meeting opened in the morning with a general session for contributed papers at which Professor P. H. Daus presided. By invitation of the Program Committee, Professor Harry Bateman of the California Institute of Technology delivered an hour address on *Some simple differential-difference equations and their related functions*. Professor G. C. Evans presided at this lecture.

The afternoon session was devoted to a Symposium on Applied Mathematics, Professor Theodore von Kármán presiding. Professor P. S. Epstein spoke on *Elastic oscillations in plates and shells*, Professor L. M. K. Boelter on *Problems in heat transfer*, and Professor J. Holmboe on *Some aspects of the circulation of viscous fluids*.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 2 and 7 were read at the morning session; those numbered 1 and 3 to 6 were read by title. Paper 7 was read by Professor Zorn.

1. A. C. Schaeffer: *On the oscillation of differential transforms*. III. (Abstract 48-3-119-*t*.)
2. P. G. Hoel: *On indices of dispersion*. (Abstract 49-1-88.)
3. Glenn James: *On equivalence of methods of summing divergent series*. (Abstract 49-1-34-*t*.)
4. J. E. Wilkins: *A note on the Weierstrass condition for multiple integrals in the calculus of variations*. (Abstract 49-1-53-*t*.)
5. Knox Millsaps: *Characterization of the abstract exponential function*. (Abstract 49-1-36-*t*.)
6. E. J. Purcell: *Flat space congruences of order one in* [*n*]. (Abstract 49-1-77-*t*.)
7. Einar Hille and Max Zorn: *Open additive semi-groups of complex numbers*. (Abstract 49-1-32.)

A. D. Michal,
*Associate Secretary*

# THE ANNUAL MEETING OF THE SOCIETY

In compliance with the request of the Office of Defense Transportation, the scientific program of the forty-ninth Annual Meeting of the American Mathematical Society, which was scheduled for December 28–30, 1942, in New York City, was cancelled. Therefore the Annual Meeting consisted of the meetings of the Council and the Board of Trustees which were held on December 27 and 28, 1942, in New York City, and at which the following members of the Society were present:

C. R. Adams, W. D. Cairns, W. B. Carver, Arnold Dresden, Nelson Dunford, Tomlinson Fort, R. M. Foster, B. P. Gill, Einar Hille, T. R. Hollcroft, E. M. Hull, J. R. Kline, Solomon Lefschetz, W. R. Longley, C. C. MacDuffee, Saunders MacLane, Marston Morse, G. W. Mullins, F. D. Murnaghan, R. G. D. Richardson, J. F. Ritt, J. D. Tamarkin, A. W. Tucker, Warren Weaver, S. S. Wilks, Oscar Zariski.

At the request of the authors, some of the papers submitted for presentation at the Annual Meeting were listed for presentation in person at future meetings of the Society in New York and Chicago. All other papers which had been submitted were read by title at the meeting of the Council; titles and cross references to the abstracts of these papers appear at the end of this report. At the meeting of the Council the Secretary reported that the Gibbs Lecture, scheduled to be delivered by Professor John von Neumann at the 1942 Annual Meeting, has been cancelled indefinitely since, because of war duties, the lecturer is not available.

The Board of Trustees met at 2:00 P.M. on December 27 in the Faculty Club of Columbia University and adjourned to reconvene at 1:00 P.M. on December 28. The meeting of the Council was held at 2:00 P.M. on December 27 in the Faculty Club of Columbia University.

The Secretary announced the election of the following twenty-two persons to membership in the Society:

Professor Albert Vinicio Baez, Wagner College, Staten Island, N.Y.;
Miss Helen P. Beard, Newcomb College, Tulane University;
Mr. Edward Griffith Begle, Yale University;
Dr. Isaiah Benjamin, Montreal, Quebec, Canada;
Mr. Albert Hosmer Bowker, Massachusetts Institute of Technology;
Mr. Paul Brock, Princeton University;
Professor James A. Cooley, University of Tennessee;
Mr. Charles Laurie Dolph, Princeton University;
Mr. Robert Dean Gordon, Douglas Aircraft Company, El Segundo, Calif.;
Dean Richard Wilson Horn, Dayton Y.M.C.A. College;
Mr. Bertram Kelsey Hovey, University of Pittsburgh;
Mr. Leroy M. Kelly, U. S. Coast Guard Academy;

Dr. Richard Milton Martin, Princeton University;
Dr. Anne Frances O'Neill, Smith College,
Mr. Charles H. Papas, Aircraft Radio Section, U. S. Navy Department, Washington, D.C.;
Mr. Sidney G. Reed, Jr., University of Notre Dame;
Mr. Edward Rosenthall, McGill University;
Mr. Robert Harris Scanlan, Fleetwings, Inc., Bristol, Pa.;
Mr. Peter Treuenfels, Brooklyn, N.Y.;
Mr. John Michael Walsh, University of Notre Dame,
Miss Rae Wyland, Eastman Kodak Company, Rochester, N.Y.;
Mr. Bertram Yood, Bayonne, N.J.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

Brown University: Dr. Lipman Bers. Messrs. Wilfred Macdonald Kincaid and Elmer Beaumont Tolsted.
University of California at Los Angeles: Messrs. Marshall Elder and Melvin Samuel Henry.
University of Chicago: Mr. Alexander Robert Jacoby, Miss Janet MacDonald, Mr. William S. Massey, Miss Mary Kathryn Toft, Mr. Daniel Zelinsky.
University of Cincinnati: Mr. David Hal Lipsich.
College of the City of New York: Mr. Herman Chernoff.
Columbia University: Messrs. Frederick Bagemihl, Bernard Russell Gelbaum, Pierre Gutmann, Miss Agnes P. Holló, Miss Louise Wilhelmina Miller, Mr. F. Steinhardt.
Duke University: Miss Eva Ann Pirkle.
University of Illinois: Miss Mary Rosalie Bear, Dr. Theodore Bedrick, Messrs. Harry Edward Cowan, Harman Leon Harter, and Ray Gartner Langebartel, Miss Jewell E. Schubert.
Indiana University: Mr. Evar Dare Nering, Miss Valentina Annette Potor.
Iowa State College: Mr. Theodore Alfonso Bancroft.
State University of Iowa: Miss Winifred Alice Asprey.
Massachusetts Institute of Technology: Messrs. Julian Himely Bigelow, Donald Laurence Thomsen, Jr., and Edward Joseph Zadina.
University of Michigan: Messrs. Lynn Ulfred Albers, Carl Allen Bennett, Leonard Gustave Johnson, and Clarence Francis Stephens.
University of Pennsylvania: Mr. Samuel Isadore Askovitz, Mrs. Ruth Eileen Goodman.
Smith College: Miss Kathleen Ethelwyn Butcher.
Syracuse University: Miss Lucille Kathryn Pinette.
Texas Technological College: Miss Nancy Ann Miller.
University of Virginia: Mr. Marion Kirkland Fort, Jr.
University of Washington: Mr. Worthie Lefler Doyle, Jr., Miss Frances Adelia Owen.

The following appointments by President Marston Morse were reported: as tellers for the 1942 annual election, Professor E. R. Lorch and Dr. W. C. Strodt; as Committee on Arrangements for the 1943 Annual Meeting in Cleveland, Ohio, Professors J. R. Musselman (Chairman), W. L. Ayres, M. G. Boyce, O. E. Brown, W. B. Carver,

E. H. Clarke, and C. C. Torrance; as new members of the Committees to Select Hour Speakers: for Annual and Summer Meetings, Professor R. L. Wilder (committee now consists of Professors J. R. Kline, Chairman, M. H. Stone, and R. L. Wilder); for Eastern Sectional Meetings, Professor Saunders MacLane (committee now consists of Professors T. R. Hollcroft, Chairman, Salomon Bochner, and Saunders MacLane); for Western Sectional Meetings, Professor R. E. Langer (committee now consists of Professors W. L. Ayres, Chairman, T. H. Hildebrandt, and R. E. Langer); for Far Western Sectional Meetings, Professor W. M. Whyburn (committee now consists of Professors A. D. Michal, Chairman, G. C. Evans, and W. M. Whyburn).

The Secretary reported that the ordinary membership in the Society is now 2528, including 213 nominees of institutional members and 71 life members. There are also 85 institutional members. The total attendance at all meetings in 1942 was 1010; the number of papers read was 319; there were 6 hour addresses, 20 symposium addresses, and 4 Colloquium Lectures; the number of members attending at least one meeting was 641.

At the annual election which closed on December 30, and at which 455 votes were cast, the following officers were elected:

*President*, Professor M. H. Stone.

*Vice-President*, Professor L. M. Graves.

*Secretary*, Professor J. R. Kline.

*Associate Secretary*, Professor T. R. Hollcroft.

*Treasurer*, Professor B. P. Gill.

*Members of the Editorial Committee of the Bulletin*, Professors Saunders MacLane and P. A. Smith.

*Member of the Editorial Committee of the Transactions*, Professor A. A. Albert.

*Members of the Editorial Committee of the Colloquium Publications*, Professors A. B. Coble and J. F. Ritt.

*Member of the Editorial Committee of Mathematical Reviews*, Professor O. E. Neugebauer.

*Member of the Editorial Committee of the American Journal of Mathematics*, Professor G. D. Birkhoff.

*Members-at-large of the Council*, Professors R. P. Agnew, E. T. Bell, Richard Courant, E. R. Hedrick, and D. H. Lehmer.

*Members of the Board of Trustees*, Professors W. R. Longley, Marston Morse, and G. W. Mullins, Dean R. G. D. Richardson, and Dr. Warren Weaver.

In an appendix to this report are excerpts from the Report of the

Treasurer for the fiscal year 1942 as verified by the Auditors. A copy of the complete report will be sent, on request, to any member of the Society. The Board of Trustees adopted a budget for 1943 showing authorized expenditures of $49,400. This includes an appropriation of $1,000 for Mathematical Reviews from the Marion Reilly Fund.

The Librarian reported that the Library of the Society now contains 10,235 volumes of which 7,344 are bound volumes.

The American Journal of Mathematics, which is published jointly by The Johns Hopkins University and the Society, and to which the Society contributes an annual subvention of $2,500, printed 772 pages during 1942.

Certain invitations to give hour addresses were announced: Professors R. H. Cameron and Nelson Dunford for the April, 1943, meeting in New York City; Professor A. E. Taylor for the April, 1943, meeting at Stanford University.

On recommendation of the Committee on Places of Meetings (Professor A. B. Coble, Chairman), it was voted to accept the invitation of Rutgers University to hold the 1943 Summer Meeting at that institution. It was also voted to accept the invitation of Hunter College for the April, 1943, meeting in New York City. The Secretary reported that, according to the Office of Defense Transportation, the final decision regarding scientific meetings lies with the officers of the group involved; the officers are requested to weigh the necessity of each meeting against the existing transportation situation.

The Bulletin Editorial Committee reported that Professors J. H. Curtiss and D. H. Lehmer had been appointed Assistant Editors, for a period of three years, replacing Professors J. L. Doob and Morgan Ward. It was also reported that, beginning in January, 1943, invited addresses will be printed in the green issues of the Bulletin.

The Transactions Editorial Committee reported that during the year 1942 it had published 1,192 pages, the largest figure in its history. As a consequence, the heavy backlog of papers has been reduced and the interval between submission and publication has been diminished from fourteen to six or eight months. It was also reported that Professors Richard Brauer, M. S. Knebelman, and J. J. Stoker have been appointed Associate Editors, to replace Professors A. A. Albert, J. L. Synge, and G. T. Whyburn.

On recommendation of the Council, the Board of Trustees set aside a revolving fund for the purpose of publishing the new series of monographs, Mathematical Surveys. It is expected that the first two volumes of this series will appear during the first half of 1943. These will be *Rings* by Professor Nathan Jacobson and *The problem of moments* by Professors J. A. Shohat and J. D. Tamarkin.

As a result of discussions concerning the manpower problem confronting the mathematicians, President Stone was authorized and requested to appoint a War Policy Committee to handle all problems which concern the most effective use of mathematicians in the war effort. He was also authorized to appoint a subcommittee to study means for alleviating the serious shortage of teachers of mathematics which will confront colleges and universities upon the inauguration of the Army and Navy programs.

Titles and cross references to the abstracts of papers read at the Council meeting follow below. Mr. Edward Rosenthall was introduced by Professor Gordon Pall.

1. R. P. Agnew: *Euler transformations.* (Abstract 49-1-21-*t*.)

2. R. P. Agnew: *On sequences with vanishing even or odd differences.* (Abstract 49-1-22-*t*.)

3. R. F. Arens: *Homeomorphism groups of a space.* Preliminary report. (Abstract 49-1-89-*t*.)

4. I. A. Barnett: *A note on skew-symmetric kernels.* (Abstract 49-1-23-*t*.)

5. E. F. Beckenbach: *The stronger form of Cauchy's integral theorem.* (Abstract 49-1-24-*t*.)

6. Stefan Bergman: *A formula for the stream function of compressible fluid flow.* (Abstract 49-1-58-*t*.)

7. G. D. Birkhoff: *Measure-preserving transformations of a planar ring without planar periodic points.* (Abstract 49-1-90-*t*.)

8. D. G. Bourgin: *On a theorem of Goldstine's.* (Abstract 49-1-91-*t*.)

9. D. G. Bourgin: *Quasi norms in linear topological spaces.* (Abstract 49-1-92-*t*.)

10. D. E. Christie: *Net homotopy for compacta.* (Abstract 49-1-93-*t*.)

11. R. F. Clippinger: *General remarks about the set of products of positive powers of n-by-n matrices and the associated manifold.* Preliminary report. (Abstract 49-1-26-*t*.)

12. R. F. Clippinger: *Matrix products of matrix powers.* Preliminary report. (Abstract 49-1-27-*t*.)

13. R. F. Clippinger: *Mean value theorems for a certain linear matrix differential equation.* Preliminary report. (Abstract 49-1-28-*t*.)

14. I. S. Cohen: *Some theorems on local rings.* (Abstract 49-1-1-*t*.)

15. H. S. M. Coxeter: *The map-coloring of unorientable surfaces.* (Abstract 49-1-94-*t*.)

16. J. H. Curtiss: *A note on the theory of moment generating functions.* (Abstract 49-1-85-*t*.)

17. John DeCicco: *Conformal geometry of second order differential equations.* (Abstract 49-1-70-*t*.)

18. Jesse Douglas: *Point transformations and isothermal families of curves.* (Abstract 49-1-71-*t*.)

19. R. M. Foster: *On the average resistance of an electrical network.* (Abstract 49-1-59-*t*.)

20. A. H. Fox: *Integral representation of the flow of a compressible fluid around a cylinder.* (Abstract 49-1-60-*t*.)

21. P. R. Halmos: *On automorphisms of compact groups.* I. (Abstract 49-1-3-*t*.)

22. A. E. Heins: *Some remarks on the solution of dual integral equations.* I. (Abstract 49-1-63-*t*.)

23. H. K. Hughes and Cleota G. Fry: *Asymptotic developments of certain integral functions.* (Abstract 49-1-33-*t*.)

24. Edward Kasner: *Trajectories in a resisting medium.* (Abstract 49-1-66-*t*.)

25. Edward Kasner and John DeCicco: *Bi-isothermal systems in pseudo-conformal geometry.* (Abstract 49-1-73-*t*.)

26. Jakob Levitzki: *On semi-nilpotent ideals.* (Abstract 49-1-6-*t*.)

27. W. H. Roever: *A new formula for the deviation in range of a projectile due to the earth's rotation.* (Abstract 49-1-69-*t*.)

28. W. H. Roever: *The axonometric method of representing the points of space on a plane.* (Abstract 49-1-78-*t*.)

29. Edward Rosenthall: *Diophantine equations in arbitrary algebraic number fields.* (Abstract 49-1-13-*t*.)

30. Edward Rosenthall: *Diophantine equations in biquadratic fields.* (Abstract 49-1-14-*t*.)

31. H. E. Salzer and Abraham Hillman: *Exact values of the first 120 factorials.* (Abstract 49-1-84-*t*.)

32. H. M. Schwartz: *On sequences of Stieltjes integrals.* II. (Abstract 49-1-42-*t*.)

33. I. M. Sheffer: *Note on a linear transformation of "analytic" type.* (Abstract 49-1-43-*t*.)

34. W. S. Snyder: *Non-parametric surfaces and inscribed polyhedra.* Preliminary report. (Abstract 49-1-44-*t*.)

35. Otto Szász: *On Abel and Lebesgue summability.* (Abstract 49-1-46-*t*.)

36. W. C. Taylor: *Asymptotic behavior of the Abel sums of the Laguerre expansion.* (Abstract 49-1-47-*t*.)

37. W. J. Trjitzinsky: *Singular nonlinear integral equations.* (Abstract 49-1-48-*t*.)

38. S. M. Ulam: *On equivalence of functions.* (Abstract 49-1-49-*t*.)

39. S. M. Ulam: *On the length of curves, the surface area and the isoperimetric problem under a general Minkowski metric.* Preliminary report. (Abstract 49-1-81-*t*.)

40. T. L. Wade: *Euclidean concomitants of the triangle.* (Abstract 49-1-15-*t*.)

41. T. L. Wade: *On conjugate tensors.* (Abstract 49-1-16-*t*.)

42. T. L. Wade: *On the factorization of rank tensors.* (Abstract 49-1-17-*t*.)

43. T. L. Wade and R. H. Bruck: *Types of symmetries.* (Abstract 49-1-18-*t*.)

44. A. D. Wallace: *On non-alternating transformations.* (Abstract 49-1-101-*t*.)

45. Alexander Weinstein: *On a general variational method for the determination of eigenvalues.* (Abstract 49-1-51-*t*.)

46. J. E. Wilkins: *Definitely self-conjugate adjoint integral equations.* (Abstract 49-1-54-*t*.)

47. Fumio Yagi: *On a certain Stieltjes integral equation.* (Abstract 49-1-55-*t*.)

48. Antoni Zygmund: *A property of the zeros of Legendre polynomials.* (Abstract 49-1-57-*t*.)

J. R. KLINE,
*Secretary*

# APPENDIX

## EXCERPTS FROM REPORT OF TREASURER

December 16, 1942

TO THE BOARD OF TRUSTEES OF THE
AMERICAN MATHEMATICAL SOCIETY

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1942, in form similar to that for the previous year.

The market value on November 30, 1942, of the securities in the Pool was $12,714.63 less than book value. This is to be compared with the total amount, $7,724.46, held in accounts "Reserve for Investment Losses" and "Profit and Loss on Sale of Securities." The market value on November 30, 1942, of securities in General and Restricted Funds was $1,459.25 less than book value. Profit on the sale of securities for General and Restricted Funds amounting to $295.73 is still being carried as a reserve against losses to these funds.

Income earned by General Fund Investments during the year amounted to $1,414.34. This represents a return of 1⅛% on amounts in savings banks, and approximately 3⅞% on securities. Investment Income earned by the Pool amounted to $7,380.12, representing a return of approximately 4½%. Total Investment Income from all sources was thus $8,794.46, corresponding to a yield of approximately 4%. This income and yield may be compared with $8,864.93 and approximately 4⅜% for the fiscal year 1941.

There appears for the first time a record of income from the Estate of Robert Henderson. The terms of Dr. Henderson's will have earlier been reported to the Trustees and to the Society. There also appears the gift of $100 from Professor R. C. Archibald for use of the library.

Respectfully submitted,

BENNINGTON P. GILL, *Treasurer*

## BALANCE SHEET

### *Assets*

| | November 30, 1942 | November 30, 1941 |
|---|---|---|
| GENERAL AND RESTRICTED FUND ASSETS: | | |
| Cash | $ 19,177.78 | $ 19,380.05 |
| Investments | 33,470.47 | 28,814.24 |
| | $ 52,648.25 | $ 48,194.29 |
| ENDOWMENT AND OTHER NONEXPENDABLE FUND ASSETS (POOL): | | |
| Cash | $ 1,409.62 | $ 1,156.70 |
| Investments | 163,195.68 | 160,395.09 |
| | $164,605.30 | $161,551.79 |
| TOTAL ASSETS | $217,253.55 | $209,746.08 |

### *Liabilities*

| | November 30, 1942 | November 30, 1941 |
|---|---|---|
| GENERAL FUNDS: | | |
| Sinking Fund | $ 515.99 | $ 420.09 |
| Profit on Sales of Securities | 295.73 | 295.73 |
| Surplus | 14,201.68 | 9,048.08 |
| | $ 15,013.40 | $ 9,763 90 |
| RESTRICTED FUNDS: | | |
| Prize Funds and other Special Funds Accumulated Income | $ 4,848.52 | $ 5,613 02 |
| Income from Henderson Estate | 2,275.00 | |
| Reilly Fund Principal | 3,968.13 | 3,968.13 |
| Reprinting Funds | 1,329.15 | 788.86 |
| Colloquium | 2,805.04 | 5,187.14 |
| Mathematical Reviews | 16,184.69 | 16,924.45 |
| International Congress | 6,050.33 | 5,943.44 |
| Library—R. C. Archibald Gift | 100.00 | |
| Miscellaneous | 73.99 | 5.35 |
| | $ 37,634.85 | $ 38,430.39 |
| | $ 52,648.25 | $ 48,194 29 |
| ENDOWMENT AND OTHER NONEXPENDABLE FUNDS (POOL): | | |
| Endowment Fund Principal | $ 60,110.68 | $ 60,110.68 |
| Prize Funds and other Special Funds | 26,935.07 | 26,935.07 |
| Life Membership and Subscription Reserve | 4,835.09 | 5,304.08 |
| Colloquium | 5,000 00 | 5,000.00 |
| Mathematical Reviews | 60,000.00 | 60,000.00 |
| Reserve for Investment Losses | 3,537.46 | 2,728.24 |
| Profit on Sales of Securities | 4,187 00 | 1,473.72 |
| | $164,605.30 | $161,551.79 |
| TOTAL LIABILITIES | $217,253.55 | $209,746.08 |

## SUMMARY STATEMENT OF INCOME AND EXPENDITURES

### 1941–1942

| | 1942 | | 1941 | |
|---|---|---|---|---|
| | *Receipts* | *Disbursements* | *Receipts* | *Disbursements* |
| GENERAL RECEIPTS | | | | |
| Dues—Ordinary Memberships | $ 15,681.19 | | $ 14,970.19 | |
| Dues—Contributing Memberships | 1,054.00 | | 1,040.00 | |
| Dues—Institutional Memberships | 7,347.35 | | 6,468.31 | |
| Initiation Fees | 665.00 | | 515.25 | |
| Investment Income | 4,643.14 | | 4,147.27 | |
| Miscellaneous | 34.73 | | 343.15 | |
| GENERAL DISBURSEMENTS | | | | |
| Secretaries | | $ 5,136.86 | | $ 6,091.29 |
| Treasurer | | 794.36 | | 750.00 |
| Officers' Traveling | | 643.61 | | 888.93 |
| Library | | 569.95 | | 824.50 |
| Committee Expenses | | 67.92 | | 101.04 |
| Furniture and Fixtures | | | | 51.70 |
| Emergency | | 5.71 | | 277.29 |
| Total | $ 29,425.41 | $ 7,218.41 | $ 27,484.17 | $ 8,984.75 |
| Excess of General Receipts | $ 22,207.00 | | $ 18,499.42 | |
| PUBLICATION | | | | |
| Bulletin | $ 1,673.13 | $12,101.37 | $ 1,755.47 | $11,037.14 |
| Bulletin Reprinting | 467.30 | | 227.62 | |
| Transactions | 5,787.17 | 9,373.59 | 4,816.11 | 8,968.10 |
| Transactions Reprinting | 274.58 | 201.59 | | |
| Colloquium | 3,814.97 | 6,197.07 | 3,437.74 | 2,969.55 |
| Mathematical Reviews | 10,809.17 | 11,548.93 | 21,467.56 | 12,969.22 |
| Semicentennial Publications | 95.20 | | 129.29 | |
| American Journal | | 2,500.00 | | 2,500.00 |
| Total | $22,922.22 | $41,922.55 | $ 31,833.79 | $38,444.01 |
| Excess of Cost of Publication | | $19,000.33 | | $ 6,610.22 |
| MISCELLANEOUS | | | | |
| Special Funds (including Congress) | $ 1,442.39 | $ 2,100.00 | $9,526.51[1] | $ 1,000.00 |
| Profit on Sales of Securities | 2,713.28 | | 103.72 | |
| Income from Henderson Estate | 2,275.00 | | | |
| Gift from R. C. Archibald | 100.00 | | | |
| Transactions Reprinting from Surplus | | 100.00 | | |
| Member-Customer Items—not allocated | 73.99 | 103.86 | | |
| Total | $ 6,604.66 | $ 2,303.86 | $ 9,630.23 | $ 1,000.00 |
| Difference | $ 4,300.80 | | $ 8,630.23 | |
| Net Change in Assets | $ 7,507.47 | | $ 20,519.43 | |
| ASSETS BEGINNING OF YEAR | $209,746.08 | | $189,226.65 | |
| ASSETS END OF YEAR | $217,253.55 | | $209,746.08 | |

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[1] Includes $7,738.78 principal of gifts.

# BOOK REVIEWS

*Algebraic topology.* By Solomon Lefschetz. (American Mathematical Society Colloquium Publications, vol. 27.) New York, American Mathematical Society, 1942. 6+389 pp. $6.00.

Since the publication of Lefschetz's *Topology* (Amer. Math. Soc. Colloquium Publications, vol. 12, 1930; referred to below as (L)) three major advances have influenced algebraic topology: the development of an abstract complex independent of the geometric simplex, the Pontrjagin duality theorem for abelian topological groups, and the method of Čech for treating the homology theory of topological spaces by systems of "nerves" each of which is an abstract complex. The results of (L), very materially added to both by incorporation of subsequent published work and by new theorems of the author's, are here completely recast and unified in terms of these new techniques. A high degree of generality is postulated from the outset. The abstract point of view with its concomitant formalism permits succinct, precise presentation of definitions and proofs. Examples are sparingly given, mostly of a simple kind, which, as they do not partake of the scope of the corresponding text, should be intelligible to an elementary student. But this is primarily a book for the mature reader, in which he can find the theorems of algebraic topology welded into a logically coherent whole.

The first chapter presents the set-theoretic considerations which will underlie both the spaces to be studied later and the algebraic machinery used to study them. Topological spaces are defined as point sets in which open sets are specified subject to the three usual axioms. Mappings (that is, continuous single-valued transformations) are next defined so that their properties may be developed along with those of the space. By using topological products of open or of closed line segments, Euclidean $n$-space, the $n$-cell and the Hilbert parallelotope are introduced. A space is called *compact* if for every covering $\{U_a\}$ by open sets there exists a finite subset of $\{U_a\}$ which covers the space. This is the property often called "bicompactness." The principal consequences of the separation axioms particularly for compact spaces precede a definition of *normality* and *characteristic function* and the Tychonoff imbedding theorem, which, via metric spaces, leads to the Urysohn metrization theorems. A set is a *directed system* if there is a relation $>$ between certain pairs $a,b$ of its elements such that $c>b$ and $b>a$ implies $c>a$, and for every pair $a,b$ of elements there is an element $c$ such that $c>a$ and $c>b$. Using these, inverse mapping systems of Hausdorff spaces and their limit spaces are defined. The chapter ends with a definition of *homotopy*, *deformation* and *retraction*.

Abelian topological groups occupy the next chapter. The standard decomposition of a group with a finite number of generators is given. The chapter as a whole leads up to duality theorems. The first is that of Pontrjagin-van Kampen which is stated for locally compact groups $G$ and $H$ one of which is the character group of the other, and is proved in case $G$ is compact, $H$ discrete (except that reference to the literature is given for the step $G \ni g \neq 0$ implies the existence of $h \in H$ such that $gh \neq 0$). Application is made to dual directed systems of groups. If a set $\Omega$ of continuous homomorphisms $\alpha$ of a group $G$ into itself forms a field (taken with discrete topology) in the sense that $\alpha g + \alpha' g = (\alpha + \alpha')g$ and $(\alpha\alpha')g = \alpha(\alpha' g)$, then $G$ is called a *vector space* and has *linear* topology if every neighborhood of the identity of $G$ is a sum of subspaces. If $G$ is finite-dimensional, linear reduces to discrete topology. When a *field character* of $G$ is defined as a homomorphism $G \to \Omega$, the field characters form a vector space $H$ called the *character space* of $G$. In case $G$ and $H$ have linear topology and are restricted by a condition analogous to compactness, a second duality theorem, analogous to the first, is proved for them.

In Chapter III, Lefschetz defines as a *complex* an abstract system based on the "abstract complex" of Tucker except that the use of negative dimensions here permits the association with each complex $X$ of a symmetric dual complex $X^*$ which carries its cohomology theory. Geometric entities like *polyhedral complexes* (made up of bounded convex regions) and *Euclidean complexes* (made up of simplexes) can then be treated as special cases. Homology and cohomology theory with general coefficient groups $G$ (in case $G$ is a discrete field the homology groups are vector spaces) and product theory are developed abstractly both for finite and for infinite complexes having local finiteness properties. It is shown, following Steenrod, that two finite complexes have the same homology groups over every $G$ if they have the same groups for either of the *universal* coefficient groups: $G$ the integers, or $G$ the reals mod 1. If $G$ is a field of characteristic $\pi$, the Betti numbers over $G$ are the same as those mod $\pi$. Using the group duality of Chapter II gives (1) the duality of the $p$th homology and cohomology groups, and (2) the Alexander duality of the $p$th cohomology group of a closed $X$ and the $(p-1)$st homology group of $Y-X$ when the $p$th and $(p-1)$st homology groups of $Y$ are the identity. After discussion of *product* and *join* of complexes, subdivision is treated by means of the *chain-mapping* induced by the transformation of one complex to another; intersections are defined as chain mappings of the product $X \times X^*$ to $X$ or, equivalently, of $X^* \times X^*$ to $X^*$. In terms of them the homology ring of Gordon and Freudenthal is set up and the theory of coincidences and fixed points is abstractly given. Combinatorial manifold is defined and the previous results are applied in this special case, the result of (1) for finite manifolds being the duality theorem of Poincaré, for infinite manifolds the theorem of (L) p. 314.

The Čech homology theory is next abstracted: the role of Alexandroff "nerves" being played by a directed set of complexes called a *net*, the role of the projection onto a covering of its refinement being taken by a chain-mapping. If whenever there is a projection it is unique, the net is called a *spectrum*, so the spectrum is the direct abstraction of the Čech situation, the net being a generalization. For a fixed dimension the homology group $\mathfrak{H}$ of a net is then the limit group of the homology groups of that dimension of the complexes of the net. The duals of these complexes yield a *conet* and cohomology groups which permit the treatment of intersections. A duality theorem like (1) is immediate; one like (2) is proved when the coefficient group is compact or a field. For a spectrum the limit groups $Z$ of the groups of cycles and $F$ of the groups of bounding cycles of the complexes of the spectrum may be defined leading to a *projective homology group* $Z/\overline{F}$ for each dimension. (The bar is the closure operator.) If the coefficient group is compact or a field, $\mathfrak{H} \approx Z/\overline{F}$. A *web* is a collection of objects, $A$, $B$, $\cdots$ between certain pairs of which a relation $\supset$ of "inclusion" has been established such that to each $A,B$ there is a $C \supset A,B$, and a $D$ with $A,B \supset D$. Thus from a web two directed sets may be formed; one *direct*, $A > B$ means $A \supset B$; one *inverse*, $A > B$ means $B \supset A$. Here the elements of the web are the subcomplexes of a complex or, more generally, the subnets of a net, and by means of the directed sets, homology theories are constructed for these webs. The principal application is to an infinite complex each of whose elements has a finite set of elements on its boundary and in which each element has a *diameter*. Then if $A,B, \cdots$ are subcomplexes made up of elements of decreasing maximum diameters one of the homology theories of the resulting web abstracts the Vietoris homology theory.

Chapters VII and VIII apply the algebraic machinery to spaces, nets being used to define homology groups and prove duality and intersection theorems for topological spaces in the manner of Čech. If the space is normal it makes no difference whether the nets are composed of nerves of open or of closed sets. Similarly webs serve for the Vietoris-Lefschetz homology theory of compact metric spaces

(L) p. 322. It is shown that the Čech and Vietoris homology groups over a discrete group $G$ are isomorphic. Special finite coverings of a topological space by closed sets whose interiors are disjunct, called *gratings*, lead to a net which is a spectrum and whose net homology theory is the same as the Kurosch homology theory by finite closed coverings. The projective theory of this spectrum is the homology theory of Alexander-Kolmogoroff. In Chapter VIII the topological space is specialized to be a polyhedral or simplicial complex $K$ and the covering to be by barycentric stars of the derived (that is, regularly subdivided) complexes of $K$. A proof of the topological invariance of the algebraic homology groups of $K$ then quickly results from the Čech theory. The manifold, intersection and fixed point theories given earlier are specialized to this case and a discussion of the singular chains which played such a large part in (L) and of continuous chains is included. Finally differentiable complexes and group manifolds are discussed. Hopf by generalizing simplicial group manifolds (group not necessarily abelian) defined a $\Gamma$-manifold. Lefschetz further generalizes to obtain a $\Gamma$-complex and proves Hopf's theorems for it: the rational homology groups and ring of a $\Gamma$-complex are isomorphic with those of a finite product of odd-dimensional spheres; and every finite product of odd-dimensional spheres is a $\Gamma$-manifold.

In Appendix A by S. Eilenberg and Saunders MacLane are proved for infinite complexes results on universal coefficient groups similar to those of Chapter III for finite complexes. The group of group extensions of a given group by another is the algebraic tool used. In Appendix B, P. A. Smith describes his application of algebraic topology to the study of the fixed points of a periodic homeomorphism $T$ of a topological space $R$ into itself. His technique is first worked out for $R$ a simplicial complex and $T$ a simplicial homeomorphism. The algebra is that of special homology groups defined by means of $T$. Then by the Čech method it is extended to compact spaces, particularly those having the homology groups of the $n$-sphere, and there yields topological results. Some unsolved problems are described at the end of this appendix.

WILLIAM W. FLEXNER

*Principles of mechanics.* By J. L. Synge and B. A. Griffith. New York and London, McGraw-Hill, 1942. 12+514 pp. $4.50.

In their preface the authors state that mechanics stands out as a model of clarity among all the theories of deductive science, and they have succeeded very well in support of that statement in the production of this book. The notation and the arrangement are good, and the scope of material covered exhibits well the methods and limitations of modern mechanics.

The book is divided into two parts, the first being entitled Plane Mechanics and the second Mechanics in Space. There is, however, in Part I considerable three-dimensional notation, the policy of the authors being to include the three-dimensional generality when undue complexity does not arise.

Vector methods are used throughout Part II as the fundamental mode of approach both to theory and to problems. They are also used in Part I, although in many of the problems of plane mechanics there is not scope for the full advantage of vector notation, and scalars are used a great deal instead.

The arrangements in the two parts to a large extent parallel each other. In each case statics precedes dynamics. The force concept is taken as the primitive one, rather than mass, and the principles of statics are developed through the principle of virtual work. Then in each case there follow chapters on the methods of dynamics and their applications. In Part II these are followed by a chapter on Lagrange's equations, and one on the special theory of relativity. These last two chapters are well adapted to the purpose of providing the student with an indication of what lies beyond the methods which have been given, and of the limitations of the postulates on which the theoretical structure of Newtonian mechanics rests.

The first chapter of Part I is entitled Foundations of Mechanics. It deals with the fundamental concepts and postulates of mechanics, but to appreciate it adequately the student needs a considerable degree of experience and maturity concerning things physical. The mass idea is illustrated by an analogy, but the amount of definite physical basis suggested for the idea is meager. In fact, the emphasis of the book is not on the development of an understanding of the fundamental concepts and postulates; a working knowledge of the fundamental ideas is presented in condensed form, chiefly in this first chapter, and then the development of the logical structure of mechanics is proceeded with. That, of course, is a legitimate choice, for it makes possible the inclusion of a greater body of theory and method in a single volume of moderate size. It makes the text unsuitable for a beginners' course, but it provides a valuable body of material for more advanced work.

There are, however, some aspects of the book to which the reviewer takes exception. A vector is defined (p. 18) as a directed segment, or any physical quantity which can be represented by a directed segment. The parallelogram law is regarded as giving the mathematical sum of two vectors, irrespective of whether it gives their equivalent when they represent physical quantities, but the theory of vectors developed is applied to physical problems. A finite rotation is explicitly classed as a vector, although it is admitted that it does not obey the parallelogram law. That forces obey the parallelogram law is regarded as an axiom, but there is no mention as to whether other specific quantities obey it. The general statement is made (p. 20) that almost all physical vectors do obey the parallelogram law but there is no justification given for the statement. This seems to be a logical weakness.

On p. 88 there is an erroneous statement, namely that "$\mu$ (the coefficient of static friction) is always less than unity." This would mean that no body or material could repose on a plane inclined to the horizontal at an angle greater than 45°, which is obviously not true.

Then in the chapters on impulsive motion there is introduced some artificial and unsatisfactory terminology. It is true that there is eminent precedent for what is done (see, for example, Lamb's *Hydrodynamics*, p. 11), but nevertheless it is open to objection. In the momentum-impulse equation

$$\Delta(m\dot{x}) = \int_{t_0}^{t_1} X dt,$$

$t_1$ is allowed to approach $t_0$ and $X$ to become infinite, the integral remaining constant. There is the statement (p. 226) that "in this limiting case of 'an infinite force acting for an infinitesimal time' there is an instantaneous change in velocity but no change in position." Later, on the same page: "no force, however large, can produce an instantaneous change in momentum." Then: "To place our new ideas on a secure foundation, we admit the concept of an ***impulsive force.***" This impulsive force has a component $\hat{X}$ given by

$$\hat{X} = \lim_{t_1 \to t_0} \int_{t_0}^{t_1} X dt,$$

and then it is stated that the impulsive force causes an instantaneous change in momentum given by

$$\Delta(m\dot{x}) = \hat{X}.$$

But the calling of $\hat{X}$ an impulsive force is unsatisfactory. This quantity is not a special kind of *force*; it is a different kind of physical quantity. The loose use of terms on this topic is conducive to confusion of ideas, and is in contrast with the general quality of the book.

At the end of the book there is a well written appendix on the theory of dimensions, but it seems unfortunate that the principles of it are not more in evidence throughout the text. There is one reference in it to the principle of significant figures. The latter is referred to as the physical way of thinking, in contrast with the mathematical way. That may be, but the main object of mechanics is to develop a theoretical structure which will describe certain aspects of nature, and the principle of significant figures has broader application than the rounding off of digits in a numerical calculation. It is a principle that is of frequent use in assessing relative values.

Thus, on p. 17 the authors express a preference for the tension of a spring rather than the weight of a body as a basis for the definition of force measurement, since the former is constant while the latter is variable. They state: "If the weight of a body is measured by means of a calibrated spring at two different latitudes, the results are not the same." But they are probably referring to an ideal calibrated spring, for it is doubtful if the best calibrated spring in use in any laboratory would detect a difference. To a beginner the testing of this statement by experiment would not be very convincing. Practice and theory with regard to the gravitational and absolute measure of force are justified with clarity by the principle of significant figures.

The momentum impulse equation is best interpreted by the same principle. The time interval does not *reach* the limit zero to make the change of momentum instantaneous, and the displacement is not zero but negligible.

In spite of these defects the general organization of the book is good, in both material and presentation. The printing is well done, and is remarkably free from typographical errors.

J. W. CAMPBELL

# NOTES

The editorial committee of the Bulletin has in the past adhered to an upper limit of ten printed pages on the length of papers published in the Bulletin, except in the case of invited addresses. In the future no fixed upper limit will be maintained. The Council of the Society has approved the general policy that the Bulletin should be reserved for papers of more general interest and the shorter notes, while the Transactions should have the deeper and longer research papers which would be read particularly by specialists. In some cases it is anticipated that a paper submitted to one of the Society's journals may be transferred to the other, particularly when it is of intermediate length, and publication may thereby be hastened. In view of the extreme demands for space in the mathematical journals, authors are urged to make their articles as concise as is consistent with clarity. In some cases consideration should be given to the possibility of combining several short notes into one memoir.

The National Research Council has announced a new quarterly publication, *Mathematical tables and aids to computation*, edited by Professor R. C. Archibald of Brown University. The publication is to serve as a clearing-house for information concerning mathematical tables and other aids to computation. The chief parts of each issue will normally be devoted to the following topics: articles, recent mathematical tables (critical reviews), mathematical tables—errata, unpublished mathematical tables, mechanical aids to computation, notes, queries, queries-replies. The price will be three dollars a year, one dollar a number. For subscriptions and further information address Division of Physical Sciences, The National Research Council, 2101 Constitution Avenue, Washington, D. C.

Professor O. P. Akers of Allegheny College has retired.

Associate Professor J. V. Atanasoff of Iowa State College has been promoted to a professorship of mathematical physics.

Mr. O. E. Bennett has been appointed to an assistant professorship at Washington College, Chestertown, Maryland.

Assistant Professor Herman Betz of the University of Missouri has been promoted to an associate professorship.

Professor J. H. Butchart of William Woods College, Fulton, Missouri, has been appointed to an assistant professorship at Grinnell College.

Mr. C. L. Buxton of Clarkson College of Technology has been promoted to an assistant professorship.

Associate Professor C. S. Carlson of St. Olaf College, St. Paul, Minnesota, has been promoted to a professorship.

Dr. G. P. Cowan of St. John's University has retired.

Dr. J. M. Dobbie of Northwestern University has been promoted to an assistant professorship.

Dr. Jesse Douglas has been appointed to an assistant professorship at Brooklyn College.

Dr. J. W. Givens of Northwestern University has been promoted to an assistant professorship.

Assistant Professor H. S. Kaltenborn of Louisiana Polytechnic Institute has been promoted to an associate professorship.

Dr. H. L. Lee of the University of Tennessee has been promoted to an assistant professorship.

Mrs. D. H. Lehmer and Mrs. C. D. Shane have been appointed to lectureships in mathematics at the University of California.

Associate Professor A. N. Lowan of Yeshiva College has been promoted to a professorship.

Professor W. A. Manning of Stanford University has retired.

Mr. E. C. Molina of the Bell Telephone Laboratories has retired.

Dr. R. H. Moorman of Tennessee Polytechnic Institute has been promoted to an assistant professorship.

Dr. J. M. H. Olmsted of the University of Minnesota has been promoted to an assistant professorship.

Assistant Professor L. D. Rodabaugh of Butler University has been appointed to a professorship at Shurtleff College, Alton, Illinois.

Assistant Professor Helen G. Russell of Wellesley College is on leave for the academic year 1942–1943 and is at the University of California.

Dr. P. C. Scott of Louisiana State University has been promoted to an assistant professorship.

Dr. Abraham Spitzbart of the University of Minnesota has been appointed to a professorship at the College of St. Thomas.

Dr. Ellen C. Stokes of the New York State College for Teachers at Albany has been appointed dean of women.

Mr. J. A. Straw of the University of Louisville has been appointed to an assistant professorship at Rose Polytechnic Institute.

Dr. S. H. Taylor has been appointed adjunct professor at the University of South Carolina.

Dr. D. L. Webb of Georgia School of Technology has been appointed to an assistant professorship at Texas Technological College.

Professor D. V. Widder has been appointed chairman of the department of mathematics at Harvard University, succeeding Professor M. H. Stone.

Associate Professor Roscoe Woods is acting head of the department of mathematics at the University of Iowa during the illness of Professor H. L. Rietz.

Dr. E. L. Welker of the University of Illinois has been made an associate.

The following appointments to instructorships are announced: University of Arkansas: Mr. R. V. Simpson; Brown University: Mr. A. F. Bartholomaz, Mr. M. E. Munroe; University of California: Mrs. R. K. Wakerling; University of California at Los Angeles: Dr. R. H. Sorgenfrey; University of Illinois: Mr. J. C. Bell; Iowa State College: Mr. J. W. Beach, Mr. F. E. Bortle, Mr. R. N. Goss; University of Kentucky: Miss Mary H. Cooper, Mr. J. C. Eaves, Miss Lydia R. Fischer; Louisiana State University: Mr. J. E. Pryor; Miami University: Miss Alberta Wolfe; Michigan College of Mining and Technology: Mr. J. C. Butler, Mr. T. R. Richards, Mr. Earl Roberts; State Teachers College, Mankato, Minnesota: Mr. C. J. Kirchen; University of Minnesota: Mr. W. D. Munro; Muskingum College, New Concord, Ohio: Mr. L. C. Knight, Jr.; University of North Carolina: Dr. I. S. Cohen; Northwestern University: Miss Louise M. Sagen; University of South Carolina: Miss Flora Dinkines; University of Southern California: Miss F. Marian Clarke, Mr. E. C. Rex, Mr. Veryl Throckmorton; Agricultural and Mechanical College of Texas: Mr. A. R. Wapple; Texas Technological College: Miss Annie N. Rowland; Tuskegee Institute: Dr. J. E. Wilkins; Williams College: Mr. E. F. Gillette; University of Wisconsin: Dr. D. T. McClay.

Dr. A. E. Mayer of London, England, died November 3, 1942. He was formerly connected with the University of Vienna.

Professor Emeritus Alfred Baker of the University of Toronto died October 27, 1942, at the age of ninety-four years. He had been a member of the Society from 1891 to 1920.

Professor W. M. Carruth of Hamilton College died January 23, 1943. He had been a member of the Society since 1907.

Professor Emeritus C. H. Currier of Brown University died January 5, 1943. He was a member of the Society from 1908 to 1939.

Professor Emeritus I. M. DeLong of the University of Colorado died September 2, 1942. He had been a member of the Society since 1891.

Professor E. L. Dodd of the University of Texas died January 9, 1943. He had been a member of the Society since 1904.

Dean Emeritus G. H. Ling of the University of Saskatchewan died October 21, 1942, at the age of sixty-eight years. He had been a member of the Society since 1894.

Assistant Professor L. I. Neikirk of the University of Washington died December 10, 1942. He had been a member of the Society since 1903.

Professor Emeritus J. H. Scarborough of Central Missouri State Teachers College died November 25, 1942. He had been a member of the Society from 1906 to 1940.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

## ALGEBRA AND THEORY OF NUMBERS

### 103. A. A. Albert: *Algebras derived by non-associative matrix multiplication.*

Let $\mathfrak{C}$ be any $J$-involutorial algebra and define its isotopes $\mathfrak{C}_\rho$, $\mathfrak{C}_\pi$, $\mathfrak{C}_{\pi\rho}$ by $x, y=x(yJ)$, $(x, y)=(xJ)y$, $[x, y]=(xJ)(yJ)$, respectively. When $\mathfrak{C}$ has a unity quantity the algebras $\mathfrak{C}_\rho$, $\mathfrak{C}_\pi$, $\mathfrak{C}_{\pi\rho}$ are all semi-simple if and only if $\mathfrak{C}$ is semi-simple, they are simple if and only if $\mathfrak{C}$ is either simple or a direct sum $\mathfrak{S}\oplus\mathfrak{S}J$ where $\mathfrak{S}$ is simple. If $\mathfrak{N}$ is the radical of $\mathfrak{C}$ then $\mathfrak{N}=\mathfrak{N}J$ forms the radical of the isotopes. The structure of row algebras, that is, subalgebras of $\mathfrak{C}_\rho$ is determined. In particular it is shown that there exist real linear spaces of matrices forming algebras under row by row but not under row by column multiplication. They are never semi-simple. (Received January 26, 1943.)

### 104. Joseph Bowden: *The quaternary permutation function and a generalisation of Newton's binomial theorem and Vandermonde's permutation theorem.*

If $a$ and $d$ are any finite numbers, $m$ any integer, $r$ a primary number and $\rho$ an integer whose elements are the primary numbers $r_1$ and $r_2$, define $(a, d)P(m, r)=\prod_{k=m+1}^{m+r}(a-(k-1)d)$ and $(a, d)P(m, \rho)=(a, d)P(m-r_2, r_1):(a, d)P(m-r_2, r_2)$. From these definitions it follows that $(a, d)P(m, 0)=1$ and $(a, d)P(m, \rho)=:[(a, d)P(m+\rho, -\rho)]$. The operation $P$ is quaternary because it acts on four operands. From the quaternary permutation function $(a, d)P(m, \rho)$ by putting $d=0$ it is found that $(a, d)P(m, \rho)=a^\rho$. By putting $d=1$ or $a=\rho d$ or $m=0$ three ternary functions are obtained. By making two of these three substitutions three binary functions are obtained. In particular if $d=1$ and $m=0$, $(a, d)P(m, \rho)=aP\rho$, the binary permutation function. By making all three of these substitutions $(\rho, 1)P\cdot(0, \rho)=\rho!$ is obtained, the unary factorial function. As examples, it is found that $0!=1$ and $\rho!=\infty$ if $\rho$ is negative. By mathematical induction the following theorem is proved, of which Newton's binomial theorem and Vandermonde's permutation theorem are special cases: If $r$ is a primary number or zero, $a$, $b$, $d$ any finite numbers, except that, if $r$ and $d$ are both zero, neither $a$ nor $b$ nor $a+b$ is zero, and $m$ and $n$ are any integers, then $(a+b, d)P(m+n, r)=\sum_{k=1}^{r+1}rC(k-1)\cdot(a, d)P(m, r-(k-1))\cdot(b, d)P(n, k-1)$. By putting $d=0$, $(a+b)^r=\sum_{k=1}^{r+1}rC(k-1)\cdot a^{r-(k-1)}\cdot b^{k-1}$, which is Newton's theorem. By putting $d=1$ and $m=0$ $(a+b)Pr=\sum_{k=1}^{r+1}rC(k-1)\cdot aP(r-(k-1))\cdot bP(k-1)$, which is Vandermonde's theorem. (Received February 1, 1943.)

### 105. L. L. Dines: *On linear combinations of quadratic forms.*

The author considers conditions under which $m$ given quadratic forms in $n$ variables admit a linear combination which is (1) definite, or (2) semi-definite. The paper will appear in full in an early issue of Bull. Amer. Math. Soc. (Received December 10, 1942.)

### 106. H. Schwerdtfeger: *Identities between skew-symmetric matrices.*

Let $P$, $Q$ be two $2m$-rowed skew-symmetric matrices, $P$ regular. Put $P^{-1}Q=A$. The determinant $|\lambda P-Q|$ equals $\kappa(\lambda)^2$ with $\kappa(\lambda)=k_0\lambda^m-k_1\lambda^{m-1}+\cdots+(-1)^m k_m$ where $k_0$, $k_m$ are the pfaffian parameters of $P$, $Q$, respectively, and $k_1, \cdots, k_{m-1}$ the rational simultaneous invariants of $P$ and $Q$. By Cayley's identity one has $\kappa(A)^2=(0)$. By means of known theorems (cf. for example, MacDuffee's *Theory of matrices*, Theorems 32.2, 32.3, and 29.3, or A. A. Bennett, Bull. Amer. Math. Soc. vol. 25 (1919) pp. 455–458) it follows that $\kappa(\lambda)$ has as a factor the highest invariant factor $h(\lambda)$ of $\lambda P-Q$, and thus the minimum polynomial of $A$. Hence follows $h(A)=(0)$ and $\kappa(A)=(0)$. This identity involving the skew-symmetric matrices $P$, $Q$ is of geometric interest; if $m=2$ one has, for instance: $k_0QP^{-1}Q=k_1Q-k_2P$ whence the elementary theory of a pair of null systems (linear complexes) in projective 3-space can be derived. (Received January 8, 1943.)

## ANALYSIS

### 107. G. E. Albert: *An extension of Korous' inequality for orthonormal polynomials.*

Let $\{q_n(x)\}$ denote the set of polynomials orthonormal on $(a, b)$ with weight functions $p(x)r(x)$ where $0\leqq p(x)$ and $0\leqq r(x)\leqq M$. If a non-negative polynomial $\pi_m(x)$ of degree $m$ can be found such that the quotient $\pi_m(x)/r(x)$ satisfies a Lipschitz condition on $(a, b)$ and if $\{p_n(x)\}$ denotes the set of polynomials orthonormal on $(a, b)$ with weight function $p(x)[\pi_m(x)]^2$ then if the polynomials $\{p_n(x)\}$ are bounded uniformly with respect to $n$ and $x$ on any subset of $(a, b)$ the same is true of the set $\{q_n(x)\}$. This result follows from an inequality that is established by the same procedure as that used on an equiconvergence theorem by L. H. Miller and the author (abstract 49-3-108). If $r(x)$ is bounded from zero and satisfies a Lipschitz condition on $(a, b)$, the inequality mentioned reduces essentially to an inequality due to Korous (G. Szego, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications vol. 23, 1939, p. 157). (Received January 19, 1943.)

### 108. G. E. Albert and L. H. Miller: *Equiconvergence of series of orthonormal polynomials.* Preliminary report.

Walsh and Wiener (Journal of Mathematics and Physics vol. 1 (1922)) found necessary and sufficient conditions for the equiconvergence of the expansions of an arbitrary function in terms of different systems of functions orthonormal on a finite interval. In the present paper these conditions are applied to the study of polynomials orthonormal relative to weight functions satisfying a variety of hypotheses. A remarkably simple proof is obtained for an equiconvergence theorem that includes one published by Szegö (*Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publication, vol. 23, 1938, Theorem 13, 1.2) and the results given by Peebles (Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) pp. 97–104). The application of the Walsh-Wiener conditions is based upon the observation that if $K_n^{(1)}(x, t)$ and $K_n^{(2)}(x, t)$ are the respective kernel polynomials for the systems orthonormal with respect to the weight functions $p(t)r_1(t)$ and $p(t)r_2(t)$, then (subject to integrability conditions) for arbitrary fixed $x$ the integral $\int_{-1}^{1}p(t)r_2(t)\left[r_1(t)r_2^{-1}(t)K_n^{(1)}(x, t)-K_n^{(2)}(x, t)\right]^2dt$ is the least squares integral of order $n$ for the function $r_1(t)r_2^{-1}(t)K_n^{(1)}(x, t)$ with respect to the system of weight $p(t)r_2(t)$. An expedient choice of a polynomial of degree $n$ to replace $K_n^{(2)}(x, t)$ and study of the result completes the proof. No asymptotic formulas of any kind are needed. (Received December 9, 1942.)

### 109. E. F. Beckenbach: *On conjugate harmonic functions.*

According to N. Cioranesco (*Sur les fonctions harmoniques conjuguées*, Bull. Sci. Math. vol. 56 (1932)), a set of $n$ conjugate harmonic functions $x_j$ of $n$ independent variables is a set such that (1) the $x_j$ are harmonic and (2) the function $-\sum(x_j+a_j)^{(2-n)/2}$ is harmonic or $-\infty$ for all values of the parameters $a_j$. It is now shown that there is a redundancy in the above characterization, for condition (1) is implied by condition (2). The result is extended by means of subharmonic functions to sets of $m$ conjugate harmonic functions of $n$ independent variables, $m \geqq n$. (Received December 31, 1942.)

### 110. R. H. Cameron and W. T. Martin: *An expression for the solution of a class of nonlinear integral equations.*

The authors give an expression for the solution of the integral equation $\phi(x) = f(x)+\int_0^x F[x, \xi, \phi(\xi)]d\xi$, where $F(x, \xi, y)$ is continuous in $0 \leqq x \leqq 1$, $0 \leqq \xi \leqq 1$, $-\infty < y < \infty$ and satisfies the uniform Lipschitz condition $|F(x, \xi, y_2)-F(x, \xi, y_1)| < M|y_2-y_1|$. For any continuous function $f(x)$ the solution $\phi(x)$ is given as the limit in the mean (in the $L_2$-sense) of the quotient of two Wiener integrals (averages) over the space of all continuous functions. The proof is carried through for more general nonlinear functional equations which include the above as a special case. (Received January 30, 1943.)

### 111. M. M. Day: *Uniform convexity.* III.

This paper fills out certain results obtained by the writer in two earlier papers (Bull. Amer. Math. Soc. vol. 47 (1941) pp. 313–317 and pp. 504–507). It also contains the following theorem: If a normed vector space is uniformly convex in the neighborhood of a single point on the unit sphere, then it is isomorphic to a uniformly convex space. (Received January 23, 1943.)

### 112. George Piranian: *On the convergence of certain partial sums of a Taylor series with gaps.*

Let $f(z)$ be defined by the series $\sum_{n=1}^{\infty}c_nz^{\lambda_n}$ where $\limsup |c_n|^{1/\lambda_n} = 1$, and let $\theta_n = \lambda_{n+1}/\lambda_n - 1$, $M(r) = \max_{|z|=r}|f(z)|$, and $S_n(z) = \sum_{p=1}^{n}c_pz^{\lambda_p}$. If $\limsup \{\log [M(1-\theta_{n_i}^2)/\theta_{n_i}]/\lambda_{n_i}\theta_{n_i}^2\} < \infty$, then $\lim S_{n_i}(z) = f(z)$ at all regular points of $f(z)$ on the circle $|z| = 1$. (Received December 12, 1942 )

### 113. R. M. Robinson: *Analytic functions in circular rings.*

The fundamental lemma on which this paper depends is the following: *If $f(z)$ is regular and single-valued in the ring $q \leqq |z| \leqq 1$, except for one simple pole on the negative real axis, and if $|f(z)| \leqq 1$ on both boundaries, then $|f(z)| < 1$ for $q < |z| < 1$, that is, on* *the radius opposite the pole.* Various applications are given, including the determination of the sharp bound in Hadamard's three circles theorem. That is, we suppose that $f(z)$ is regular and single-valued for $q \leqq |z| \leqq 1$, that $|f(z)| \leqq p$ for $|z| = q$ and $|f(z)| \leqq 1$ for $|z| = 1$, and find the largest possible value for $|f(z_0)|$, where $z_0$ is some point within the ring. A formula for the bound is given in terms of theta functions, and the problem is also discussed geometrically. In particular, if $q < p < 1$, then the maximum value of $|f(z_0)|$ is attained by a function $f(z)$ which is univalent in $q < |z| < 1$, and maps this ring on $|w| < 1$ excluding an arc of $|w| = p$. (Received January 23, 1943.)

### 114. Raphael Salem: *Sets of uniqueness and sets of multiplicity.*

An algebraic integer $\alpha$ having the property that all its conjugates have their moduli inferior to 1 will be called a "Pisot number" ($\alpha$ is necessarily real and greater than 1). The following theorems are proved: I. Let $0 < \xi < 1$. If the Fourier-Stieltjes transform $\prod_{k=0}^{\infty} \cos \pi u \xi^k$ does not tend to zero for $u \to \infty$, then $1/\xi$ is a Pisot number. II. Let $0 < \xi < 1/2$, and let $P$ be the symmetrical perfect set of Cantor type and of constant ratio of dissection $\xi$ constructed on $(0, 2\pi)$ (relative length of the black intervals is $1 - 2\xi$). Then $P$ is a set of uniqueness for trigonometrical series if (and only if) $1/\xi$ is a Pisot number. III. There exist Pisot numbers of the form $2 + \epsilon$, $\epsilon$ being positive and arbitrarily small; hence, there exist sets of uniqueness which are of Hausdorff dimensionality as near to 1 as desired. (Received January 11, 1943.)

### 115. Gabor Szegö: *On the oscillation of differential transforms.* IV. *Jacobi polynomials.*

Let $\alpha \geqq 0$, $\beta \geqq 0$, $c \geqq 0$. In a recent paper (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 463–497, cf. p. 489), E. Hille proved the following two theorems: (A) The differential operator $\vartheta - c = (1 - x^2)D^2 + [\beta - \alpha - (\alpha + \beta + 2)x]D - c$, $D = d/dx$, does not diminish the number of the sign changes of a function in $-1 < x < +1$; (B) If the number of the sign changes of $(\vartheta - c)^k f(x)$ remains less than or equal to $N$ for all $k$, $k = 1, 2, 3, \cdots$, then $f(x)$ is a polynomial of degree less than or equal to $N$. The purpose of the present note is the extension of Theorem A to $\alpha > -1$, $\beta > -1$ and of Theorem B to arbitrary real values of $\alpha$ and $\beta$, in the latter case with the modification that the possible degree of the polynomial $f(x)$ is less than or equal to $N + \gamma$, $\gamma = \gamma(\alpha, \beta, c)$. (Received January 20, 1943.)

## APPLIED MATHEMATICS

### 116. Stefan Bergman: *A formula for the stream function in compressible fluid flow.*

Using the hodograph method and a general representation for the stream function of a flow of an incompressible fluid (see Bergman, *Hodograph method in the theory of compressible fluid*, Publication of Brown University, 1942) the author gives an explicit formula for the stream functions of flows of certain types. (Received January 27, 1943.)

### 117. Nathaniel Coburn: *Boundary value problems in plane plasticity.* Preliminary report.

The following problem is discussed in this paper: given an infinite plate of perfectly plastic material bounded by the $x$-axis; to determine the stresses within the plate when the stresses on the boundary are known. First, the equation of plasticity (yield condition) is linearized. It is shown that, in this case, the above problem reduces to finding a solution $F(x, y)$ for the hyperbolic equation under given Cauchy data. Next, it is shown that for certain types of boundary conditions, the stresses corresponding to the linear problem are equal to the stresses corresponding to the original nonlinear problem over a series of equally spaced lines parallel to the boundary ($x$-axis). The situation is analogous to two surfaces which do not coincide but do intersect in curves whose projections on the $xy$-plane are parallel lines. The method may be modified to give approximate solutions of the nonlinear problem throughout the plate. Some examples are worked. An indication is given of how the method may be applied to the finite rectangular plate. (Received January 2, 1943.)

### 118. H. J. Greenberg: *Application of a summability method in solving boundary value problems.*

In order to apply the method of particular solutions for solving boundary value problems for the elliptic linear partial differential equation $L(U)=0$, S. Bergman (Duke Math. J. vol. 6 (1940) p. 541) has introduced the complete set $P_{2n-\alpha}(z)$ of such solutions, by means of which every function $U$, $L(U)=0$, regular in the circle $x^2+y^2<R^2$, $R>0$, can be developed in the series $S$: $U(z)=\sum a_k P_k(re^{i\phi})$. By studying the "associated function" $\sum a_k\Gamma(1/2)\cdot\Gamma(k+1/2)z^k/\Gamma(k+1)$ necessary and sufficient conditions are given for the series $S$ to be convergent on $|z|=R$ and, what is more important for applications, for $S$ to be $(C, 1)$ summable on this circle of convergence. Conditions are given in each case under which the values thus obtained are the boundary values of the function $U(z)$. Using these results a method is given for the actual solution of boundary value and characteristic value problems for the equation $L(U)=0$. (Received January 29, 1943.)

## GEOMETRY

### 119. John DeCicco: *Extensions of certain dynamical theorems of Halphen and Kasner.*

The theorem of Halphen which characterizes central fields of force and the general theorem of Kasner concerning one-third the curvatures is extended to generalized fields of force in space, which depend upon the position of the point and direction. The number of generalized fields of force whose dynamical trajectories are all plane curves is $\infty^{f(3)+f(4)+2f(5)}$. The $\infty^5$ generalized trajectories consist of $\infty^2$ systems of $\infty^3$ generalized plane trajectories, each system lying in a plane tangent to a given surface $\Sigma$. In an arbitrary positional field of force, Kasner showed that the rest trajectory and line of force through a given point $O$ have the same osculating plane and that the ratio $\rho$ of the curvature of the rest trajectory to that of the line of force is 1/3. For generalized fields of force this theorem is no longer valid. All generalized fields of force such that the rest trajectory and the line of force through any point $O$ of the space have the same osculating plane are determined; and, also in this class, the subclass of all generalized fields of force for which $\rho=1/3$. (Received January 2, 1943.)

### 120. Edward Kasner and John DeCicco: *Generalized dynamical trajectories in space.*

The differential geometry of positional fields of force has been developed in *Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications vol. 3, 1913. In abstract 48-11-329, the authors began the study of generalized fields of force in the plane which depend not only upon the position of the point but also upon the direction through the point. In this paper, the geometry of the dynamical trajectories of such generalized fields of force in space is studied. It results that the $\infty^3$ dynamical trajectories are completely characterized by two geometric properties as follows. For each of the $\infty^1$ curves of the $\infty^3$ trajectories which pass through a given lineal element $E$, construct the osculating plane and sphere at $E$. The two properties are: (I) The $\infty^1$ trajectories all have the same osculating plane, and (II) The locus of the centers of the osculating spheres is a straight line. Finally velocity systems are defined and shown to be curvature trajectories. (Received December 28, 1942.)

## STATISTICS AND PROBABILITY

### 121. Henry Scheffé: *On solutions of the Behrens-Fisher problem, based on the t-distribution.*

The problem is to find confidence intervals for the difference of the means of two normal populations when the ratio of their variances is unknown. Certain disadvantages inhere in solutions hitherto proposed: Some data are completely discarded when the sample sizes are unequal (Neyman), or the confidence coefficient is not known exactly (Fisher), or existing tables are inadequate if the confidence coefficient is to be 95 or 99 per cent (Wilks). The present solution is as follows: Let the samples be $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_n)$, where the $x$'s and $y$'s are mutually independently normally distributed, the former with mean $a_x$ and variance $\sigma_x^2$, the latter with mean $a_y$ and variance $\sigma_y^2$. Assume $m \leq n$. Let $\bar{x}$, $\bar{y}$ be the sample means, and $\delta = a_x - a_y$. Then $(\bar{x} - \bar{y} - \delta)/Q^{1/2}$ has the Fisher $t$-distribution with $m-1$ degrees of freedom if $m(m-1)Q = \sum_{i=1}^{m}(u_i - \bar{u})^2$, $u_i = x_i - (m/n)^{1/2}y_i$, $\bar{u} = \sum_{i=1}^{m} u_i/m$. This leads immediately to confidence intervals for $\delta$, and it is shown that of a certain class these have the minimum expected length. (Received January 7, 1943.)

## TOPOLOGY

### 122. M. G. Ettlinger: *On irreducible continuous curves.*

It is proved that, in a connected space satisfying Axioms 0–2 of R. L. Moore's *Foundations of point set theory* (Amer. Math. Soc. Colloquium Publications, vol. 13, 1932), every compact and closed point set is a subset of a compact continuous curve, and every compact and closed point set having no continuum of condensation is a subset of a compact hereditary continuous curve. It is shown that if, in a space satisfying Axioms 0–1, $M$ is a locally compact continuous curve which is an irreducible continuum about a closed subset $T$ of $M$, then every continuum of condensation of $M$ is a continuum of condensation of $T$. It is then demonstrated that in a connected space satisfying Axioms 0–2, every compact and closed point set having no continuum of condensation is a subset of a compact continuous curve having no continuum of condensation. (Received January 21, 1943.)

### 123. O. G. Harrold: *A decomposition theorem for certain compacta.*

Let $\psi_n$ denote the property of being locally connected in dimension $n$ in the sense of homotopy $(n-LC)$ at a point $p$ of a compactum $X$. Let $\Delta_n$ denote the $\psi_n$-singular points of $X$. Let $\mathfrak{C}_n^{n+1}$ denote the class of compacta which are $LC^n$ $(i-LC$, $i=0, 1, \cdots, n)$ and such that small singular $(n+1)$-spheres bound. If $X \in \mathfrak{C}_n^{n+1}$, either $\bar{\Delta}_{n+1}$ is vacuous or contains nondegenerate connected sets. Also, analogous to results of R. L. Wilder (*Decompositions of compact metric spaces*, Amer. J. Math. vol. 63 (1941) pp. 691–697), if $X\in\mathfrak{C}_n^{n+1}$ and the $\psi_{n+1}$-prime parts of $X$ are simply-$i$-connected, $i=1, 2, \cdots, n+1$, then the space $X'$ of $\psi_{n+1}$-prime parts of $X$ is $LC^{n+1}$. For $n\geqq 1$, by use of a well known theorem due to Hurewicz, these theorems may be stated in terms of Vietoris cycles with integer coefficients. (Received January 30, 1943.)

### 124. A. H. Stone: *Connectedness and coherence.*

This paper is concerned with the incidence geometry of a topological space $S$, that is, the relations between the numbers of components of sets and their frontiers, unions and intersections, expressed in terms of the zero-dimensional Betti number $b_0$ and the coherence $c=r+1$ (Eilenberg, Fund. Math. vol. 27 (1936)). Among others, the following theorems are obtained. The coherence of a space may be defined using open sets instead of closed ones. For any set $A$, $b_0(\mathcal{J}(A))\leqq b_0(\bar{A})+b_0(\overline{\mathcal{C}(A)})+c(S)-1$, and $\sum b_0(\mathcal{J}(C))\leqq b_0(\bar{A})+c(S)-1$, the summation extending over the components $C$ of the complement of $A$. A theorem of Eilenberg on coverings of order 3 is extended and sharpened, yielding the corollary that if $S$ is unicoherent and $\mathcal{J}(A)\cap\mathcal{J}(B)\subset A\cap B\cap\mathfrak{I}(A\cup B)$ then $b_0(A\cap B)+b_0(A\cup B)=b_0(A)+b_0(B)$. Several other theorems, as well as extensions of some known results, are obtained for unicoherent spaces, and some of them are extended to the multicoherent case. Most of the theorems apply to sets which need not be open or closed. $S$ is assumed to be normal; but if $S$ is unicoherent, methods are developed which dispense completely with separation axioms. Applications are made to spaces of coherence 2, and to euclidean space. (Received January 20, 1943.)

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WREN, F. L. See OLDS, E. G.

# A MANUAL FOR AUTHORS OF MATHEMATICAL PAPERS*

The purpose of this manual is to help mathematicians in preparing papers for publication. A reasonable amount of attention on the part of authors to the rules and suggestions given below will save the time of editors, referees, printers, and readers, and will help keep the cost of publication at a minimum. Although most of the discussion will apply to mathematical papers in general, special rules will occasionally be given for papers specifically intended for the Bulletin or Transactions of the American Mathematical Society.

**Style.** In the course of writing a mathematical paper intended for publication, the author must settle such questions of usage and arrangement as (1) the organization of the paper into chapters, sections, paragraphs, propositions, theorems, proofs, examples, remarks, and the use of appropriate headings; (2) the adoption of a system of enumeration for the sections, theorems; (3) the treatment of footnotes, references and bibliography; (4) the use of capital or small initial letters in such adjectives as abelian, euclidean, jacobian; (5) the use or non-use of hyphens in such technical terms as sub-harmonic, fixed-point; (6) the questions of style which arise when words in the text are replaced by symbols.

Concerning the first three questions—particularly the third—the author's decisions must to some extent be governed by the routine stylistic requirements of the journal for which his paper is intended. For guidance, the author should consult current numbers of that journal; if his paper is intended for the Bulletin or Transactions, he may consult, instead, the Special Rules given below.

Concerning the remaining questions, perhaps the only general rule that can be offered is the Rule of Uniformity: Whatever the usage adopted by an author in a given paper, *that usage should be maintained consistently throughout the paper*. It may be remarked that the reasons for urging a careful observance of this rule are economic as well as stylistic. Every paper is carefully checked by an editorial assistant before it is sent to the printer. An author who has written "euclidean" part of the time and "Euclidean" the rest of the time may have done so inadvertently. But the assistant cannot guess this.

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* The writers of this manual have received invaluable advice from members of the office staff of the American Mathematical Society and have consulted freely the following excellent pamphlets: *Notes on the preparation of mathematical papers* (The London Mathematical Society); *Typographic suggestions to authors* (George Banta Publishing Co.); *Author's manual* (Duke Mathematical Journal).

Nor can one guess whether the author prefers the capital or the small letters. The correspondence between author and editorial staff which must take place in order to settle the matter causes delay and expense which could easily have been avoided.

The use of mathematical symbols in the solid text is not only permissible but is frequently encouraged by editors as a means for securing greater economy of space on the printed page. Authors, however, should consider carefully the questions of style which arise from this practice, particularly in the use of "$<$" and "$=$". Such phrases as "assume $x>y$" and "assume that $x$ is $>y$" are in common use and there appears to be no widely accepted ground for preferring one to the other. On the other hand, no one would write "assume that $x$ and $y$ are $=$" although the objection appears to be largely visual. Some writers do not hesitate to begin a sentence with a mathematical symbol, but others regard this as a barbarism.[1] In the complete absence of agreement on these questions, we offer only the following general rules in addition to the Rule of Uniformity:

1. Symbols in the text should be used in such a way that their translation into words or phrases will require no effort on the part of the reader.

2. The resulting translation should conform to the standards of good literary composition.

Rules 1 and 2 are admittedly not decisive in all cases. Thus "let $\epsilon \geqq 0$ be a number such that $\cdots$" would be objectionable on both counts, but is probably preferable, in appearance at least, to "let $\epsilon$ be a number $\geqq 0$ such that $\cdots$." One could, of course, avoid both forms by writing "let $\epsilon$ be a non-negative real number $\cdots$." Again, "$\cdots$ the expression on the left is $<$ the smallest value of $\cdots$" satisfies rules 1 and 2. Yet the substitution of two words for "$<$" would result in a far better appearance. This improvement is surely worth the slight extra effort involved.

**The printing of mathematical symbols.** There are unfortunately many useful arrangements of symbols which must be avoided in a manuscript intended for publication, on account of the difficulties which would have to be overcome by the compositor. A type consists of a solid block or "body" with a raised character on one of its rectangular faces. The bodies are normally laid side by side by a machine and form a row of uniform width. Besides the ordinary alphabets, most printers have on hand a number of "dashed letters" ($\bar{A}$, $\bar{a}$, · · · ) and "tilde letters" ($\tilde{A}$, · · · ). If a dashed letter, say $\bar{a}$, is not in the printer's possession, it does not follow that this letter cannot be printed. But the typesetting involved is much more complicated and includes a number of operations by hand. If $\bar{a}$ occurs in the solid text, the body of an ordinary $a$ must be reduced in size by filing, to make room for a thin body carrying the bar. The two bodies must then be placed in position by hand. The thin body may or may not stay in place and thus the alignment of the bar may or may not be correct. Moreover, the bars, when printed, may not always be uniform. Thus to the extra expense involved in the formation and typesetting of the symbol $\bar{a}$, it is necessary to add the cost of re-alignment or re-setting in case of mishaps during the printing of proof.

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[1] There is a similar division of opinion concerning the use of the "hanging theorem" in which the word THEOREM is simultaneously the first word of a paragraph and the last word of the sentence which precedes it.

The typesetting of inferior and superior symbols (subscripts and superscripts) in the solid text offers no difficulty when those symbols are available as *inferiors and superiors* (see Appendix C) and when no symbol is to be placed directly above or below another. Thus, $(a+bx)^{3/2}$, $A^2$ can be set by machine, but $x_{\bar{i}}$, $A_3^2$ require hand work.[2] Subscripts to superscripts and superscripts to subscripts, all belonging to a single term, are extremely difficult to set, even by hand.

In displayed formulas, the insertion of bars, subscripts and the like is somewhat less difficult since there is space above and below, whereas in the solid text, the compositor has only a single line with which to work. Thus the symbol $A_3^2$ no longer requires hand work since the superscript 2 can be set by machine in the subscript position of the line above the line of the $A$. Note however that when set in this manner, the superscript will be slightly higher: $A_3^{\,2}$ rather than $A_3^2$.

**Special rules.** Authors of papers intended for the Bulletin or Transactions should give special attention to the following rules. A reasonably strict observance of these rules will help keep printing costs down, prevent an uneconomical use of space, insure a certain uniformity of appearance, and will greatly simplify the task of final preparation of manuscripts for the printer.

1. Use only such characters as the printer has in stock. The list in Appendix C includes all special types now available for the Bulletin and Transactions. Additions to this list are made from time to time. Note particularly what letters and numbers are available in sub-superscript (inferior to superior) and super-subscript positions.

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[2] These remarks are not intended to imply that symbols such as $A_3^2$ must never be used in the solid text, rather that the author should exercise a measure of discretion.

Note also that many symbols are not available in footnote size. For this reason, if for no other, complicated formulas should not be put into the footnotes. In this connection it might be remarked that an excessive number of footnotes sometimes gives the undesirable impression that the paper is being "written in the footnotes." Careful organization, however, always makes it possible to reduce the number and length of footnotes, even to eliminate them entirely.

2. Do not put into the solid text any formula which would require spreading (uneven spacing) of lines.

3. In the solid text, write $\sum_{i=1}^{n}$; $\prod_{i=1}^{n}$; $\lim_{n\to\infty}$; $\int_a^b$. In the first two examples the $n$ should be directly over the $i$. In the integral the $b$ should be directly over the $a$; it is not feasible to place the $a$ nearer the integral sign since the latter is on a rectangular face: $\boxed{\int}$. In displayed formulas, the limits should occupy only one line. For example, write

$$\sum_{i=i_0,\, j=j_0}^{\infty} \quad \text{instead of} \quad \sum_{\substack{i=i_0\\ j=j_0}}^{\infty}.$$

4. In the text, replace $e^{(\ )}$ by exp ( ) if the expression in the parentheses is complicated.

5. Use $C_{m,n}$ (binomial coefficient) instead of $\binom{m}{n}$.

6. In the text, write $\sim_{\epsilon}$, in display $\underset{\epsilon}{\sim}$.

7. Use $\in$ to mean "is a member of," $\epsilon$ for epsilon. The symbol for "is not a member of" is $\notin$.

8. Use fractional exponents in place of radicals. A fraction occurring as subscript or superscript should be formed with a solidus, for example 2/3. In display, the number of lines occupied by a fraction should be as small as possible. Frequently the number of lines can be reduced by a simple rearrangement. For example:

$$\frac{\cos\dfrac{1}{x}}{\sqrt{a+\dfrac{b}{x}}} \quad \text{can be written} \quad \frac{\cos(1/x)}{(a+b/x)^{1/2}}.$$

9. Long formulas, or expressions which might break awkwardly at the end of a line of text, should be displayed.

10. In numbering sections, theorems, displayed formulas, and so on, the author should adopt a system which is simple and self-explanatory. A system based on such symbols as II.3.6.1a is unnecessarily complicated.

11. Footnotes should be assembled at the end of the paper, typed double spaced, and numbered with arabic numbers. The footnote symbols currently in use are illustrated as follows: "It was proved[1] that · · · " (Bulletin); "It was proved (1) that · · · " (Transactions). A footnote symbol should not follow a mathematical symbol. Footnotes should be numbered in the order in which they appear, beginning with 1 and continuing throughout the paper. Do not use any symbol at all for the footnote "Presented to · · · " which frequently appears on the title page.

12. Mathematical works which are referred to in the text may be listed at the end of the paper under the heading "Bibliography" or "References." The following examples illustrate the two methods of arrangement and enumeration used in the Bulletin and Transactions:

Form I

Bibliography [or References]

1. R. R. Roe, *Canonical systems*, Amer. J. Math. vol. 60 (1938) pp. 1042–1100.
2. ———, *An elementary solution of the fundamental problem of mathematics*, J. of Metamath. vol. 17 (1939) pp. 1–100.
3. D. D. Doe, *The theory of topological hypergroups*, New York, 1940.

Form II

Bibliography [or References]

R. R. Roe
  1. *Canonical systems*, Amer. J. Math. vol. 60 (1938) pp. 1042–1100.
  2. *An elementary solution of the fundamental problem of mathematics*, J. of Metamath. vol. 17 (1939) pp. 1–100.

D. D. Doe
  1. *The theory of topological hypergroups*, New York, 1940.
  2. *A generalization of the generalized Doe-differential*, Duke Math. J. vol. 48 (1940) pp. 1100–1200.

The following examples illustrate the various methods of referring to the bibliography. The basic marks of enclosure are brackets: [ ]. With a small amount of care, the use of parentheses as additional marks of enclosure can be kept at a minimum and the appearance of the printed page thereby improved.

For form I: [1, Theorem 3.7]; · · · proved by Roe [2, Lemma 4] · · · ; (see [2, Lemma 4]).

For form II: [Roe 1, Theorem 3.7]; · · · proved by Roe [2, p. 100] · · · ; (cf. Roe [2, p. 100]).

The use of abbreviations for the names of the more obscure scientific journals often creates difficulties for readers and librarians. The Bulletin and Transactions, therefore, make it a rule to print such titles without any abbreviations. *Cooperation on the part of the authors would diminish greatly the work involved in this connection.* The names of the standard journals, however, should be abbreviated. Appendix B contains a list of abbreviations which have been adopted by the Bulletin and Transactions.

**Preparation of the manuscript.** A compositor is not a trained mathematician and cannot take the responsibility of altering so much as the position of a single dot in the manuscript. It is obvious, therefore, that the manuscript, when it is sent to the printer, must be exact in every detail. Since it is impossible to produce a manuscript in which all characters are facsimiles of printed characters, there will be many details which can be made clear to the printer only by special indication such as underlining or marginal notes.

The task of preparing the manuscript for the printer is generally shared between author and editorial staff. In the case of the Bulletin and Transactions, indications concerning routine matters of style, such as the spacing of material, the choice of type in titles, paragraph headings, and so on, are inserted by an editorial assistant. All indications of a non-routine nature must be made by the author and must be complete and explicit. The most important rules in this connection are the following:[3]

1. The manuscript should be typewritten. The text should be double spaced and displayed material should not be crowded. Plenty of space should be left for the characters which are to be filled in by hand. It is well to make at least one carbon copy of the manuscript, but the *original*—not the carbon copy—should be sent to the editor.

2. Subscripts or superscripts may be filled in by hand; if typed they should be a half-space above or below the line.

3. Handwritten characters should be made with great care and each character must be distinct in shape from every other. In cases where the author's intention may be in doubt, identification, such as "Greek chi," should be written in pencil in the margin. Capital letters may be indicated by "cap," small letters by "l.c." (lower case). Letters and symbols which are particularly liable to confusion are:

$$o\,O\,0 \quad \gamma\,r\,n\,\nu \quad \rho\,p \quad s\,S\,5 \quad {}_{1}, \quad {}^{1}{}' \quad h\,k\,\kappa\,K\,\mathrm{K}$$
$$z\,Z\,2 \quad \psi\,\chi\,x\,\times\,4 \quad \ell\,l\,1 \quad V\,\cup \quad \in\,\epsilon$$

4. German and script letters may be handwritten if they can be made with sufficient accuracy. A far more satisfactory procedure is to type the corresponding Roman letters, then to underline or encircle

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[3] See also rules 3-8 in the section Special Rules.

each letter with colored pencil, say green for German, blue for script. In fact, all alphabets other than the Roman and Greek are best treated in this manner. The meaning of the colored marks should be stated at appropriate places in the margin or on a separate sheet of "instructions for the printer."

5. No mark meaning "Greek alphabet" should be attached to the symbols $\sum$ (summation), $\prod$ (product), $\partial$ (in partial derivatives). Notice the difference in the size and style between the first two symbols and the Greek capital letters $\Sigma$, $\Pi$.

6. Italic type is indicated by underlining, preferably with a blue pencil. In the Bulletin and Transactions, the text of a theorem or corollary is always in italics and the necessary indications may be left to an editorial assistant.

7. Omissions should be denoted by *exactly* three slightly raised dots, such as $ab \cdots c$. If commas are required, write $a_1, \cdots, a_n$ or $a_1, a_2, \cdots$, never $a_1, \cdots a_n$ or $a_1, a_2 \cdots$. A handwritten comma must be carefully made so that it will not be taken for subscript 1.

8. The number one is ordinarily typed 1. The letter "ell" is denoted by ł (the dash may be written). Experience shows that the use of the solidus (/) in place of the dash may lead to errors in typesetting.

9. Drawings and diagrams should be made with particular care since they cannot be changed or corrected in galley proofs or page proofs. Drawings should be made in black India ink on Bristol board and should be two or three times larger than they are to be when printed. When inserting letters or symbols, allowance should be made for the reduction in size. For drawings which are to appear in the Bulletin or Transactions, all letters and symbols should be drawn lightly in pencil since they will eventually be replaced by type letters and symbols. The reproduction of even the simplest drawing involves considerable extra expense, and the author, therefore, should include no drawing which is not really essential for a clear understanding of his paper.

10. A displayed formula should be numbered only if referred to elsewhere in the text. If displayed formulas are numbered indiscriminately, it is impossible to combine several formulas into a single line or to run an occasional formula into the text to facilitate the breaking up of galleys into pages.

**Proof reading.** It is important that galley proofs be carefully read and corrected by the author, since it is only the author who can detect errors which are due to an imperfect manuscript.

Alterations in type which has been set are delicate and costly to make. The author should call for only such changes as are really essential. The insertion or deletion of a comma, for example, should be based on a compelling reason and should not be merely a matter of whim. If a number of consecutive words or symbols must be deleted, they should if possible be replaced by words or symbols occupying the same amount of space. If this is not done, the compositor often must tear down and reset a number of preceding or following lines in order to make the necessary adjustments. This is not only costly but makes possible the introduction of new errors.

Galley proofs of papers for the Bulletin or Transactions are read in the office of the American Mathematical Society before they are sent to the author. During the reading, there may arise questions which can be answered only by the author. Such questions are written in the margins, and it would seem unnecessary to point out that they should be answered were it not for the fact that they are frequently ignored.

The list in Appendix A contains most of the standard signs used in this country in correcting proof. A consistent use of these signs by authors would simplify greatly the work of printer and editorial staff. In any case, authors should make sure that their corrections are clear and explicit and should refrain from using elaborate "systems" of their own invention. Joint authors should be particularly careful not to use two non-equivalent systems of signs in a single set of galleys.

The example in Appendix A illustrates the standard method of using the proof reading signs. It is important to note that all indications for corrections must appear in margins and for each correction a corresponding mark must appear in the text. The signs in the margins should retain the order of the corresponding errors in the line and should be separated from each other by |.

# APPENDIX A

## SIGNS USED IN CORRECTING PROOFS

| | |
|---|---|
| ₰ | Delete; take out |
| ⁀ | Close up |
| ^ | Insert |
| # | Insert space. |
| ⊓ | Raise |
| ⊔ | Lower |
| [ | Move to left |
| ] | Move to right |
| \|\| | Straighten type line at side of page |
| // | Straighten lines |
| ¶ | Paragraph |
| center | Put in middle of page or line |
| ∽ | Transpose |
| Tr. | Transpose |
| 9 | Turn inverted letter right side up |
| × | Change broken letter |
| stet | Let it stand as set |
| …… | Let it stand as set |
| w.f. | Wrong font, size or style |
| lc. | Lower case, not capitals |
| rom. | Use Roman letter |
| bf. | Use black type letters |
| ⊙ | Period |
| $\overset{,}{\wedge}$ | Comma |
| $\overset{,}{\vee}$ | Apostrophe |
| $\overset{2}{\vee}$ | Superior figure |
| $\underset{2}{\wedge}$ | Inferior figure |
| =/ | Hyphen |
| sc. | Use small capitals |
| caps | Use capitals |
| ital. | Use italics |

## EXAMPLE SHOWING THE USE OF PROOF READING SIGNS

of $B_1$ and $d_1d_s'$ divides $d_1d_s'+1$. Hence B is reducible to the from ita
(11.5) with diagonal terms $d_1, d_1d_2', \cdots, d_1d_s'$ which proved (11.4).

**12. Groups with a finite number of generators.** We shall discuss certain properties of these groups culminating in the basic product decomosition (12.5).

(12.1) DEFINITION Let $B=\{g_1,\cdots,g_n\}$, $B'=\{g_1',\cdots g_n'\}$ be *two sets of elementsof G containing the same number n of elements. By a unimodular transformation* $\tau:B\rightarrow B'$ *is meant a system of elations* s.c. lc

$$(12.2)\qquad g_i'=\sum a_{ij}g_j,\qquad \|a_{ij}\|\ \textit{unimodular.}$$

The following proposition shows in how natural a manner unimodular transformations make their appearance in the theory of groups with finite bases. rom

(12.3) *Let G be a group with a finite base* $B=\{g_1,\cdots,g_n$. *In order that* $B'=\{g_1',\cdots,g_n'\}$ *be a base for G it is necessary and sufficient that B' be obtainable from B by a unimodular transformation.*

For any given set $B'=\{g_1',\cdots,g_n'\}$ of elements of $G$ there exist relations

$$(12.4)\qquad g_i=\sum c_{ij}g_j,\qquad C=\|c_{ij}\|$$

A necessary and sufficient condition ~~in order~~ that $\{g_i'\}$ be as base is that the $G_j$ be expressible as linear combinations of the $g_i'$, or that there exist relations B'

$$(12.5)\qquad g_i=\sum d_{ij}g_j',\qquad D=\|d_{ij}\|.$$ center

From this follows

$$g_i=\sum d_{ij}c_{jk}g_k.$$

Hence since $B$ is a base we must have $DC=1$.
(This matrix relation yields $|D|\,|C|=1$, and since the determinants are integers we must have $|C|=\pm1$. Thus in order that $B'$ be a base $C$ must be unimodular, or the condition of (12.3) must be fulfilled. Conversely, if it is fulfilled, $C$ is unimodular and (12.5) holds with $D=C^{-1}$, from which it follows readily that $b'$ is a base. run in (12.3) cap

## THE PRECEDING PASSAGE PRINTED WITH ALL CORRECTIONS MADE

of $B_1$ and $d_1d'_p$ divides $d_1d'_{p+1}$. Hence $B$ is reducible to the form (11.5) with diagonal terms $d_1, d_1d'_2, \cdots, d_1d'_p$ which proves (11.4).

12. **Groups with a finite number of generators.** We shall discuss certain properties of these groups culminating in the basic product decomposition (12.5).

(12.1) Definition. *Let* $B = \{g_1, \cdots, g_n\}$, $B' = \{g'_1, \cdots, g'_n\}$ *be two sets of elements of* $G$ *containing the same number* $n$ *of elements. By a unimodular transformation* $\tau: B \to B'$ *is meant a system of relations*

$$(12.2) \qquad g'_i = \sum a_{ij}g_j, \qquad \|a_{ij}\| \ \textit{unimodular.}$$

The following proposition shows in how natural a manner unimodular transformations make their appearance in the theory of groups with finite bases.

(12.3) *Let* $G$ *be a group with a finite base* $B = \{g_1, \cdots, g_n\}$. *In order that* $B' = \{g'_1, \cdots, g'_n\}$ *be a base for* $G$ *it is necessary and sufficient that* $B'$ *be obtainable from* $B$ *by a unimodular transformation.*

For any given set $B' = \{g'_1, \cdots, g'_n\}$ of elements of $G$ there exist relations.

$$(12.4) \qquad g'_i = \sum c_{ij}g_j, \qquad C = \|c_{ij}\|.$$

A necessary and sufficient condition that $B' = \{g'_i\}$ be a base is that the $g_j$ be expressible as linear combinations of the $g'_i$, or that there exist relations

$$(12.5) \qquad g_i = \sum d_{ij}g'_j, \qquad D = \|d_{ij}\|.$$

From this follows

$$g_i = \sum d_{ij}c_{jk}g_k.$$

Hence since $B$ is a base we must have $DC = 1$. This matrix relation yields $|D| \cdot |C| = 1$, and since the determinants are integers we must have $|C| = \pm 1$. Thus in order that $B'$ be a base $C$ must be unimodular, or the condition of (12.3) must be fulfilled. Conversely, if (12.3) is fulfilled, $C$ is unimodular and (12.5) holds with $D = C^{-1}$, from which it follows readily that $B'$ is a base.

# APPENDIX B

## ABBREVIATIONS USED IN THE BULLETIN AND TRANSACTIONS

| | |
|---|---|
| Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität<br>(formerly . . . Hamburgischen Universität) | Abh. Math. Sem. Hansischen Univ. |
| Academy of Sciences of the USSR. Journal of Physics | Acad. Sci. USSR. J. Phys. |
| Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum | Acta Univ. Szeged. |
| Acta Mathematica | Acta Math. |
| American Journal of Mathematics | Amer. J. Math. |
| The American Mathematical Monthly | Amer. Math. Monthly |
| American Mathematical Society Colloquium Publications | Amer. Math. Soc. Colloquium Publications |
| American Statistical Association Bulletin | Amer. Statist. Assoc. Bull. |
| Annales de la Faculté des Sciences de l'Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques | Ann. Fac. Sci. Univ. Toulouse |
| Annales Scientifiques de l'École Normale Supérieure | Ann. École Norm. |
| The Annals of Mathematical Statistics | Ann. Math. Statist. |
| Annals of Mathematics | Ann. of Math. |
| Atti della Reale Accademia delle Scienze di Torino. Classe di Scienze fisiche, Matematiche e Naturali | Atti Accad. Sci. Torino |
| Il Bollettino di Matematica | Boll. Mat. |
| Bulletin de l'Académie des Sciences de l'URSS. Série Mathématique | Bull. Acad. Sci. URSS. Sér. Math. |
| Bulletin de la Société Mathématique de France | Bull. Soc. Math. France |
| Bulletin de la Société Physico-Mathématique de Kazan et de l'Institut de Mathématiques et Mécanique | Bull. Soc. Phys.-Math. Kazan |
| Bulletin des Sciences Mathématiques | Bull. Sci. Math. |
| Bulletin Mathématiques de l'Université de Moscou. Série Internationale | Bull. Math. Univ. Moscou |
| Bulletin of the American Mathematical Society | Bull. Amer. Math. Soc. |
| Bulletin of the Calcutta Mathematical Society | Bull. Calcutta Math. Soc. |
| Commentarii Mathematici Helvetici | Comment. Math. Helv. |
| Compositio Mathematica | Compositio Math. |
| Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS | C. R. (Doklady) Acad. Sci. URSS. |
| Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences | C. R. Acad. Sci. Paris |
| Doklady Akademii Nauk SSSR<br>See Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS | |
| Duke Mathematical Journal | Duke Math. J. |
| Fundamenta Mathematicae | Fund. Math. |
| Izvestia Akademii Nauk SSSR<br>See Bulletin de l'Académie des Sciences de l'URSS | |

| | |
|---|---|
| Jahresbericht der Deutschen Mathematiker Vereinigung | Jber. Deutschen Math. Verein. |
| Japanese Journal of Mathematics. Transactions and Abstracts | Jap. J. Math. |
| Journal de l'École Polytechnique | J. École Polytech. |
| Journal de Mathématiques Pures et Appliquées | J. Math. Pures Appl. |
| Journal fur die reine und angewandte Mathematik | J. Reine Angew. Math. |
| The Journal of Symbolic Logic | J. Symbolic Logic |
| Journal of the Faculty of Science. Imperial University of Tokyo. Section I. Mathematics, Astronomy, Physics, Chemistry | J. Fac. Sci. Imp. Univ. Tokyo. Sect. I |
| The Journal of the Indian Mathematical Society | J. Indian Math. Soc. |
| The Journal of the London Mathematical Society | J. London Math. Soc. |
| Koninklijke Nederlandsche Akademie van Wetenschappen. Indagationes Mathematicae ex Actis Quibus Titulus. Proceedings of the Section of Sciences (sometimes referred to as the "Amsterdam Proceedings") | Neder. Akad.. Wetensch. |
| Matematičeskij Sbornik<br>See Recueil Mathématique | |
| Mathematische Annalen | Math. Ann. |
| Mathematische Zeitschrift | Math. Zeit. |
| Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse. Neue Folge. Fachgruppe I. Nachrichten aus der Mathematik | Nachr. Ges. Wiss. Göttingen |
| Periodico di Matematiche | Period. Math. |
| Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences | Philos. Trans. Roy. Soc. London. Ser. A. |
| Proceedings of the Cambridge Philosophical Society | Proc. Cambridge Philos. Soc. |
| Proceedings of the Imperial Academy | Proc. Imp. Acad. Tokyo |
| Proceedings of the London Mathematical Society | Proc. London Math. Soc. |
| Proceedings of the National Academy of Sciences of the United States of America | Proc. Nat. Acad. Sci. U.S.A. |
| Proceedings of the Royal Society. Series A. Mathematical and Physical Sciences | Proc. Roy. Soc. London. Ser. A. |
| The Quarterly Journal of Mathematics Oxford Series | Quart. J. Math. Oxford Ser. |
| Recueil Mathématique. Nouvelle Série | Rec. Math. (Mat. Sbornik) N.S. |
| Rendiconti del Circolo Matematico di Palermo | Rend. Circ. Mat. Palermo |
| Sitzungsberichte der Berliner Mathematischen Gesellschaft | Berlin Math. Ges. Sitzungsber. |
| Sitzungsberichte der Preussischen Akademie der Wissenschaften | Preuss. Akad. Wiss. Sitzungsber. |
| The Tôhoku Mathematical Journal | Tôhoku Math. J. |
| Transactions of the American Mathematical Society | Trans. Amer. Math. Soc. |

# APPENDIX C

## A LIST OF SIGNS AND SPECIAL CHARACTERS AVAILABLE FOR THE BULLETIN AND TRANSACTIONS

Lightface Greek—$\alpha$ $\beta$ $\gamma$ . . (all) Α Β Γ . . . (all).

★ Lightface Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma}$ · · · (all except $o$).

★ Lightface Greek Inferiors—$_{\Delta\ \Lambda\ \Sigma\ \Omega}$ and $_{\alpha\ \beta\ \gamma}$ . . . (all except $o$).

* Boldface Greek—**α β γ δ ε ζ η θ κ λ μ ν ξ π ρ σ τ φ ω Δ Π** and **Ω.**

* Lightface German—𝔄 𝔅 ℭ . . . (all) 𝔞 𝔟 𝔠 . . . (all). No superiors or inferiors.

* Lightface dashed German—$\bar{\mathfrak{p}}$ $\bar{\mathfrak{a}}$ $\bar{\mathfrak{B}}$ $\bar{\mathfrak{U}}$ $\bar{\mathfrak{R}}$ $\bar{\mathfrak{S}}$ $\bar{\mathfrak{G}}$

* Boldface German—𝖉 𝕬 𝕭 𝕯

Script (special font) 𝒜 ℬ 𝒞 . . . (all). No lower case manufactured.

* Hebrew—א א א כּ ב troublesome to use. No superior or inferior.

★ Dashed Italics—$\bar{A}$ $\bar{a}$ $\bar{B}$ $\bar{b}$ $\bar{C}$ $\bar{c}$ . . . (all)

★ Tilde Italics—$\tilde{A}$ $\tilde{a}$ $\tilde{e}$ $\tilde{N}$ $\tilde{n}$ $\tilde{O}$ $\tilde{r}$ $\tilde{u}$ $\tilde{g}$ $\tilde{s}$ $\tilde{M}$ $\tilde{T}$ $\tilde{B}$ $\tilde{K}$

★ Tilde Greek— $\tilde{\alpha}$ $\tilde{\epsilon}$ $\tilde{\eta}$ $\tilde{\iota}$ $\tilde{o}$ $\tilde{\upsilon}$ $\tilde{\omega}$

★ Dashed Greek—$\bar{\alpha}$ $\bar{\beta}$ $\bar{\gamma}$ $\bar{\delta}$ $\bar{\eta}$ $\bar{\theta}$ $\bar{\iota}$ $\bar{\mu}$ $\bar{\nu}$ $\bar{\rho}$ $\bar{\omega}$ $\bar{\Gamma}$ $\bar{\lambda}$ $\bar{\zeta}$ $\bar{\sigma}$ $\bar{\psi}$ $\bar{\xi}$ $\bar{\epsilon}$

★ Dotted Italic—$\dot{a}$ $\ddot{a}$ $\dot{e}$ $\ddot{e}$ $\dot{g}$ $\dot{l}$ $\dot{m}$ $\dot{n}$ $\dot{q}$ $\dot{r}$ $\ddot{r}$ $\dot{u}$ $\ddot{u}$ $\dot{x}$ $\ddot{x}$ $\dot{y}$ $\ddot{y}$ $\dot{z}$ $\ddot{z}$

★ Dotted Greek—$\ddot{\epsilon}$ $\dot{\eta}$ $\ddot{\eta}$ $\dot{\theta}$ $\ddot{\theta}$ $\dot{\xi}$ $\dot{\zeta}$ $\dot{\psi}$ $\ddot{\psi}$ $\ddot{\omega}$ (single dotted $\beta$ $\gamma$ $\delta$ $\phi$; double dotted $\gamma$ readily available).

★ Superiors—[illegible]

★ Inferiors—[illegible]

Signs—Superior—$^{-\ +\ -\ \infty\ \pm\ (\ )\ [\ ]\ .\ ,\ \prime\ /\ |}$

Signs—Superior to Superior—$^{-\ +\ -}$

Signs—Inferior—$_{-\ -\ +\ \infty\ \pm\ \nparallel\ (\ )\ [\ ]\ .\ ,\ /\ |\ \prime}$

Signs—Inferior to Inferior—$_{+\ -\ -}$

Special—Superior to Superior—[illegible]

Special—Inferior to Inferior—[illegible]

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* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders. Normally they would necessitate a delay of from four to eight weeks and average expense of $4.50 per matrix. However, because of wartime conditions delivery dates are uncertain and prices subject to change without notice.

| No. | 6 | 8 | 10 | 12 | No. | 6 | 8 | 10 | 12[4] |
|---|---|---|---|---|---|---|---|---|---|
| 1. − | √ | √ | √ | √ | 40. ∨ |  | √ | √ |  |
| 2. ·× | √ | √ | √ | √ | 41. ⊃ |  |  | √ |  |
| 3. ꭓ |  | √ |  |  | 42. ⊂ |  |  | √ |  |
| 4. ÷ | √ | √ | √ | √ | 43. ⊅ |  |  | √ |  |
| 5. + | √ | √ | √ | √ | 44. ∪ | √ |  |  |  |
| 6. † |  |  | √ |  | 45. ∩ |  | √ |  |  |
| 7. ± |  | √ |  |  | 46. ∴ |  |  | √ |  |
| 8. ± | √ | √ | √ | √ | 47. △ |  |  | √ |  |
| 9. ∓ |  |  | √ |  | 48. ⟁ |  |  | √ |  |
| 10. = | √ | √ | √ | √ | 49. ◬ |  | √ |  |  |
| 11. ≡ |  | √ | √ |  | 50. ▲ |  | √ |  |  |
| 12. ⩲ | √ |  | √ |  | 51. ∠ |  | √ | √ |  |
| 13. ≅ | √ |  | √ |  | 52. ∡ |  |  | √ |  |
| 14. ≌ |  | √ |  |  | 53. □ |  |  | √ |  |
| 15. ≭ | √ | √ | √ |  | 54. ⊡ |  | √ |  |  |
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[4] These figures refer to the size of type. The text of Bulletin and Transactions is 10 point type and footnotes 8 point type.

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# THE ELEMENTARY DIVISOR THEOREM FOR CERTAIN RINGS WITHOUT CHAIN CONDITION

OLAF HELMER

1. **Introduction.** The Elementary Divisor Theorem is known to hold in principal ideal rings and in rings with Euclidean algorithm.[1] It is an open question whether it holds in every *Prüfer ring*, that is, in every domain of integrity in which every ideal with finite basis is a principal ideal.[2] (A Prüfer ring can also be characterized as a domain of integrity in which the greatest common factor of any finite number of elements can be represented as a linear combination of these elements.) The Elementary Divisor Theorem will here be proved for what will be called "adequate" rings. They are special Prüfer rings, but not restricted by any equivalent to a chain condition, so that they comprise considerably more than just the principal ideal rings.[3]

2. **Definition of adequate rings.** Let $R$ be a domain of integrity, $a$, $b$ in $R$, and $a \neq 0$. By a *relatively prime part* of $a$ with respect to $b$, written $RP(a, b)$, we shall understand a factor $a_1$ of $a$ such that, if $a = a_1 \cdot a_2$,

(i) $(a_1, b) = 1$,

(ii) $(a_3, b) \neq 1$ for any non-unit factor $a_3$ of $a_2$.[4]

$RP(a, b)$ may or may not exist; if it does, it is, in a sense, a largest factor of $a$ that is relatively prime to $b$.

We now define $R$ to be an *adequate ring* if

(i) $R$ is a Prüfer ring,

(ii) $RP(a, b)$ exists for all $a$, $b$ in $R$ with $a \neq 0$.

3. **Relationship to Prüfer rings and principal ideal rings.** By definition every adequate ring is a Prüfer ring. On the other hand, every

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[1] B. L. van der Waerden, *Moderne Algebra*, vol. 2, Berlin, 1931, p. 122. For further reference, see the papers by J. H. M. Wedderburn (J. Reine Angew. Math. vol. 167 (1932)), N. Jacobson (Ann. of Math. (2) vol. 38 (1937)), and O. Teichmuller (Preuss. Akad. Wiss. Sitzungsber. 1937).

[2] Rings of this kind were considered by H. Prufer in *Untersuchungen uber Teilbarkeitseigenschaften in Korpern*, J. Reine Angew. Math. vol. 168 (1932).

[3] My thanks are due to Professor Reinhold Baer for several helpful suggestions.

[4] Notation throughout this paper: "$(a, b, \cdots)$" for "greatest common factor of $a, b, \cdots$," "$a | b$" for "$a$ is a factor of $b$," "$a = b$" for "$a$ equals $b$ except possibly for a unit factor." (This last convention serves the purpose of replacing statements about principal ideals simply by statements about their generating elements.)

principal ideal ring is adequate; for in a ring of that kind we have unique prime power decomposition, and hence the existence of $RP(a, b)$ for $a \neq 0$ is obvious. An example of an adequate ring which is not a principal ideal ring is furnished by the set of integral functions with coefficients in a field $F$. That a ring of this type is a Prüfer ring has been proved in a previous paper.[5] The existence of $RP(a, b)$ is an immediate consequence of the unique factorization theorem for integral functions (Theorem 6 of the same paper). Theorem 8 of that paper, finally, states that the ring is not a principal ideal ring.

4. **Six lemmas.** Now let $R$ be an adequate ring. Among the properties of such a ring are those stated in the six lemmas below. The first four are well known, and hold in any Prüfer ring; sketches of their proofs have been included, however, for reasons of completeness.

LEMMA 1. *If $f \mid ab$ and $(f, a) = 1$, then $f \mid b$.*

PROOF. Let $hf + ka = 1$; then $f \mid abk = b(1 - hf) = b - bhf$ and hence $f \mid b$.

LEMMA 2. *An element $a$ has an inverse $a^*$* mod $b$,

$$a \cdot a^* \equiv 1 \pmod{b}, \tag{1}$$

*if and only if $(a, b) = 1$. Moreover, $a^*$, if it exists, is uniquely determined* mod $b$.

PROOF. If $aa^* \equiv 1 \pmod{b}$, then $aa^* = 1 + bk$ and hence $(a, b) = 1$. Conversely, if $(a, b) = 1$, then $h$ and $k$ exist such that $ha + kb = 1$, and hence $ah \equiv 1 \pmod{b}$ and $a^* = h$. Finally, if $aa^* \equiv aa^{**} \equiv 1 \pmod{b}$, then $a^* \equiv a^*(aa^{**}) \equiv (aa^*)a^{**} \equiv a^{**} \pmod{b}$.

LEMMA 3. *The congruences*

$$x \equiv a_i \pmod{m_i}, \qquad i = 1, 2, \cdots, n, \tag{2}$$

*have a simultaneous solution $x$, provided any two $m_i$ are relatively prime. Moreover, the solution is uniquely determined* mod $m$, *where $m = m_1 m_2 \cdots m_n$.*

PROOF. Let $m = m_i M_i$. Since any two $m_i$ are relatively prime it follows that $(m_i, M_i) = 1$. Hence, by Lemma 2, the $M_i$ possess inverses $M_i^*$ mod $m_i$. It is seen at once that

$$x \equiv a_1 M_1 M_1^* + \cdots + a_n M_n M_n^* \pmod{m} \tag{3}$$

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[5] Theorem 9 in *Divisibility properties of integral functions*, Duke Math. J. vol. 6 (1940) p. 351.

is a required solution, and the uniqueness is obvious.

LEMMA 4. *Let* $a_1, a_2, \cdots, a_n$ *not be all zero, and*

$$(a_1, a_2, \cdots, a_n) = d; \tag{4}$$

*then there exist elements* $a_{ij}$ $(i=2, \cdots, n; j=1, \cdots, n)$ *such that*

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = d. \tag{5}$$

PROOF. If all but one $a_k$ are zero, then $d = a_k$. Choose $a_{2k} = \cdots = a_{nk} = 0$, and the unity matrix for the rest of the $a_{ij}$; the resulting determinant will be $d$ or $-d$, if $-d$, a change of one 1 into $-1$ will put things right. If, on the other hand, no more than $n-2$ of the $a_k$ are zero, we can proceed by induction. For $n = 1$ the assertion is trivially correct. For $n > 1$ let

$$d' = (a_2, \cdots, a_n); \tag{6}$$

by induction we can find $a_{ij}$ $(i=2, \cdots, n-1; j=2, \cdots, n)$ such that

$$\begin{vmatrix} a_2 & \cdots & a_n \\ a_{22} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot \\ a_{n-1,2} & \cdots & a_{n-1,n} \end{vmatrix} = d'. \tag{7}$$

Suppose now that

$$d = (a_1, d') = h\cdot a_1 + k\cdot d'. \tag{8}$$

We can then complete our determinant as follows:

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & a_{n-1,2} & \cdots & a_{n-1,n} \\ (-1)^{n-1}k & (-1)^n h a_2/d' & \cdots & (-1)^n h a_n/d' \end{vmatrix} = h\, a_1 + k\cdot d' = d. \tag{9}$$

LEMMA 5. $RP(a, b)$ *is uniquely determined except for a unit factor.*

PROOF. Let

$$a = a_1\cdot a_2 = a'\cdot a'', \tag{10}$$

where $a_1$ and $a'$ are both largest factors of $a$ that are relatively prime to $b$. We show first that $(a_1, a')=1$. Suppose that $(a_1, a')=d\neq 1$; then $d|a_1$, and by definition of $a_1$, $(d, b)\neq 1$; but $d|a'$ implies $(d, b)|(a', b)$, so that $(a', b)\neq 1$, which is impossible by definition of $a'$. Hence $(a_1, a')=1$, and we can now apply Lemma 1: since $a_1|a'a''$, it follows that $a_1|a''$. Similarly $a''|a_1$, so that $a_1$ and $a''$ can only differ by a unit factor.

Lemma 6. *Let $RP(a, b)=r$, and let $d|a$ with $(d, r)=1$; then $RP(d, b)=1$.*

Proof. Let $a=r\ s$. By assumption and Lemma 1, $d|s$. Hence, by definition of $r$ and $s$, $(\delta, b)\neq 1$ for every non-unit factor $\delta$ of $d$, and consequently $RP(d, b)=1$.

## 5. A preparatory theorem.

The main result of this paper will be a consequence of the following theorem.

Theorem 1. *Let*

$$M = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1k} \\ a_{21} & a_{22} \cdots a_{2k} \\ \cdot & \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ a_{n1} & a_{n2} \cdots a_{nk} \end{pmatrix} \tag{11}$$

*be a matrix with coefficients in an adequate ring $R$, let*

$$1 < n = \text{rank } M \leqq k \tag{12}$$

*and*

$$(a_{11}, a_{12}, \cdots, a_{1k}, a_{21}, \cdots, a_{nk}) = d; \tag{13}$$

*then there exist $t_1, t_2, \cdots, t_{n-1}$ in $R$ such that*

$$(A_1, A_2, \cdots, A_k) = d, \tag{14}$$

*where*

$$A_i = a_{1i}t_1 + a_{2i}t_2 + \cdots + a_{n-1,i}t_{n-1} + a_{ni}, \qquad i = 1, 2, \cdots, k. \tag{15}$$

Proof. (a) Suppose the theorem has been proved for $d=1$. Then the general case results as follows. Let

$$N = (1/d)\cdot M; \tag{16}$$

the coefficients

$$b_{ij} = a_{ij}/d \tag{17}$$

of the matrix $N$ are then relatively prime, and the theorem is considered proved for this case; there are, therefore, quantities $t_1, t_2, \cdots, t_{n-1}$ in $R$ such that

$$(B_1, B_2, \cdots, B_k) = 1, \tag{18}$$

where

$$B_i = b_{1i}t_1 + b_{2i}t_2 + \cdots + b_{n-1,i}t_{n-1} + b_{ni} = A_i/d. \tag{19}$$

Hence

$$(A_1, A_2, \cdots, A_k) = d\cdot(B_1, B_2, \cdots, B_k) = d. \tag{20}$$

(b) For $d=1$ the proof will be given by induction with respect to $n$. First, let $n=2$. We have to find $t$ so that the

$$A_i = a_{1i}t + a_{2i}, \qquad i = 1, 2, \cdots, k, \tag{21}$$

have the greatest common factor 1. Note that any common factor of $A_i$ and $A_j$ is also a factor of

$$D_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} = a_{1i}A_j - a_{1j}A_i. \tag{22}$$

If, therefore, we set

$$D = (D_{12}, D_{13}, D_{23}, D_{14}, \cdots, D_{k-1,k}), \tag{23}$$

then, however $t$ be chosen,

$$(A_1, A_2, \cdots, A_k) \mid D. \tag{24}$$

On account of (12), not all $D_{ij}$ are zero; hence $D\neq 0$, and we may define:

$$\left\{\begin{aligned} D_1 &= RP(D, a_{11}),\\ D_2 &= RP\left(\frac{D}{D_1}, a_{12}\right),\\ &\cdots\cdots\cdots\cdots,\\ D_k &= RP\left(\frac{D}{D_1D_2\cdots D_{k-1}}, a_{1k}\right).\end{aligned}\right. \tag{25}$$

We then have, obviously,

$$(D_i, D_j) = 1, \qquad i \neq j, \tag{26}$$

because a largest relatively prime factor of a quantity is always prime to the supplementary factor. Consequently, by Lemma 3, there exists a simultaneous solution $t$ of the congruences

(27) $$t \equiv \overset{*}{a}_{1i} \cdot (1 - a_{2i}) \pmod{D_i}, \qquad i = 1, 2, \cdots, k,$$

where $a^*_{1i}$ is an inverse of $a_{1i}$ mod $D_i$:

(28) $$\overset{*}{a}_{1i} \cdot a_{1i} \equiv 1 \pmod{D_i}$$

(the existence of $a^*_{1i}$ being guaranteed by Lemma 2, since $(a_{1i}, D_i) = 1$ by virtue of (25)). With $t$ chosen thus, we shall now show that

(29) $$(A_1, A_2, \cdots, A_k) = 1.$$

From (21) and (27) we have

(30) $$A_i = a_{1i}t + a_{2i} \equiv 1 \pmod{D_i}.$$

Suppose, now, that the $A_i$ have some non-unit factor $c$ in common; then, by (24), it would follow that $c|D$. This, together with (30), implies that

(31) $$(c, D_i) = 1;$$

this means that $c$ is a factor of $D$ that is prime to all $D_i$, and we can apply Lemma 6, with the result that

(32) $$RP(c, a_{1i}) = 1, \qquad i = 1, 2, \cdots, k.$$

But then, as $c$ was not supposed to be a unit, none of its non-unit factors is prime to any $a_{1i}$. In particular,

(33) $$\begin{cases} (c, a_{11}) = c_1 \neq 1, \\ (c_1, a_{12}) = c_2 \neq 1, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ (c_{k-1}, a_{1k}) = c_k \neq 1; \end{cases}$$

the $a_{1i}$ thus have a common non-unit factor $c_k$, and, since $c_k | A_i$, it follows that $c_k$ is also a factor of $A_i - a_{1i}t = a_{2i}$. But this is incompatible with the assumption that $d = 1$.

(c) Suppose now that the theorem has been proved for values less than $n$; we then have to show that it holds for $n$ provided $d = 1$. Let

(34) $$d_1 = (a_{21}, a_{22}, \cdots, a_{2k}, a_{31}, \cdots, a_{nk});$$

applying induction, we can find $t_2, t_3, \cdots, t_{n-1}$ so that

(35) $$(B_1, B_2, \cdots, B_k) = d_1$$

where

(36) $$B_i = a_{2i}t_2 + a_{3i}t_3 + \cdots + a_{n-1,i}t_{n-1} + a_{ni}, \qquad i = 1, 2, \cdots, k,$$

and it remains to determine $t_1$ suitably. We have

$$A_i = a_{1i}t_1 + B_i \tag{37}$$

with

$$\begin{aligned}(a_{11}, B_1, a_{12}, B_2, \cdots, a_{1k}, B_k) &= (a_{11}, a_{12}, \cdots, a_{1k}, d_1) \\ &= (a_{11}, a_{12}, \cdots, a_{1k}, a_{21}, \cdots, a_{nk}) = 1.\end{aligned} \tag{38}$$

Therefore, using induction with $n=2$, we can find $t_1$ such that

$$(A_1, A_2, \cdots, A_k) = 1, \tag{39}$$

and the proof of Theorem 1 is completed.

6. **The case of a square matrix with nonzero determinant.** Before taking up the general case of the Elementary Divisor Theorem we shall first treat the case of a nonsingular matrix.

THEOREM 2. *Let $R$ be an adequate ring, $M$ a square matrix with coefficients in $R$, and $|M| \neq 0$. Then there exist two matrices $X$ and $Y$ with coefficients in $R$, such that*

$$|X| = |Y| = 1 \tag{40}$$

*and*

$$XMY = \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e_n \end{pmatrix}, \tag{41}$$

*where the $e_i$ are the elementary divisors of $M$.*

PROOF. For $n=1$ the theorem is trivially true. For $n>1$ assume the theorem proved for all values less than $n$. It is sufficient to consider the case where $e_1$ (which is the greatest common factor of the coefficients of $M$) equals 1; for otherwise we first deal with the matrix $M/e_1$, whose elementary divisors are $1, e_2/e_1, \cdots, e_n/e_1$; hence if

$$X\frac{M}{e_1}Y = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e_2/e_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e_n/e_1 \end{pmatrix}, \tag{42}$$

we have at once

$$XMY = \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e_n \end{pmatrix}. \tag{43}$$

Let now $e_1=1$ and $M=(a_{ij})$. Applying Theorem 1, with $k=n$ and $d=1$, we can find $t_1, t_2, \cdots, t_{n-1}$ in $R$ such that

$$(A_1, A_2, \cdots, A_n) = 1, \tag{44}$$

where

$$A_i = a_{1i}t_1 + a_{2i}t_2 + \cdots + a_{n-1,i}t_{n-1} + a_{ni}, \qquad i = 1, 2, \cdots, n. \tag{45}$$

Applying Lemma 4 to these $A_i$, a square matrix $A$ with coefficients in $R$ can be found whose first row is $A_1, A_2, \cdots, A_n$ and whose determinant is 1. If $k_i$ is the minor belonging to $A_i$, we have

$$k_1A_1 + k_2A_2 + \cdots + k_nA_n = 1. \tag{46}$$

As $|A| = 1$, we can form, in $R$, the inverse matrix

$$Q = A^{-1}. \tag{47}$$

Note that the first column of $Q$ is $k_1, k_2, \cdots, k_n$. If, now, we set

$$q_i = k_1a_{i1} + k_2a_{i2} + \cdots + k_na_{in}, \qquad i = 1, 2, \cdots, n, \tag{48}$$

we can define

$$P = P_1 \cdot P_2, \tag{49}$$

where

$$P_1 = \begin{pmatrix} 1 & 0 \cdots & 0 \\ q_1 & 0 \cdots & -1 \\ \cdot & \cdots\cdot & \cdot \\ q_{n-1} & -1 \cdots & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} t_1 \cdots t_{n-1} & 1 \\ 0 \cdots 1 & 0 \\ \cdot\ \cdots\ \cdot & \cdot \\ 1 \cdots 0 & 0 \end{pmatrix}. \tag{50}$$

Then

$$|P_1| = |P_2| = (-1)^{n(n-1)/2} \tag{51}$$

so that

$$|P| = |P_1| \cdot |P_2| = 1. \tag{52}$$

Multiplying out, we obtain

$$(53)\qquad P = \begin{pmatrix} t_1 & t_2 & \cdots & t_{n-1} & 1 \\ q_1t_1 - 1 & q_1t_2 & \cdots & q_1t_{n-1} & q_1 \\ q_2t_1 & q_2t_2 - 1 & \cdots & q_2t_{n-1} & q_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ q_{n-1}t_1 & q_{n-1}t_2 & \cdots & q_{n-1}t_{n-1} - 1 & q_{n-1} \end{pmatrix}.$$

Notice that we also have

$$(54)\qquad |Q| = 1.$$

We now multiply $M$ from the left by $P$ and from the right by $Q$:

$$(55)\qquad \begin{aligned} PMQ = &\begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ q_1A_1 - a_{11} & q_1A_2 - a_{12} & \cdots & q_1A_n - a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ q_{n-1}A_1 - a_{n-1,1} & q_{n-1}A_2 - a_{n-1,2} & \cdots & q_{n-1}A_n - a_{n-1,n} \end{pmatrix} \\ &\cdot \begin{pmatrix} k_1 & * \cdots * \\ k_2 & * \cdots * \\ \cdot & \cdots\cdots \\ k_n & * \cdots * \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots 0 \\ 0 & * & \cdots * \\ \cdot & \cdot & \cdots \\ 0 & * & \cdots * \end{pmatrix} \end{aligned}$$

where the zeros in the first row of this matrix arise from multiplying the row $A_1, A_2, \cdots, A_n$ by a row of minors of $A$ not belonging to that row, while the zero in the $(i+1)$st place of the first column results as follows:

$$(56)\qquad \begin{aligned} &(q_iA_1 - a_{i1})k_1 + \cdots + (q_iA_n - a_{in})k_n \\ &\quad = q_i(A_1k_1 + \cdots + A_nk_n) - (a_{i1}k_1 + \cdots + a_{in}k_n) \\ &\quad = q_i \cdot 1 - q_i = 0. \end{aligned}$$

We can now apply induction to the matrix $M_1$ of order $n-1$ that appears in the lower right corner of $PMQ$. Let

$$(57)\qquad X_1M_1Y_1 = \begin{pmatrix} e_1' & 0 & \cdots & 0 \\ 0 & e_2' & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & e_{n-1}' \end{pmatrix}$$

where $|X_1| = |Y_1| = 1$ and where $e_1', e_2', \cdots, e_{n-1}'$ are the elementary divisors of $M_1$. Then, if we set

$$X = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & X_1 \\ 0 & \end{pmatrix} \cdot P, \qquad Y = Q \cdot \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & Y_1 \\ 0 & \end{pmatrix}, \tag{58}$$

we obtain immediately

$$XMY = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e_1' & 0 & \cdots & 0 \\ 0 & 0 & e_2' & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & e_{r-1}' \end{pmatrix}, \tag{59}$$

and all that remains to be shown is that

$$e_i = e_{i-1}', \qquad i = 2, 3, \cdots, n. \tag{60}$$

Now, clearly, the determinant divisors of $XMY$ are $1, e_1', e_1'e_2', \cdots, e_1'e_2' \cdots e_{n-1}'$; hence its elementary divisors are $1, e_1', e_2', \cdots, e_{n-1}'$, and the rest follows from the fact that "equivalent" matrices (here $M$ and $XMY$) have the same elementary divisors (proof literally as for the ring of rational integers).

## 7. The general case.

We can now prove the general Elementary Divisor Theorem:

THEOREM 3. *Let $R$ be an adequate ring, and $M$ a matrix of type $(m, n)$ with coefficients in $R$. Then there exist two matrices $X$ and $Y$ with coefficients in $R$ and of types $(m, m)$ and $(n, n)$ respectively, such that*

$$|X| = |Y| = 1 \tag{61}$$

*and*

$$XMY = \begin{pmatrix} e_1 & 0 & & \cdots & & 0 \\ 0 & e_2 & & & & \cdot \\ \cdot & & \ddots & & & \cdot \\ & & & e_r & & \\ \cdot & & & & 0 & \cdot \\ \cdot & & & & & \ddots \\ 0 & \cdot & & \cdots & & 0 \end{pmatrix} \tag{62}$$

*where $e_1, e_2, \cdots, e_r$ are the elementary divisors of $M$.*

PROOF. (a) Let $M=(a_{ij})$, of rank $r$. If $r=0$, we can set $X=E_m$ and $Y=E_n$. Otherwise not all $a_{ij}$ are zero, and we have

$$(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, \cdots, a_{mn}) = d \neq 0. \tag{63}$$

The case where $m=n=r$ is covered by Theorem 2. If $r<m$ or $r<n$ or both, it will be shown below that $M$ can be transformed unitarily into a matrix of the form

$$\begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} \tag{64}$$

where $N$ is of type $(r, r)$ (and hence $|N|\neq 0$). Once this has been done, Theorem 2 can be applied to $N$; and if, say, $X_1$ and $Y_1$ take $N$ into the desired normal form, then

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & E_{m-r} \end{pmatrix}, \quad Y = \begin{pmatrix} X_2 & 0 \\ 0 & E_{n-r} \end{pmatrix} \tag{65}$$

will do the same for $M$.

(b) Let $r<m$. The $m$ rows of $M$ must then be linearly dependent in $R$:

$$j_1a_{1i} + j_2a_{2i} + \cdots + j_ma_{mi} = 0, \qquad i = 1, 2, \cdots, n, \tag{66}$$

where the $j_i$ may be assumed to be relatively prime. By Lemma 4 we can find elements $j_{ik}$ in $R$ $(i=1, \cdots, m-1;\ k=1, \cdots, m)$ such that the matrix

$$J = \begin{pmatrix} j_{11} & j_{12} \cdots & j_{1m} \\ \cdot & \cdots\cdot & \cdot \\ j_{m-1,1} & \cdots & j_{m-1,m} \\ j_1 & j_2 \cdots & j_m \end{pmatrix} \tag{67}$$

has determinant 1; and if we multiply $M$ on the left by $J$, we obtain a matrix whose last row consists of zeros.

Clearly, if $r<n$, an analogous multiplication of $M$ on the right by a unitary matrix will generate a last column of zeros.

This procedure can be applied repeatedly. If

$$X_1MX_2 = \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{68}$$

where $M_1$ is of type $(m_1, n_1)$ with $r<m_1$ or $r<n_1$, a further row or column of $M_1$ (and hence of $M$) can be turned into zeros. This process will break off only when the form

$$\begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} \tag{69}$$

with $N$ of type $(r, r)$ has been reached. This, according to (a), completes the proof.

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# THE GEOMETRY OF VELOCITY SYSTEMS

EDWARD KASNER AND JOHN DeCICCO

1. **Introduction.** In connection with the investigation of the differential geometric properties of positional fields of force, Kasner introduced certain important systems of curves which he termed *velocity systems*.[1] In this paper, we propose to present some of the old and many new geometric properties of any velocity system.

Since velocity systems serve to characterize the conformal group, many of our results are logically theorems of conformal geometry. This may be contrasted with the study of all the trajectories of positional fields of force, which is essentially differential projective geometry.

2. **Velocity systems.** We shall find it most convenient for our study to use the minimal coordinates $u=x+iy$, $v=x-iy$, $p=dv/du$, $q=d^2v/du^2$, where, of course, $(x, y)$ are the cartesian coordinates of the plane.

In the plane, consider a particle of unit mass moving in any field of force whose components parallel to the $u$-axis and the $v$-axis are $d(u, v)$ and $c(u, v)$. The equations of motion are

$$\frac{d^2u}{dt^2} = d(u, v), \qquad \frac{d^2v}{dt^2} = c(u, v), \tag{1}$$

where $t$ is the time. Now if $r$ is the radius of curvature along a trajectory, we have

$$\omega^2 = rN, \tag{2}$$

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[1] Kasner, *Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications vol. 3 (1912); Trans. Amer. Math. Soc. vols. 9–10 (1908–1909).

where $\omega = p^{1/2}du/dt$ is the velocity (speed) and $N$ is the principal normal component of the force given by

$$N = \frac{c - dp}{2ip^{1/2}}. \tag{3}$$

Since (1) defines a three-parameter family of curves, the path is uniquely determined if the initial position, the initial direction, and the initial velocity are given.

If only the initial position and the initial direction of motion are given, then the initial radius of curvature will depend on the initial velocity. In the actual trajectory, the velocity $\omega$ varies from point to point. If now the square of the velocity $\omega^2$ is replaced by some constant, say 1/const., the resulting equation is

$$q = \text{const. } p(c - dp). \tag{4}$$

The curves satisfying this differential equation, they are not in general trajectories, are called *velocity curves.*

For any field, a curve is a velocity curve corresponding to the speed $\omega_0$, provided any particle starting from a lineal element of the curve with that speed, describes a trajectory which is initially osculated by that curve. In a given field of force, there are in all $\infty^3$ trajectories and $\infty^3$ velocity curves.

If the constant is given, there are $\infty^2$ velocity curves. In particular, if the constant and hence the velocity is taken to be unity, the following result is obtained.

THEOREM 1. *Any system of $\infty^2$ curves defined by a differential equation of the form*[2]

$$q = p(c - dp), \tag{5}$$

*may be regarded as the totality of velocity curves corresponding to unit velocity in the unique field of force whose u-component is $d(u, v)$ and whose v-component is $c(u, v)$.*

Throughout this paper, we shall understand a velocity system to be the totality of velocity curves with unit velocity. That is, any velocity system is given by the preceding differential equation.

THEOREM 2. PROPERTY A. *A set of $\infty^2$ curves of the plane is a velocity system if and only if the $\infty^1$ osculating circles of the $\infty^1$ curves of the set passing through any point $(u, v)$, constructed at $(u, v)$, form a linear pencil of circles.*

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[2] The corresponding cartesian equation of the general velocity system is $y'' = (1+y'^2)(\chi - y'\phi)$ where $\phi$ and $\chi$ are any functions of $(x, y)$.

Of course, the centers of these circles describe a straight line. The other point of intersection $(U, V)$ of this pencil of circles is

$$(6) \qquad U = u + 2/c, \qquad V = v + 2/d.$$

We shall call this the *associated correspondence* of the velocity system.

Obviously any arbitrary point transformation except identity, may serve as the associated correspondence of a unique velocity system.

Special important types of velocity systems are natural families, isogonal families, and $\Gamma$ families of curves. Now we shall consider these sets of curves (special wexes).

3. **Natural families.** A natural family consists of the extremals connected with the variation problem of the form

$$(7) \qquad \int F(u, v)ds = \text{minimum},$$

where $F$ is a point function and $ds = p^{1/2}du$ is the element of arc. Any natural family is a velocity system for which $c_v = d_u$.

Kasner has already indicated the many problems which are connected with natural families. Thus natural families arise in the study of (a) the trajectories of a conservative field of force, (b) general catenaries, (c) the paths of light in an isotropic medium, and (d) the geodesics of a surface.[1] (See papers by Lipka, Douglas, Blaschke, Fialkow, DeCicco, Struik.)

A natural family of curves is a special type of velocity system. Hence in addition to the Property A, it must possess an additional differential property.

THEOREM 3. PROPERTY B. *At any point $(u, v)$, there exist two lineal elements $E_1$ and $E_2$, each of which is converted by the associated correspondence of the velocity system into a cocircular element at the point $(U, V)$. Our velocity system is a natural family if and only if $E_1$ and $E_2$ are orthogonal.*

Thus any system of $\infty^2$ curves with the properties $A$ and $B$ is a natural family.

4. **Isogonal families.** Any nonminimal simple family of $\infty^1$ curves is given by a differential equation of the first order $p = e^{f(u,v)}$. The set of *all* simple families of curves each of which makes a constant angle with all the curves of the given simple family is called a complete *isogonal family*. Any isogonal family is a velocity system for which $c = f_u$ and $d = -f_v$ so that $c_v = -d_u$.

THEOREM 4. PROPERTY C. *At any point* $(u, v)$, *there exist two lineal elements* $E_1$ *and* $E_2$, *each of which is converted by the associated correspondence of the velocity system into an element orthogonal to the cocircular element at the point* $(U, V)$. *Our velocity system is an isogonal family if and only if* $E_1$ *and* $E_2$ *are othogonal.*

Thus the conditions necessary and sufficient for an isogonal family are properties A and C. Thus the famous theorem of Cesàro-Scheffers is completed.

5. $\Gamma_0$ **systems.** The search for those velocity systems which are both natural and isogonal yields the following result.

THEOREM 5. *For a velocity system to be both natural and isogonal (that is, to possess properties* A, B, *and* C), *it is necessary and sufficient that it be the complete set of isogonal trajectories of an isothermal family.*

This family may be called a *conformal rectilinear wex.*[3] That is, it is conformally equivalent to the $\infty^2$ straight lines of the plane. Any such system shall be denoted by $\Gamma_0$.[4] Any system $\Gamma_0$ may be characterized geometrically as a velocity system whose associated point correspondence is direct conformal.

6. **The Γ families.** Any set of $\infty^2$ curves which is conformally equivalent to the set of $\infty^2$ circles orthogonal to a fixed proper circle (or a fixed straight line) is called a Γ *family.*[5]

The finite form of a Γ family is

$$a_0\phi(u)\psi(v) + b_0[\phi(u) + \psi(v)] + c_0 = 0. \tag{8}$$

Upon eliminating the constants, $a_0$, $b_0$, $c_0$, we discover that a Γ family is a velocity system where the basic functions

$$c = \frac{\phi_{uu}}{\phi_u} - \frac{2\phi_u}{\phi - \psi}, \qquad d = \frac{\psi_{vv}}{\psi_v} + \frac{2\psi_v}{\phi - \psi}. \tag{9}$$

Now we proceed to find the conditions on the functions $c$ and $d$ that a velocity system be a Γ family. Solving the first equation for $\psi$, and the second for $\phi$, we find

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[3] Kasner has introduced the word *wex* to represent any system of $\infty^2$ curves. Thus a wex is given by a differential equation of the second order $y'' = f(x, y, y')$. A web denotes $n$ $\infty^1$ curves.

[4] The authors have determined all curvature element transformations which preserve the class of $\Gamma_0$ systems. See Kasner and DeCicco, *Transformation theory of the isogonal trajectories of an isothermal family*, Proc. Nat. Acad. Sci. U.S.A., vol. 28 (1942).

[5] The Γ and $\Gamma_0$ systems are special subsets of Ω families. Any Ω family is conformally equivalent to the $\infty^3$ circles. See Kasner and DeCicco, *Families of curves conformally equivalent to circles*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 378–391.

$$(10) \qquad \psi = \phi + \frac{2\phi_u}{c - \phi_{uu}/\phi_u}, \qquad \phi = \psi + \frac{2\psi_v}{d - \psi_{vv}/\psi_v}.$$

By differentiating the first equation of (10) with respect to $u$ and the second with respect to $v$, we obtain

$$(11) \quad 2c_u - c^2 = \frac{2\phi_u\phi_{uuu} - 3\phi_{uu}^2}{\phi_u^2}, \quad 2d_v - d^2 = \frac{2\psi_v\psi_{vvv} - 3\psi_{vv}^2}{\psi_v^2}.$$

Substituting the values of $\phi$ and $\psi$ as determined by (10) into these equations, we find

$$(12) \qquad dc_v = c_{vv}, \qquad cd_u = d_{uu}.$$

THEOREM 6. PROPERTY D. *A velocity system represents a $\Gamma$ family of curves if and only if the functions $c$ and $d$ with $c_v \neq 0$ and $d_u \neq 0$ satisfy either one of the three equivalent systems of partial differential equations of the second order.*

$$(13.1) \qquad c_{uv} = cc_v, \qquad c_{vv} = dc_v;$$

$$(13.2) \qquad d_{uv} = dd_u, \qquad d_{uu} = cd_u;$$

$$(13.3) \qquad c_v = d_u, \qquad c_{uv} = cc_v, \qquad d_{uv} = dd_u.$$

The last set of equations demonstrates that every $\Gamma$ family is a natural family. But the converse is not true. The only possible $\Gamma$ families which are isogonal are the $\Gamma_0$ systems.

7. **Velocity systems which possess isothermal families.** We shall now classify velocity systems according to the number of isothermal families they contain. For the velocity system (5) to contain the simple family $p = e^{f(u,v)}$, it is necessary and sufficient that

$$(14) \qquad f_u + e^f f_v = c - e^f d.$$

If the simple family is isothermal, then, as deduced from the theorem of Lie, $f_{uv} = 0$. Differentiating the above partially with respect to $v$ and simplifying, we find

$$(15) \qquad f_{vv} + f_v^2 + df_v = e^{-f} c_v - d_v.$$

Differentiating this partially with respect to $u$ and making use of (14), we obtain

$$(16) \qquad (c_v - d_u)f_v = (d_{uv} - dc_v) - (c_{uv} - cc_v)e^{-f}.$$

From these equations, it is concluded that a velocity system will contain *exactly* $\infty^2$ isothermal systems if and only if it is a $\Gamma$ family. This is a characteristic property of $\Gamma$ families (including $\Gamma_0$).

Suppose next our velocity system is natural but *not* a $\Gamma$ family. Then there exists possibly *one* isothermal family of curves. This possible isothermal family is

$$p = \frac{c_{uv} - cc_v}{d_{uv} - dc_v} \quad \text{where} \quad c_v = d_u. \tag{17}$$

Finally let us suppose that our family is *not* natural. Such a velocity system may possess at most $\infty^1$ isothermal families. These are defined by (14) and (16).

Differentiating the equation (16) with respect to $u$ and eliminating $f_u$ and $f_v$ from the result, we find

$$e^f = \frac{\left(\dfrac{\partial}{\partial u}\dfrac{c_{uv} - cc_v}{c_v - d_u}\right) - \left(\dfrac{c_{uv} - cc_v}{c_v - d_u}\right)^2 - \dfrac{c(c_{uv} - cc_v)}{c_v - d_u}}{\left(\dfrac{\partial}{\partial u}\dfrac{d_{uv} - dc_v}{c_v - d_u}\right) - \dfrac{(c_{uv} - cc_v)(d_{uv} - dc_v)}{(c_v - d_u)^2} - \dfrac{d(c_{uv} - cc_v)}{c_v - d_u}}. \tag{18}$$

Again by differentiating (16) with respect to $v$ and eliminating $f_u, f_v, f_{vv}$ from the result, we find

$$e^f = -\frac{\left(\dfrac{\partial}{\partial v}\dfrac{c_{uv} - cc_v}{c_v - d_u}\right) + \dfrac{(c_{uv} - cc_v)(d_{uv} - dc_v)}{(c_v - d_u)^2} + \dfrac{d(c_{uv} - cc_v)}{c_v - d_u} + c_v}{\left(\dfrac{\partial}{\partial v}\dfrac{d_{uv} - dc_v}{c_v - d_u}\right) + \left(\dfrac{d_{uv} - dc_v}{c_v - d_u}\right)^2 + \dfrac{d(d_{uv} - dc_v)}{c_v - d_u} + d_v}. \tag{19}$$

THEOREM 7. *A velocity system of curves may contain exactly* $\infty^2$, $\infty^1$, *one, or no isothermal systems as described below.*

(1) A velocity system will contain *exactly* $\infty^2$ isothermal families if and only if it is a $\Gamma$ family (including $\Gamma_0$).

(2) A nonnatural velocity system will contain *exactly* $\infty^1$ isothermal systems if and only if the functions $c$ and $d$ satisfy the four partial differential equations of the third order obtained by setting equal to zero the numerators and denominators of the fractions in (18) and (19).

(3) A nonnatural velocity system will contain exactly *one* isothermal system if the functions $c$ and $d$ satisfy one partial differential equation of the third order (obtained by the equalities of the right-

hand sides of (18) and (19), one of the fourth order (the family (18) or (19) lies in the velocity system), and one of the fifth order (the family (18) or (19) is isothermal).

A natural family will contain exactly one isothermal family if the functions $c$ and $d$ satisfy identically one partial differential equation of third order (the family (17) must lie in the velocity system), and another of fourth order (the family (17) must be isothermal).

(4) All other types of velocity systems contain no isothermal families. This is of course the general case.

THEOREM 8. PROPERTY E. *A velocity system is a* $\Gamma$ *or* $\Gamma_0$ *family if and only if it contains* $\infty^1$ *isothermal families.*

Properties A and E are necessary and sufficient conditions for the characterization of a $\Gamma$ or $\Gamma_0$ family.

8. **Reciprocal velocity systems.** Let $\Sigma$ be the associated point correspondence (6) of a velocity system $S$. The unique velocity system $S^*$ whose associated point correspondence is the inverse $\Sigma^{-1}$ of $\Sigma$ is said to be the *reciprocal*[9] or *conjugate* of $S$.

By (7), we find that the reciprocal velocity system $S^*$ of $S$ is

$$Q = P(\gamma - \delta P), \tag{20}$$

where

$$\begin{aligned} &\gamma = -c, \qquad \delta = -d, \\ &\frac{\partial\gamma}{\partial U} = -\frac{c^2}{J}\left[c_u(d^2 - 2d_v) + 2c_v d_u\right], \qquad \frac{\partial\gamma}{\partial V} = -\frac{c^2d^2c_v}{J}, \\ &\frac{\partial\delta}{\partial U} = -\frac{c^2d^2du}{J}, \qquad \frac{\partial\delta}{\partial V} = -\frac{d^2}{J}\left[2d_uc_v + d_v(c^2 - 2c_u)\right], \\ &J = (c^2 - 2c_u)(d^2 - 2d_v) - 4d_uc_v. \end{aligned} \tag{21}$$

THEOREM 9. *The reciprocal velocity system* $S^*$ *of a given velocity system* $S$ *is natural or isogonal according as* $S$ *is natural or isogonal.*

The reciprocity is not valid in general for $\Gamma$ systems (of course excluding $\Gamma_0$). We now wish to find those particular velocity systems whose reciprocals are $\Gamma$ families.

Before proceeding, let us note the following formulas in addition to those given by (21)

---

[9] Scheffers, Math. Ann. vol. 60 (1905) pp. 491–531.

$$
(22)\qquad
\begin{aligned}
\frac{\partial^2\gamma}{\partial U\partial V}\bigg/\frac{\partial\gamma}{\partial V} &= \frac{2c}{J}\left[c_u(d^2-2d_v)+2c_vd_u\right]+\frac{2c^2dd_u}{J}\\
&\quad+\frac{c^2}{Jc_v}\left[c_{uv}(d^2-2d_v)+2d_uc_{vv}\right]\\
&\quad-\frac{c^2}{J^2}\left[J_u(d^2-2d_v)+2J_vd_u\right],\\
\frac{\partial^2\delta}{\partial U\partial V}\bigg/\frac{\partial\delta}{\partial U} &= \frac{2cd^2c_v}{J}+\frac{2d}{J}\left[2c_vd_u+d_v(c^2-2c_u)\right]\\
&\quad+\frac{d^2}{Jd_u}\left[2c_vd_{uu}+d_{uv}(c^2-2c_u)\right]\\
&\quad-\frac{d^2}{J^2}\left[2J_uc_v+J_v(c^2-2c_u)\right].
\end{aligned}
$$

By (21) and (22), we may prove the following result.

THEOREM 10. *A velocity system $S$ is the reciprocal of a $\Gamma$ family $S^*$ if and only if its functions $c$ and $d$ with $c_v \neq 0$ and $d_u \neq 0$ satisfy the partial differential equations of third order*

$$
(23)\qquad c_v = d_u,\qquad \frac{J_u}{J} = c+\frac{c_{uv}}{c_v},\qquad \frac{J_v}{J} = d+\frac{c_{vv}}{c_v}\cdot
$$

In the *real* domain, it may be proved that the only velocity systems such that both $S$ and $S^*$ are $\Gamma$ families must consist of the $\infty^2$ circles orthogonal to a fixed circle. In that event, $S$ and $S^*$ coincide.

**9. The conformal covariants of a velocity system.**[7] Upon extending a conformal transformation twice, we find the formulas

$$
(24)\qquad U = \phi(u),\ V = \psi(v),\ P = \frac{\psi_v}{\phi_u}p,\ \frac{Q}{P} = \frac{q}{\phi_u p}+\frac{\psi_{vv}p}{\phi_u\psi_v}-\frac{\phi_{uu}}{\phi_u^2},
$$

which have many applications.

THEOREM 11. *Every velocity system $S(c, d)$ is converted into a velocity system $\Sigma(C, D)$ by a conformal transformation. The transformation formulas between corresponding velocity systems $S$ and $\Sigma$ are*

---

[7] For the analogous conformal geometry of a simple family of curves, see DeCicco, *The two conformal covariants of a field*, Revista di Matematica vol. 2(1941) pp. 59–66. A paper on *Isothermal families* will appear in the forthcoming Pastor commemoration volume edited by Beppo Levi.

$$(25)\qquad C = \frac{c}{\phi_u} - \frac{\phi_{uu}}{\phi_u^2}, \qquad D = \frac{d}{\psi_v} - \frac{\psi_{vv}}{\psi_v^2}.$$

(It may be proved that if a quasi-contact transformation of curvature elements carries every velocity system into a velocity system, then it must be a conformal transformation.)

We now inquire: what are the necessary and sufficient conditions that the velocity systems $S(c, d)$ and $\Sigma(C, D)$ be equivalent by a conformal transformation? That is, given the four functions $c(u, v)$, $d(u, v)$, $C(\phi, \psi)$, $D(\phi, \psi)$, can we find $\phi(u)$ and $\psi(v)$ such that the equations (25) are satisfied?

Firstly upon differentiating the first equation (25) with respect to $v$ and the second with respect to $u$, we find

$$(26)\qquad \phi_u \psi_v = \frac{c_v}{C_\psi} = \frac{d_u}{D_\phi}.$$

Differentiating this with respect to $u$ and $v$ and eliminating $\phi_{uu}$ and $\psi_{vv}$ by (25), we obtain

$$(27)\quad \phi_u = \left(\frac{c_{uv}}{c_v} - c\right) \Big/ \left(\frac{C_{\phi\psi}}{C_\psi} - C\right), \quad \psi_v = \left(\frac{d_{uv}}{d_u} - d\right) \Big/ \left(\frac{D_{\phi\psi}}{D_\phi} - D\right).$$

Now these equations will determine appropriate functions $\phi_u$ and $\psi_v$ if and only if the derivative of the first and second fractions with respect to $v$ and $u$, respectively, are zero. Accordingly we find that the velocity system $S(c, d)$ possesses the two absolute covariants

$$(28)\qquad \begin{gathered} \left[\frac{\partial}{\partial v} \log\left(\frac{c_{uv}}{c_v} - c\right)\right] \Big/ \left(\frac{d_{uv}}{d_u} - d\right) \quad \text{and} \\ \left[\frac{\partial}{\partial u} \log\left(\frac{d_{uv}}{d_u} - d\right)\right] \Big/ \left(\frac{c_{uv}}{c_v} - c\right). \end{gathered}$$

But these values as given by (27) must also satisfy the equations (25) and (26). Hence our velocity system $S(c, d)$ possesses the three absolute covariants

$$(29)\qquad \begin{gathered} \left(\frac{c_{uv}}{c_v} - c\right)\left(\frac{d_{uv}}{d_u} - d\right) \Big/ c_v, \\ \left[c - \frac{\partial}{\partial u} \log\left(\frac{c_{uv}}{c_v} - c\right)\right] \Big/ \left(\frac{c_{uv}}{c_v} - c\right) \quad \text{and} \\ \left[d - \frac{\partial}{\partial v} \log\left(\frac{d_{uv}}{d_u} - d\right)\right] \Big/ \left(\frac{d_{uv}}{d_u} - d\right). \end{gathered}$$

Summarizing, we have this theorem.

THEOREM 12. *The velocity system $S(c, d)$ possesses the six absolute covariants*

$$
\begin{gathered}
\frac{c_v}{d_u}, \qquad \left(\frac{c_{uv}}{c_v} - c\right)\left(\frac{d_{uv}}{d_u} - d\right) \Big/ c_v, \\
\left[\frac{\partial}{\partial v} \log\left(\frac{c_{uv}}{c_v} - c\right)\right] \Big/ \left(\frac{d_{uv}}{d_u} - d\right), \\
\left[\frac{\partial}{\partial u} \log\left(\frac{d_{uv}}{d_u} - d\right)\right] \Big/ \left(\frac{c_{uv}}{c_v} - c\right), \\
\left[c - \frac{\partial}{\partial u} \log\left(\frac{c_{uv}}{c_v} - c\right)\right] \Big/ \left(\frac{c_{uv}}{c_v} - c\right), \\
\left[d - \frac{\partial}{\partial v} \log\left(\frac{d_{uv}}{d_u} - d\right)\right] \Big/ \left(\frac{d_{uv}}{d_u} - d\right).
\end{gathered} \tag{30}
$$

*Any conformal covariant of a velocity system is a function of these and their conformal partial derivatives. These are defined by*

$$
\frac{\partial}{\partial u} \Big/ \left(\frac{c_{uv}}{c_v} - c\right) \quad \textit{and} \quad \frac{\partial}{\partial v} \Big/ \left(\frac{d_{uv}}{d_u} - d\right). \tag{31}
$$

Let now $S(c, d)$ and $\Sigma(C, D)$ be such that there exist functions $\phi$ and $\psi$ which satisfy the resulting covariant equations (30). If the operations on these six equations with the conformal derivatives (31) are reduced to identities by these functions $\phi$ and $\psi$, then $S(c, d)$ and $\Sigma(C, D)$ are conformally equivalent.

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# A COMPARISON OF ALGEBRAIC, METRIC, AND LATTICE BETWEENNESS

M. F. SMILEY

**Introduction.** We propose to investigate here the consequences of the identity of each pair chosen from three important generalizations of the relation of betweenness on a line, namely, algebraic betweenness [1, p. 27],[1] metric betweenness [3, p. 36], and lattice betweenness [7, Part II]. We shall also find an interpretation of metric betweenness in the Banach space of all continuous functions defined on the interval $0 \leqq t \leqq 1$ which can be used to establish the fact that this relation satisfies no strong four or five point transitivity [7, Part I] except $t_1$ and $t_2$.

We note first that algebraic betweenness implies metric betweenness and lattice betweenness. We find that algebraic betweenness and metric betweenness coincide in a seminormed real vector space[2] if and only if it is strictly convex in the sense of Clarkson [4, p. 404]. We then show that the coincidence of metric and lattice betweenness in a semimetric space [3, p. 38] which is also a lattice [2, p. 16] leads to a system which is a metric lattice (in the sense of G. Birkhoff [2, p. 41]). It follows that a complete seminormed real vector lattice is equivalent to an $(L)$-space [6] if and only if its metric and lattice betweenness relations are identical. Finally, we prove that algebraic and lattice betweenness coincide in a real vector lattice if and only if it is equivalent to the system of all real numbers. We conclude by giving the interpretation of metric betweenness in the space[3] $C[0, 1]$.

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[1] References to the bibliography at the end of the paper will be in brackets.

[2] We shall use these terms as follows. A *seminormed real vector space* is a vector space over the field of all real numbers together with a real non-negative single-valued function $\|a\|$; called the "norm of $a$," satisfying (i) $\|\lambda a\| = |\lambda| \|a\|$, and (ii) $\|a\| = 0$ *if and only if* $a = 0$. A *normed real vector space* satisfies in addition (iii) $\|a\| + \|b\| \geqq \|a+b\|$. A *real vector lattice* is a vector space over the field of all real numbers which is also a lattice [2, p. 16] with respect to a partial ordering relation "$\geqq$" such that (i) $a \geqq b$ *and* $\lambda \geqq 0$ *implies* $\lambda a \geqq \lambda b$, and (ii) $a \geqq b$ *implies* $a + c \geqq b + c$ *for every* $c$. A *(semi)normed real vector lattice* is a real vector lattice which is also a (semi)normed real vector space; it is *complete* if evey fundamental sequence has a limit. A complete normed real vector space is usually called a (real) *Banach space.*

[3] The notation $C[0, 1]$ (sometimes simply $C$) is currently used to designate the space described in the concluding sentence of the preceding paragraph.

1. **The three betweenness relations.** We shall be interested in the following relations:

$$(a\ b\ c)_A \equiv b = \lambda a + (1-\lambda)c, \qquad \lambda \text{ a real number, } 0 \leqq \lambda \leqq 1,$$
$$(a\ b\ c)_M \equiv \delta(a, b) + \delta(b, c) = \delta(a, c),$$
$$(a\ b\ c)_L \equiv (a \cap b) \cup (b \cap c) = b = (a \cup b) \cap (b \cup c),$$

to which we shall refer as $R_A$ (algebraic betweenness), $R_M$ (metric betweenness), and $R_L$ (lattice betweenness). The relation $R_A$ applies to a real vector space, $R_M$ to a semimetric space, and $R_L$ to a lattice. Our purpose is to find those systems to which each two of these three relations apply and have the same meaning.

The following three lemmas are known or evident.

LEMMA 1. *If $S$ is a seminormed real vector space, then we have $(a\ b\ c)_M$ if and only if $\|a-b\|+\|b-c\|=\|a-c\|$.*

LEMMA 2. *If $S$ is a distributive lattice, then $(a\ b\ c)_L$ if and only if $a\cap c \leqq b \leqq a\cup c$.*

PROOF. This is Lemma 9.1 of [7].

LEMMA 3. *If $S$ is a seminormed real vector space, then $(a\ b\ c)_A \rightarrow (a\ b\ c)_M$ for every $a, b, c \in S$.*

LEMMA 4. *If $S$ is a real vector lattice, then $(a\ b\ c)_A \rightarrow (a\ b\ c)_L$ for every $a, b, c \in S$.*

PROOF. It is well known that a vector lattice is distributive [2, p. 108], and hence by Lemma 2 it will suffice to show that $a\cap c \leqq b \leqq a\cup c$ is a consequence of $(a\ b\ c)_A$. We have the equations

$$\begin{aligned}(a \cup c) - b = (a-b) \cup (c-b) &= (1-\lambda)(a-c) \cup -\lambda(a-c) \\ &= ((a-c)\cup 0) - \lambda(a-c).\end{aligned}$$

Note that $(a-c)\cup 0 \geqq \lambda((a-c)\cup 0)$, since $\lambda \leqq 1$; and that $\lambda((a-c)\cup 0) = \lambda(a-c)\cup 0 \geqq \lambda(a-c)$, since $\lambda \geqq 0$. Hence $(a\cup c)-b \geqq 0$, that is, $a\cup c \geqq b$. The other inequality is dual.

2. **Coincident metric and algebraic betweenness.** Clarkson [4, p. 404] calls a normed real vector space *strictly convex* if the equality $\|a\|+\|b\|=\|a+b\|$ for nonzero $a$ and $b$ implies that $a=\mu b$ for some $\mu>0$. The following theorem then tells us when metric and algebraic betweenness coincide.

THEOREM 1. *A seminormed real vector space $S$ is strictly convex if and only if algebraic and metric betweenness coincide in $S$.*

PROOF. Let $S$ be a strictly convex seminormed real vector space. By Lemma 3, we need only show that $(a\ b\ c)_M \rightarrow (a\ b\ c)_A$ for every $a, b, c \in S$. We may clearly suppose that $a-b$ and $b-c$ are nonzero. But then the strict convexity of $S$ implies that $a-b=\mu(b-c)$ for some $\mu>0$. The choice $\lambda=1/1+\mu$ is effective for $(a\ b\ c)_A$. Conversely, if $(a\ b\ c)_M \rightarrow (a\ b\ c)_A$ for every $a, b, c \in S$, the equality $\|a\|+\|b\|=\|a+b\|$ for nonzero $a$ and $b$ implies $(a\ 0\ -b)_A$, and consequently that $0=\lambda a-(1-\lambda)b$ for real $\lambda$ between 0 and 1. The choice $\mu=(1-\lambda)/\lambda$ is then effective in the definition of the strict convexity of $S$. The proof is complete.

REMARK 1. Clarkson [4, p. 413] also shows that every *separable* Banach space may be renormed to secure strict convexity.

3. **Coincident metric and lattice betweenness.** Our definition of lattice betweenness arose as a generalization of the metric betweenness of a metric lattice [2, p. 41; 5, 9]. We shall show in this section that, conversely, a semimetric space which is also a lattice has identical lattice and metric betweenness relations only if it is a metric lattice.[4] We apply this result to give a characterization of $(L)$-spaces in terms of betweenness relations.[5]

Let us consider a semimetric space $M$ with distance function $\delta(a, b)$ which is also a lattice and in which lattice betweenness and metric betweenness coincide.

LEMMA 5. *If $a, b, c \in M$ and $a \leq b \leq c$, then $(a\ b\ c)_M$. If $a, b \in M$ then $(a\ a\cup b\ b)_M$ and $(a\ a\cap b\ b)_M$.*

PROOF. This is clear from Lemma 8.1 of [7].

LEMMA 6. *If $a, b \in M$, then $\delta(a, b)=\delta(a\cup b, a\cap b)$, and $\delta(a\cap b, b)=\delta(a, a\cup b)$.*

PROOF. Consider two elements $a, b \in M$. By Lemma 5 we have the relations $(a\ a\cup b\ b)_M$, $(a\ a\cap b\ b)_M$, $(a\cap b\ b\ a\cup b)_M$, and $(a\cap b\ a\ a\cup b)_M$. Hence we have the following equations:

---

[4] It should be noted that the results of §2 remain valid if we merely require the values of $\delta(a, b)$ to be in a real vector lattice, provided the terms *modular functional*, *metric lattice*, and so on, are given proper interpretations.

[5] G. Birkhoff's remark that $m[a]=\|a_+\|-\|a_-\|$ is a sharply positive modular functional in an $(L)$-space (for proof, see [8]) aided the author materially in securing the present form of Theorem 2. The functional $m[a]$ is also *linear*. Its *homogeneity* follows readily from that of $\|a\|$, while its *additivity* is a simple consequence of the identity $m[a]-m[b]=\|a-b\|$ for $a \geqq b$. When the condition (IX) of Kakutani [6] holds, or, equivalently, when $m[a\cup b]-m[a\cap b]=\|a-b\|$, we find that $m[a]$ is the value of the Lebesgue-Stieltjes integral of the function corresponding to $a$.

$$\delta(a, a \cup b) + \delta(a \cup b, b) = \delta(a, b) = \delta(a, a \cap b) + \delta(a \cap b, b),$$
$$\delta(a \cap b, b) + \delta(b, a \cup b) = \delta(a \cap b, a \cup b) = \delta(a \cap b, a) + \delta(a, a \cup b).$$

Subtracting these equations we see that

$$\begin{aligned}\delta(a, a \cup b) - \delta(a \cap b, b) &= \delta(a, b) - \delta(a \cup b, a \cap b)\\ &= \delta(a \cap b, b) - \delta(a, a \cup b).\end{aligned}$$

It follows that the relations stated in the lemma hold.

REMARK 2. Note that the first result of Lemma 5 together with the results of Lemma 6 imply that the second result of Lemma 5 is valid. This is clear since $\delta(a, b) = \delta(a \cup b, a \cap b) = \delta(a \cup b, a) + \delta(a, a \cap b) = \delta(a, a \cap b) + \delta(a \cap b, b)$, and dually.

LEMMA 7. *The lattice $M$ is modular.*

PROOF. We base our proof on the fact that if $M$ fails to be modular then it contains as a sublattice the simplest non-modular lattice [**2**, p. 34]. Let the elements of this sublattice be $O$, $a$, $b$, $c$, $I$, with $I$ the greatest, $O$ the least element, and $b > c$. Since $O = a \cap c$ and $O < a < I$, Lemma 5 gives $\delta(O, c) = \delta(a, I)$. Since $O < c < b < I$, Lemma 5 and this result gives $\delta(a, I) + \delta(c, b) + \delta(b, I) = \delta(O, I)$. Again, since $I = a \cup b$, we have $\delta(O, I) + \delta(c, b) = \delta(O, I)$, and consequently $\delta(b, c) = 0$, contrary to $b > c$. Hence $M$ cannot fail to be modular.

LEMMA 8. *If $a, b, c \in M$ and $a \geqq b$, then*

$$\delta(a \cup c, b \cup c) + \delta(a \cap c, b \cap c) = \delta(a, b).$$

PROOF. Consider elements $a, b, c \in M$ with $a \geqq b$. By Lemma 6 we have

$$\delta(a \cup c, b \cup c) = \delta(a \cup b \cup c, b \cup c) = \delta(a, a \cap (b \cup c)).$$

Noting that $a \geqq a \cap (b \cup c) \geqq b$, we find from Lemma 5 that

$$\delta(a \cup c, b \cup c) = \delta(a, b) - \delta(b, a \cap (b \cup c)).$$

Using Lemmas 7 and 6 we have

$$\begin{aligned}\delta(a \cup c, b \cup c) &= \delta(a, b) - \delta(b, b \cup (a \cap c))\\ &= \delta(a, b) - \delta(a \cap c, b \cap a \cap c).\end{aligned}$$

Since $a \geqq b$, this proves the lemma.

LEMMA 9. *If $a^* \in M$ the functional $m[a] = \delta(a \cup a^*, a^*) - \delta(a^*, a \cap a^*)$ is a sharply positive modular functional and $\delta(a, b) = m[a \cup b] - m[a \cap b]$ for every $a, b \in M$.*

PROOF. From Lemma 6 and the fact that $\delta(a, b) > 0$ for $a \neq b$, it will suffice to show that

$$\delta(a, b) = m[a] - m[b] \tag{1}$$

for every $a, b \in M$ such that $a \geqq b$. Hence consider elements $a, b \in M$ for which $a \geqq b$. Lemma 8 yields the equation

$$\delta(a \cup a^*, b \cup a^*) + \delta(a \cap a^*, b \cap a^*) = \delta(a, b). \tag{2}$$

Since $a \cup a^* \geqq b \cup a^* \geqq a^*$, and dually, Lemma 5 yields the following equations

$$\delta(a \cup a^*, a^*) - \delta(b \cup a^*, a^*) = \delta(a \cup a^*, b \cup a^*),$$
$$\delta(a^*, b \cap a^*) - \delta(a^*, a \cap a^*) = \delta(a \cap a^*, b \cap a^*).$$

Adding these equations and using the equation (2) we obtain (1) from the definition of the functional $m[a]$. The proof is complete.

REMARK 3. The validity of Lemmas 7–9 depends only on Lemma 6 and the first result of Lemma 5.

LEMMA 10. *The distance function $\delta(a, b)$ is a metric for $M$.*

PROOF. This follows from Lemma 9 and Theorem 3.10 of [2].

THEOREM 2. *If $S$ is a semimetric space with distance function $\rho(a, b)$ which is also a lattice, then metric betweenness and lattice betweenness coincide in $S$ if and only if:*

(i) *For every $a, b, c \in S$ the inequalities $a \leqq b \leqq c$ imply that $(a\ b\ c)_M$.*

(ii) *For every $a, b \in S$, $\rho(a, b) = \rho(a \cup b, a \cap b)$ and $\rho(a, a \cup b) = \rho(b, a \cap b)$.*

*The conditions* (i) *and* (ii) *hold if and only if for each $a^* \in M$ the functional $m[a] \equiv \rho(a \cup a^*, a^*) - \rho(a^*, a \cap a^*)$ is a sharply positive modular functional and $S$ is a metric lattice with metric $\rho(a, b) = m[a \cup b] - m[a \cap b]$.*

PROOF. This is clear from Lemmas 5–9, Remarks 2 and 3, and Theorem 10.1 of [7].

We also have a specialization of Theorem 2.

THEOREM 3. *A complete seminormed real vector lattice $S$ is equivalent to an $(L)$-space if and only if lattice and metric betweenness coincide in $S$.*

PROOF. Let $S$ be a complete seminormed real vector lattice in which lattice and metric betweenness coincide. By Theorem 2, Lemma 6, and Lemma 10, $S$ is a complete normed real vector lattice satisfying the conditions,

(VIII) if $a, b \in S$ and $a \geqq 0$, $b \geqq 0$, then $\|a\| + \|b\| = \|a+b\|$;

(IX) if $a, b \in S$ and $a \cap b = 0$, then $\|a+b\| = \|a-b\|$;

of Kakutani [6]. Kakutani's fundamental Theorem 7 [6] together with the fact that his condition (V) is not needed (in his Theorem 7) [8] now shows that $S$ is equivalent to an $(L)$-space.

Conversely, if $S$ is equivalent to an $(L)$-space, it is easy to verify the conditions (i) and (ii) of Theorem 2. Hence lattice betweenness and metric betweenness coincide in $S$. The proof is complete.

REMARK 4. We may restate Theorem 3 as follows. *A complete semi-normed real vector lattice is equivalent to an $(L)$-space if and only if it is a metric lattice.*

## 4. Coincident lattice and algebraic betweenness.

We include the following theorem for the sake of completeness.

THEOREM 4. *If $S$ is a real vector lattice, then lattice betweenness and algebraic betweenness coincide in $S$ if and only if the dimension of $S$ (as a vector space) is one.*

PROOF. We need only show that $(a\ b\ c)_L \rightarrow (a\ b\ c)_A$ for every $a, b, c \in S$ implies that $S$ is one-dimensional. It is easy to verify that $R_A$ has the fundamental transitivity $t_1$ [7]. By Theorem 9.7 of [7] every pair of elements of $S$ are comparable by "$\geqq$." If, then, there were two linearly independent elements $a, b \in S$, we should have (by symmetry) $a > b > 0$ or $a > 0 > b$; and hence [7, Lemma 8.1] $(a\ b\ 0)_L$ or $(a\ 0\ b)_L$. By hypothesis, we would have $b = \lambda a$ or $0 = \lambda a + (1-\lambda)b$, which clearly contradicts the linear independence of $a$ and $b$. Thus $S$ is one-dimensional.

## 5. Metric betweenness in the space $C[0, 1]$.

The question of whether or not the metric betweenness of a normed real vector lattice possesses any more than the required transitivities (that is, $(\alpha)$, $(\beta)$, $t_1$, and $t_2$ of [7]) can be settled by the space $C[0, 1]$. We give in this section an interpretation of $R_M$ in $C[0, 1]$ and leave to the reader the construction of simple examples which show that this relation has only the required transitivities. This result should be contrasted with the result of Clarkson mentioned in Remark 1.

LEMMA 11. *In the space $C[0, 1]$ the relation $(a\ 0\ b)_M$ holds if and only if there is a point $t_0$ on $[0, 1]$ at which each of the functions $|a(t)|$, $|b(t)|$, and $|a(t) - b(t)|$ attains its maximum value and the relation $(a(t_0)\ 0\ b(t_0))_M$ holds.*

PROOF. Suppose that the relation $(a\ 0\ b)_M$ holds in $C[0, 1]$. We may clearly assume that $a$ and $b$ are nonzero. Let $t_0$ be a point on $[0, 1]$ at which the function $|a(t)-b(t)|$ attains its maximum value. Then we have

$$\begin{aligned}\|a\|+\|b\| &= \|a-b\| = |a(t_0)-b(t_0)| \\ &\leq |a(t_0)|+|b(t_0)| \leq \|a\|+\|b\|.\end{aligned}$$

It follows that $|a(t_0)|+|b(t_0)|=\|a\|+\|b\|$ and that each of the functions $|a(t)|$ and $|b(t)|$ attains its maximum at $t_0$. Consequently neither $a(t_0)$ nor $b(t_0)$ can be zero. If they should both have the same sign we would find that $|a(t_0)-b(t_0)|<|a(t_0)|+|b(t_0)|$. This is impossible, and we conclude that the relation $(a(t_0)\ 0\ b(t_0))_M$ holds.

Conversely, if there is a point $t_0$ on $[0, 1]$ at which each of the functions $|a(t)|$, $|b(t)|$, and $|a(t)-b(t)|$ attains its maximum value and the relation $(a(t_0)\ 0\ b(t_0))_M$ holds, we see that

$$\|a-b\| = |a(t_0)-b(t_0)| = |a(t_0)|+|b(t_0)| = \|a\|+\|b\|.$$

The proof is complete.

THEOREM 5. *In the space* $C[0, 1]$ *the relation* $(a\ b\ c)_M$ *holds if and only if there is a point* $t_0$ *on* $[0, 1]$ *at which each of the functions* $|a(t)-b(t)|$, $|b(t)-c(t)|$, *and* $|a(t)-c(t)|$ *attains its maximum value and the relation* $(a(t_0)\ b(t_0)\ c(t_0))_M$ *holds.*

PROOF. This is an immediate consequence of Lemma 11 and the fact that $(a\ b\ c)_M$ if and only if $(a+x\ b+x\ c+x)_M$ for every $a, b, c, x \in C[0, 1]$.

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# ABSTRACT POLYNOMIALS IN NON-ABELIAN GROUPS

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**Introduction.** The aim of this note is to give some generalizations for groups of the theories of abstract polynomials as developed by Fréchet, Gateaux, Martin, Mazur, Michal, Orlicz and more recently, Van der Lijn.[1] Although the theories are equivalent for functions with arguments and values in abelian groups,[2] this equivalence is not the case when the argument and value groups are non-abelian.[3]

In §1, a calculus of finite differences for functions with arguments and values in non-abelian groups which contain no elements of finite order is defined, and the fundamental definitions of polynomials and monomials are stated. In §2, the homogeneity in the increment of the $n$-difference of a polynomial of degree $n$ is proved, and the theorem on unique pseudo-decomposition is proved after giving some preliminary theorems on the structure of differences of arbitrary functions and polynomials. In §3, a brief discussion of the extensions to non-abelian groups as value spaces is given.

I should like to thank Professor A. D. Michal for his helpful suggestions and constructive criticisms during the preparation of this note.

## 1. Definitions.

For the purposes of polynomial theory the value groups are restricted to contain no elements of finite order.

To construct the calculus of finite differences for functions with arguments and values in non-abelian groups, we define

$$
{}_{\omega}\Delta[\alpha_1;\beta_1]f(x) \equiv \begin{cases} f(x\omega)f^{-1}(x) & \text{if} \quad \alpha_1 = 1 \quad \text{and} \quad \beta_1 = 1, \\ f(\omega x)f^{-1}(x) & \text{if} \quad \alpha_1 = 1 \quad \text{and} \quad \beta_1 = 2, \\ f^{-1}(x)f(x\omega) & \text{if} \quad \alpha_1 = 2 \quad \text{and} \quad \beta_1 = 1, \\ f^{-1}(x)f(\omega x) & \text{if} \quad \alpha_1 = 2 \quad \text{and} \quad \beta_1 = 2, \end{cases}
$$

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[1] References to these theories will be found in the bibliography. This list will be referred to by numbers in brackets.

[2] The equivalence of some of these definitions was proved by Martin in his California Institute of Technology thesis, 1932, and of the remaining definitions by Van der Lijn. A summary has been given by Van der Lijn [1, pp. 78–80].

[3] If the abstraction of additivity is multiplicative distributivity, then the generalizations of the definitions of Mazur and Orlicz [1, p. 63] and Van der Lijn, [1, pp. 60–61] are not equivalent; this is easily seen by considering $f(x) = x^2$.

and inductively

$$
\begin{aligned}
&{}_{\omega}^{n}\Delta[\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n]f(x) \\
&\qquad = {}_{\omega}\Delta[\alpha_n; \beta_n]\, {}_{\omega}^{n-1}\Delta[\alpha_1, \cdots, \alpha_{n-1}; \beta_1, \cdots, \beta_{n-1}]f(x).
\end{aligned}
$$

DEFINITION OF A MONOMIAL. *A* $[\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n]$ *monomial*[4] *is a function* $f(x)$ *which satisfies for all* $x$ *and* $\omega$ *the following functional equation*

$$
{}_{\omega}^{n}\Delta[\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n]f(x) = f^{n'}(\omega).
$$

The obviously unique value of $n$ is termed the degree of the monomial.

DEFINITION OF A POLYNOMIAL. *A function* $f(x)$ *which satisfies the following functional equation for all* $x$ *and* $\omega$

$$
{}_{\omega}^{n+1}\Delta[\alpha_1, \cdots, \alpha_{n+1}; \beta_1, \cdots, \beta_{n+1}]f(x) = 1
$$

*will be called a* $[\alpha_1, \cdots, \alpha_{n+1}; \beta_1, \cdots, \beta_{n+1}]$ *polynomial.*[5]

The least value of $n$ for which the above equation holds will be termed the degree of the polynomial.

The binomial coefficients are denoted in the usual manner; $\Gamma^{i}_{k,n}$ is defined by

$$
\Gamma^{i}_{k,n} = \sum \frac{n!}{\alpha_1!\alpha_2! \cdots \alpha_{k+1}!}
$$

where $\sum_{i=1}^{k+1}\alpha_i = n$, $\sum_{q=2}^{k+1}(q-1)\alpha_q = i$, $i \neq nk$, and $i \neq 0$. An arbitrary element of the central subgroup will be denoted by $\vartheta$. To simplify notation, the dropping of unnecessary indices implies that the value of a product of factors is independent of the order of the particular factors controlled by the dropped indices.

The next equation is a generalization of an identity due to Marchaud[6]

$$
{}_{\omega^{k+1}}^{n}\Delta[\beta_1, \cdots, \beta_n]f(x) = \sum_{i=0}^{nk} \sum_{j} {}_{\omega}^{n}\Delta[\beta_1, \cdots, \beta_n]f[P_{i,j}],
$$

---

[4] The inner automorphisms of a group are interesting examples of a [1; 2] monomial; similarly, the canonical transformations of quantum mechanics.

[5] If the elements of a group are taken to be the $n \times n$ matrices whose elements are in a commutative field and whose determinants do not vanish, and if the group operation is defined as row by column multiplication, then $f[\|x^i_j\|] = \|a^i_k x^k_m x^m_n b^n_j\|$ is a [1, 1; 2, 1] polynomial.

[6] Marchaud [1, p. 368].

where $j=1, \cdots, \Gamma^i_{1,n}$ and $P_{i,j}$ denotes a particular permutation of $\omega, \omega, \cdots, \omega$ ($i$ times) and $x$.

2. **Fundamental theorems.** Let $A(x)$ denote a function with arguments in a non-abelian group and values in an abelian group. The theorems of Van der Lijn for functions with arguments and values in abelian groups can be extended with a few immediate changes to hold for differences of the type ${}^n_\omega\Delta A(x)$. This section is devoted to abstractions for differences of the kind ${}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x)$, where $\omega$ is an arbitrary element.

THEOREM I. *If $A(x)$ is a $[\beta_1, \cdots, \beta_{n+1}]$ polynomial of degree $n$, and if $k$ is an integer, then*

$$ {}^{n}_{\omega^k}\Delta[\beta_1, \cdots, \beta_n]A(x) = k^n\, {}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x). $$

PROOF. For $k=0$ or 1, the theorem is trivial. For $k>1$, we hypothetically have

$$ {}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x, \omega^i) - {}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x, \omega^{i-1}) = 0. $$

By a few manipulations and Marchaud's identity we get

$$ \begin{aligned} {}^{n}_{\omega^k}\Delta[\beta_1, \cdots, \beta_n]A(x) &= \sum_{i=0}^{n(k-1)} \Gamma^i_{k-1,n}\, {}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x) \\ &= k^n\, {}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x). \end{aligned} $$

For $k<0$, we evidently have

$$ \begin{aligned} {}^{n}_{\omega^k}\Delta[\beta_1, \cdots, \beta_n]A(x) &= {}^{n}_{(\omega^{-1})^{-k}}\Delta[\beta_1, \cdots, \beta_n]A(x) \\ &= (-k)^n\, {}^n_{\omega^{-1}}\Delta[\beta_1, \cdots, \beta_n]A(x) \\ &= (-)^n(-k)^n\, {}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x) \\ &= k^n\, {}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x). \end{aligned} $$

THEOREM II. *If ${}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x)=g(\omega)$, where $g(\omega)$ is independent of $x$, then $g(\omega)$ is a $[\beta_1, \cdots, \beta_n]$ monomial of degree $n$ in $\omega$.*

THEOREM III. *If $A(x)$ is a $[\beta_1, \cdots, \beta_{n+1}]$ polynomial of degree $n$, then ${}^n_\omega\Delta[\beta_1, \cdots, \beta_n]A(x)$ is independent of $x$.*

PROOF. If $n+2$ elements of the argument group are denoted by $x_i$, where

$$ x_{i+1} = \prod(x_i, \xi), \qquad i = 0, \cdots, n+1, $$

then by steps roughly analogous to the usual proof of similar theorems we can derive

$$\sum_{p=0}^{n}\sum_{i=0}^{n+1} C_{n+1,i}C_{n,p}(-)^{i}i^{n-p}k^{p}\,{}_{i}^{n}\Delta[\beta_1, \cdots, \beta_n]A(x_i) = 0.$$

In the last equation we see that $\sum_{p=0}^{n} C_{n,p}k^{p}$ is a simple polynomial. From this observation we deduce[7]

$$\sum_{i=0}^{n+1} (-)^{i}C_{n+1,i}i^{n-p}\,{}_{i}^{n}\Delta[\beta_1, \cdots, \beta_n]A(x_i) = 0, \qquad p = 0 \cdots, n.$$

If we consider one difference as given and the remaining $n+1$ differences as unknown, then the unique solutions are

$${}_{i}^{n}\Delta[\beta_1, \cdots, \beta_n]A(x_i) = {}_{0}^{n}\Delta[\beta_1, \cdots, \beta_n]A(x_0), \qquad i = 1, \cdots, n + 1,$$

for the determinant of the system of equations is given by

$$(-)^{\delta(n)}\prod_{i=1}^{n+1} C_{n+1,i}V \neq 0$$

where $V$ is a Vandermonde determinant and $\delta(n) = 1$, if $n \equiv 1, 2 \bmod 4$; $\delta(n) = 2$, if $n \equiv 0, 3 \bmod 4$.

THEOREM IV. *If $A(x)$ is a $[\beta_1, \cdots, \beta_{n+1}]$ polynomial of degree $n$, and if $\rho_n = \prod_{i=0}^{n-1}(n-i)!$, then we have the unique pseudo-decomposition*

$$\rho_n A(x) = \sum_{i=0}^{n} M_i(x),$$

*where $M_i(x)$ are $[\beta_1, \cdots, \beta_i]$ monomials of degree $i$.*

3. **Further discussion.** With the generalized definition of a polynomial an extension of Theorem I for functions with arguments and values in non-abelian groups can be made. The trivial converse of Theorem III can be proved in an obvious manner. The non-abelian analogue of Theorem IV does not hold, for counterexamples can be exhibited.[8]

In conclusion, it is remarked that the difficulty with the non-abelian valued case is an inability to solve explicitly a system of group equations.

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[7] It can be shown that if a simple polynomial $\sum_{i=0}^{n} a_i x^i$ of degree $n$ vanishes for $n+1$ distinct values of $x$, then the coefficients vanish.

[8] With the group defined in (5), a trivial, but sufficient, example is $\|x_j'\| = \|x_j'\|\,\|\delta_j'\| = \|\delta_j'\|\,\|x_j'\|$.

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# ON THE QUADRIC OF LIE

SU-CHENG CHANG

The most important quadric attached to an ordinary point of a non-ruled and nondegenerate surface is, perhaps, the quadric of Lie. The characteristic curve of the quadric of Lie varying along an asymptotic curve of the surface decomposes into an asymptotic tangent and two edges of the quadrilateral of Demoulin.[1] In this note we propose to determine whether the characteristic curve of the quadric of Lie may decompose into two conics when the quadric of Lie varies along certain curves of the surface. The answer is positive.

Let $(u, v)$ be the asymptotic net of a surface $(M)$ and $(M, M_1, M_2, M_3)$ its normal tetrahedron of Cartan, $MM_1$, $MM_2$ being the two asymptotic tangents and $MM_3$, $M_1M_2$ being the directrices of Wilczynski. Except for a projective transformation the surface $(M)$ is determined by the system

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[1] A. Demoulin, *Sur la théorie des lignes asymptotiques*, C. R. Acad. Sci. Paris vol. 147 (1908) pp. 413–415, *Sur la quadrique de Lie*, ibid. pp. 493–496, *Sur quelques propriétés des surfaces courbes*, ibid. pp. 565–568.

$$
(1)\qquad
\begin{aligned}
\frac{\partial M}{\partial u} &= (1/2)\frac{\partial \log \gamma}{\partial u} M + M_1,\\
\frac{\partial M_1}{\partial u} &= B^2 M - (1/2)\frac{\partial \log \gamma}{\partial u} M_1 + \beta M_2,\\
\frac{\partial M_2}{\partial u} &= K M + (1/2)\frac{\partial \log \gamma}{\partial u} M_2 + M_3,\\
\frac{\partial M_3}{\partial u} &= A^2\beta M + K M_1 + B^2 M_2 - (1/2)\frac{\partial \log \gamma}{\partial u} M_3,\\
\frac{\partial M}{\partial v} &= (1/2)\frac{\partial \log \beta}{\partial v} M + M_2,\\
\frac{\partial M_1}{\partial v} &= \overline{K} M + (1/2)\frac{\partial \log \beta}{\partial v} M_1 + M_3,\\
\frac{\partial M_2}{\partial v} &= A^2 M - (1/2)\frac{\partial \log \beta}{\partial v} M_2 + \gamma M_1,\\
\frac{\partial M_3}{\partial v} &= B^2\gamma M + A^2 M_1 + \overline{K} M_2 - (1/2)\frac{\partial \log \beta}{\partial v} M_3,
\end{aligned}
$$

where $M$ denotes a point of the surface with the coordinates $M^i$ $(i=1, 2, 3, 4)$ and

$$
2K = \beta\gamma - \frac{\partial^2 \log \beta}{\partial u \partial v}, \qquad 2\overline{K} = \beta\gamma - \frac{\partial^2 \log \gamma}{\partial u \partial v}.
$$

The conditions of integrability of the system (1) are found to be

$$
(2)\qquad
\begin{gathered}
\frac{\partial A^2}{\partial u} = K\frac{\partial(\log \beta K)}{\partial v}, \qquad \frac{\partial B^2}{\partial v} = \overline{K}\frac{\partial(\log \gamma\overline{K})}{\partial u},\\
A\frac{\partial(A\beta)}{\partial v} = B\frac{\partial(B\gamma)}{\partial u}.
\end{gathered}
$$

As the coordinates of any point in space can be expressed in the form

$$
(3)\qquad P = y_1 M + y_2 M_1 + y_3 M_2 + y_4 M_3,
$$

$(y_1, y_2, y_3, y_4)$ being the local coordinates of $P$ with respect to the tetrahedron $\{M, M_1, M_2, M_3\}$, we have the equation of the quadric of Lie at $M$, namely,

$$
(4)\qquad y_1 y_4 - y_2 y_3 = 0.
$$

Suppose that the point $P$ be fixed in space, from (1) and (3) it follows that

$$
\begin{aligned}
\frac{\partial y_1}{\partial u} &= -(1/2)\frac{\partial \log \gamma}{\partial u} y_1 - B^2 y_2 - K y_3 - A^2 \beta y_4, \\
\frac{\partial y_2}{\partial u} &= -y_1 + (1/2)\frac{\partial \log \gamma}{\partial u} y_2 - K y_4, \\
\frac{\partial y_3}{\partial u} &= -\beta y_2 - (1/2)\frac{\partial \log \gamma}{\partial u} y_3 - B^2 y_4, \\
\frac{\partial y_4}{\partial u} &= -y_3 + (1/2)\frac{\partial \log \gamma}{\partial u} y_4, \\
\frac{\partial y_1}{\partial v} &= -(1/2)\frac{\partial \log \beta}{\partial v} y_1 - \overline{K} y_2 - A^2 y_3 - B^2 \gamma y_4, \\
\frac{\partial y_2}{\partial v} &= -(1/2)\frac{\partial \log \beta}{\partial v} y_2 - \gamma y_3 - A^2 y_4, \\
\frac{\partial y_3}{\partial v} &= -y_1 + (1/2)\frac{\partial \log \beta}{\partial v} y_3 - \overline{K} y_4, \\
\frac{\partial y_4}{\partial v} &= -y_2 + (1/2)\frac{\partial \log \beta}{\partial v} y_4.
\end{aligned}
\tag{5}
$$

Differentiating (4) along a curve $v = v(u)$ of the surface $(M)$ and making use of (5), we obtain that the characteristic of the quadric of Lie along this curve is given by (4) and

$$\beta(y_2^2 - A^2 y_4^2) + \frac{dv}{du}\gamma(y_3^2 - B^2 y_4^2) = 0. \tag{6}$$

*The latter represents two planes when and only when*

$$\frac{dv}{du} = -\frac{\beta A^2}{\gamma B^2}. \tag{7}$$

Therefore *there are curves $L$ of a one-parameter system on a surface along each of which two consecutive quadrics of Lie intersect in two conics $C_1$ and $C_2$*. It is easily seen that the planes of these conics at $M$ are

$$B y_2 - A y_3 = 0 \tag{$8_1$}$$

and

$$(8_2) \qquad By_2 + Ay_3 = 0,$$

respectively.

In order that *the curves $L$ should be indeterminate it is necessary and sufficient that the surface in consideration is of Demoulin and Godeaux.*[2]

In general *the conics $C_1$ and $C_2$ have $M$ and $M_3$ in common.*

Noticing that the vertices of the quadrilateral of Demoulin are

$$(9) \qquad D_{\epsilon\epsilon'} = \epsilon\epsilon' ABM + \epsilon AM_1 + \epsilon' BM_2 + M_3,$$

where $\epsilon = \pm 1$, $\epsilon' = \pm 1$ and that the planes $(8_1)$ and $(8_2)$ pass through the two diagonals of the quadrilateral of Demoulin, respectively, we infer that *the two conics $C_1$ and $C_2$ can be constructed by means of the quadric of Lie and the quadrilateral of Demoulin.*

The second quadric $\Phi_1$, namely, the associate quadric,[3] in the sequence of Godeaux[4] originally defined in virtue of the representation in $S_5$, is given by the equation

$$(10) \qquad y_1^2 - B^2y_2^2 - A^2y_3^2 + A^2B^2y_4^2 + \frac{N}{\beta\gamma}(y_1y_4 - y_2y_3) = 0$$

where

$$N = B(B\gamma)_u = A(A\beta)_v.$$

It is obvious that *each of the planes $(8_i)$ $(i = 1, 2)$ intersects the associate quadric in a conic which is tangent to the conic $C_i$ $(i = 1, 2)$ at two opposite vertices of the quadrilateral of Demoulin.*

*When the point $M$ varies along a curve $L$ of the system the conics $C_1$, $C_2$ and their consecutive conics always lie on the quadric of Lie at $M$.*

In fact, a point on the conic $C_1$ at $M$ can be given parametrically in the form $(AB\rho^2, A\rho, B\rho, 1)$, where $\rho$ is a parameter. By means of (5) we can expand the local coordinates of the points on the consecutive conic of $C_1$ into the following series:

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[2] L. Godeaux, *Sur une classe de surfaces*, Bulletin de l'Académie royale de Belgique vol. 18 (1932) pp. 1015–1025.

[3] B. Su, *On the surfaces whose asymptotic curves belong to linear complexes*, I, Tôhoku Math. J. vol. 40 (1935) pp. 408–420; II, ibid. pp. 433–448; III, ibid. vol. 41 (1935) pp. 1–9; IV, ibid. pp. 203–215; V, Science Reports of the Tôhoku Imperial University vol. 25 (1935) pp. 601–633; VI, ibid. pp. 634–642; *On the surface whose Lie quadrics all touch a fixed plane*, Science Reports of the National University of Chekiang vol. 2 (1936) pp. 39–51. See also L. Godeaux, *Remarques sur les quadriques associées aux points d'une surface*, Journal of the Chinese Mathematical Society vol. 2 (1937) pp. 1–5.

[4] L. Godeaux, *La théorie des surfaces et l'espace réglé*, Paris, 1934. A new definition of the sequence of Godeaux is given by the present author. See S. C. Chang, *Some theorems on ruled surfaces*, appear in Science Record, Academia Sinica vol. 1 (1942).

$$
\begin{aligned}
y_1 &= AB\rho^2 + \Bigg[ -(1/2)\left(\frac{\partial \log \gamma}{\partial u} - \frac{\beta A^2}{\gamma B^2}\frac{\partial \log \beta}{\partial v}\right) AB\rho^2 - AB^2\rho \\
&\qquad - KB\rho - \frac{\beta A^2}{\gamma B^2}(\overline{K}A\rho + A^2B\rho)\Bigg]\Delta u + (\Delta u^2), \\
y_2 &= A\rho + \Bigg[ -AB\rho^2 + (1/2)\,\frac{\partial \log \gamma}{\partial u}A\rho - K \\
&\qquad - \frac{\beta A^2}{\gamma B^2}\left(-(1/2)\,\frac{\partial \log \beta}{\partial v}A\rho - B\gamma\rho - A^2\right)\Bigg]\Delta u + (\Delta u^2), \\
y_3 &= B\rho + \Bigg[ -\beta A\rho - (1/2)\,\frac{\partial \log \gamma}{\partial u}B\rho - B^2 \\
&\qquad - \frac{\beta A^2}{\gamma B^2}\left(-AB\rho^2 + (1/2)\,\frac{\partial \log \beta}{\partial v}B\rho - \overline{K}\right)\Bigg]\Delta u + (\Delta u^2), \\
y_4 &= 1 + \Bigg[ -B\rho + (1/2)\frac{\partial \log \gamma}{\partial u} \\
&\qquad - \frac{\beta A^2}{\gamma B^2}\left(-A\rho + (1/2)\frac{\partial \log \beta}{\partial v}\right)\Bigg]\Delta u + (\Delta u^2).
\end{aligned}
\tag{11}
$$

Substituting (11) into (4) gives that the coefficients of the terms $(\Delta u)^0$ and $(\Delta u)^1$ vanish for any value of $\rho$. A similar result holds for the conic $C_2$, which completes the proof.

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# TRANSFORMATIONS OF MULTIPLE FOURIER SERIES

L. B. HEDGE

**1. Introduction.** The object of the present paper is the study and characterization of certain classes of factor sequence transformations of multiple Fourier series. A recent moment problem solution[1] by the author and a scheme of summation of multiple Fourier series developed by Bochner[2] are used in the study. The results include and extend known results for single Fourier series.

**2. Definitions and notation.** Let $n$ be a positive integer, fixed but arbitrary. $R^n$ will denote the euclidean $n$-space. $(x)$, $(y)$, and so on will denote $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$, and so on, points of $R^n$. $\nu$, $\tau$, $j$, $k$, $s$ will be used for non-negative integers, and $(\nu)$, $(\tau)$, and so on will be used for $(\nu_1, \nu_2, \cdots, \nu_n)$, $(\tau_1, \tau_2, \cdots, \tau_n)$, lattice points of $R^n$. $(0)$ will mean $(0, 0, \cdots, 0)$, and $(x)=(y)$ will mean $x_j=y_j$, $j=1, 2, \cdots, n$. $(k\cdot x)$ will stand for the number $k_1x_1+k_2x_2+\cdots+k_nx_n$, $|x|$ for the number $(x_1^2+x_2^2+\cdots+x_n^2)^{1/2}$. $A$, $I$, and $\lambda$ will be used for functions defined on the lattice points of $R^n$, and $I$ will be the characteristic function of the lattice points of $R^n$. $E$ will be the set $E_{(x)}(-\pi\leqq x_j<\pi, j=1, 2, \cdots, n)$. $R$ and $t$ will be used for real numbers. $(x+y)$ will stand for $(x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$, and $B(n)$ for a real constant depending only on $n$. $\Phi$ will be used for a function $U^*$ of bounded variation in the sense of Saks, and if $\Phi(H)=\Phi_1(H)+\Phi_2(H)$ for any Borel set $H$ with $\Phi_1(H)\geqq 0\geqq\Phi_2(H)$ we will write $\int_H f(x)|d\Phi(E)|$ for $\int_H f(x)d\Phi_1(E)-\int_H f(x)d\Phi_2(E)$. When $\Phi$ is the Lebesgue measure function we will write $\int_H f(x)dx$ for

$$\int_H f(x)d\Phi(E).$$

We will write $f\in L$ to indicate that $\int_E f(x)dx$ exists, and $f\in C$ to indicate that $f$ is continuous on $\bar{E}$ and $f(x)=f(x+y)$ for all combinations of $y_j=0$ or $2\pi$, $j=1, 2, \cdots, n$. A function $f$ defined over $E$ will be defined over $R_n$ by the extension $f(x)=f(x+y)$ with $y_j=0$ or $2\pi$, $j=1, 2, \cdots, n$.

Let

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[1] L. B. Hedge, *Moment problem for a bounded region*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 282–285. Referred to later as MP.

[2] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 175–207.

$$\mathfrak{S}(\lambda, A, x) \simeq \sum_{(k)} \lambda(k)A(k)e^{i(k\cdot x)}, \tag{1}$$

and

$$\mathfrak{S}_\nu(\lambda, A, x) = \sum_{|k|^2\leqq\nu} \lambda(k)A(k)e^{i(k\cdot x)}. \tag{2}$$

If

$$\lambda(k)A(k) = (2\pi)^{-n}\int_E f(x)e^{-i(k\cdot x)}dx \tag{3}$$

or

$$\lambda(k)A(k) = (2\pi)^{-n}\int_E e^{-i(k\cdot x)}d\Phi(E), \qquad \int_E |\, d\Phi(E)\,| < \infty, \tag{4}$$

we will write $\mathfrak{S}(f, x)$ or $\mathfrak{S}(d\Phi, x)$, respectively, for the left side of equation (1), and similarly alter equations (2) and (5).

We write

$$S_R(\lambda, A, x) = \sum_{(k)} \Psi(\,|\, k\,|\,/R)\lambda(k)A(k)e^{i(k\cdot x)}, \tag{5}$$

where

$$\Psi(t) = e^{-t^2},$$

for the Bochner-Gauss[2] spherical means of the sequence (2), and

$$K_R(u) = \sum_{(k)} \Psi\left(\frac{|\,K\,|}{R}\right)e^{-i(k\cdot u)} = S_R(I, I, u) \tag{6}$$

for the corresponding kernel.

3. **Spherical summation.** We proceed to some modification (largely notational) of the Bochner summation theory. The transformation which takes (2) into (5) is given by the matrix $T:\|a_{R,\nu}\|$, where $a_{R,\nu} = \Psi(R_\nu/R) - \Psi(R_{\nu+1}/R)$, and $\{R_\nu\}$ is a subsequence of $\{0, 1, 2^{1/2}, 3^{1/2}, 4^{1/2}, \cdots, k^{1/2}, \cdots\}$. We have at once

$$a_{R,\nu} \geqq 0, \qquad \lim_{R\to\infty} a_{R,\nu} = 0, \qquad \sum_{\nu=0}^{\infty} a_{R,\nu} = 1,$$

whence $T$ is a regular Toeplitz transformation.[3]

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[3] See, for example, A. Zygmund, *Trigonometrical series*, Monografje Matematyczne, vol. 5, Warsaw, 1935, pp. 79–86. That $R$ is a continuous variable is unimportant.

The following properties of the summation scheme follow immediately from Bochner's work:

$$(7) \qquad 0 \leqq K_B(u) \leqq M_B < \infty,$$

$$(8) \qquad S_B(f, x) = B(n)\int_E f(u)K_B(u - x)du, \qquad \text{if } f \in L,$$

and,

$$(9) \qquad S_B(f, x) \to f(x)$$

at every point $(x)$ of continuity of $f$, and uniformly on $E$ if $f \in C$.

4. **Classes of multiple Fourier series.** The theorem of MP may now be given in the following form:

THEOREM. *In order that* $\mathfrak{S}(\lambda, A, x) = \mathfrak{S}(d\Phi, x)$ *with*

$$1. \quad \int_E |d\Phi| \leqq M, \qquad \text{or} \qquad 2. \quad \Phi \geqq 0,$$

*it is necessary and sufficient that*

$$1. \quad \int_B |S_B(\lambda, A, x)|\, dx \leqq M, \qquad \text{or} \qquad 2. \quad S_B(\lambda, A, x) \geqq 0,$$

*and in order that* $\mathfrak{S}(\lambda, A, x) = \mathfrak{S}(f, x)$ *with*

$$3. \quad f \in L, \qquad \text{or} \qquad 4. \quad |f| \leqq M, \qquad \text{or} \qquad 5. \quad f \in C,$$

*it is necessary and sufficient that* 3. $\{S_B(\lambda, A, x)\}$ *converge in the mean with exponent* 1, *or* 4. $|S_B(\lambda, A, x)| \leqq M$, *or* 5. $\{S_B(\lambda, A, x)\}$ *converge uniformly in* $B$.

We shall write $\mathfrak{S}(\lambda, A, x) \in S$ to indicate that $\mathfrak{S}(\lambda, A, x) = \mathfrak{S}(d\Phi, x)$ with $\int_E |d\Phi| \leqq M$, and $\mathfrak{S}(\lambda, A, x) \in L$, $M$, or $C$, if $\mathfrak{S}(\lambda, A, x) = \mathfrak{S}(f, x)$ with $f \in L$, $f \in M$ or $f \in C$ respectively. We will write $\lambda \in (P, Q)$ to indicate that $\mathfrak{S}(I, A, x) \in P$ implies $\mathfrak{S}(\lambda, A, x) \in Q$.

5. **Transformations.** We begin with the following lemma.

LEMMA 1. $\mathfrak{S}(I, I, x) \in S$.

PROOF. Let $f \in C$. Since $S_B(I, I, x) = K_B(x)$, we have

$$S_B(f, x) = B(n)\int_E f(u)S_B(I, I, u - x)du,$$

but $f$ is continuous, the left-hand member converges uniformly to $f$, and for any sequence of values of $R$, $\{|S_R(f, x)|\}$ is bounded uniformly. $\{S_R(f, x)\}$ is a sequence of linear operations whose norms are

$$\int_E |S_R(I, I, x)|\,dx,$$

and by a simple corollary of a theorem of Banach and Steinhaus[4] these must be uniformly bounded, or $\mathfrak{S}(I, I, x) \in S$.

We now have this theorem.

THEOREM 1. *The transformation classes* $(S, S)$, $(M, M)$, $(L, L)$, *and* $(C, C)$ *are identical, and* $\lambda$ *belongs to each of them if and only if* $\mathfrak{S}(\lambda, I, x) \in S$, *or*

$$\int_E |S_R(\lambda, I, x)|\,dx \leqq M. \tag{10}$$

(A) $\lambda \in (S, S) \rightarrow (10)$. This follows immediately from Lemma 1.

(B) $(10) \rightarrow \lambda \in (S, S)$. From (5) we have

$$\begin{aligned} S_R(\lambda, A, x) &= \sum_{(k)} \Psi\left(\frac{|K|}{R}\right)\lambda(k)A(k)e^{i(k\cdot x)} \\ &= \sum_{(k)} \Psi\left(\frac{|K|}{R}\right)\lambda(k)(2\pi)^{-n}\int_{E(u)} e^{i(k\cdot x-u)}\,d\Phi(E)^{5} \\ &= (2\pi)^{-n}\int_{E(u)} S_R(\lambda, I, x-u)f(u)\,d\Phi(E), \end{aligned}$$

whence

$$\int_E |S_R(\lambda, A, x)|\,dx \leqq (2\pi)^{-n}\cdot M \cdot \int_E |d\Phi|.$$

(C) $\lambda \in (M, M) \rightarrow (10)$ and $\lambda \in (C, C) \rightarrow (10)$.

We have immediately in both cases

$$S_R(\lambda, A, x) = B(n)\int_{E(u)} S_R(\lambda, I, x-u)f(u)\,du$$

and the boundedness of the set $\{S_R(\lambda, A, 0)\}$ implies (10).

(D) $(10) \rightarrow \lambda \in (M, M)$, $(10) \rightarrow \lambda \in (C, C)$, and $(10) \rightarrow \lambda \in (L, L)$.

From (5) and (10) we have

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[4] S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne, vol. 1, Warsaw, 1932, p. 80.

[5] The notation $\int_{E(u)} f(x, u)\,d\Phi(E)$ is used to indicate the integral over $E$ of $f$ regarded as a function of a point $(u)$.

$$S_E(\lambda, A, x) = \sum_{(k)} \Psi\left(\frac{|K|}{R}\right) A(k)(2\pi)^{-n} \int_{E(u)} e^{-i(k\cdot x-u)} d\Phi(E)$$

$$= (2\pi)^{-n} \int_{E(u)} S_E(I, A, x-u) d\Phi(E),$$

and the boundedness, uniform convergence, or mean convergence of $\{S_E(I, A, x)\}$ implies the same for $\{S_E(\lambda, A, x)\}$.

(E) $\lambda \in (L, L) \rightarrow (10)$.

Suppose (10) does not hold. Then there is a sequence $\{R_m\}$ and a sequence of sets $\{G_m\}$, each $G$ being a finite sum of nonoverlapping cubes of $E$, such that

$$\int_{G_m} S_{E_m}(\lambda, A, x)dx = (2\pi)^{-n} \int_{E(u)} f(u) \left\{ \int_{G_m} S_{E_m}(\lambda, I, x-u)dx \right\} du$$

is not bounded. But this is a sequence of linear functionals defined over $L$, of norm

$$\max_u \left| \int_{G_m} S_{E_m}(\lambda, I, x-u)dx \right|,$$

and by the theorem of Banach and Steinhaus[4] there is an $f \in L$ such that

$$\left\{ \int_E S_{E_m}(\lambda, A, x)dx \right\}$$

is unbounded, and hence $\mathfrak{S}(\lambda, A, x) \notin S$, which obviously implies $\mathfrak{S}(\lambda, A, x) \notin L$, contrary to hypothesis.

THEOREM 2. *The transformation classes* $(S, L)$ *and* $(M, C)$ *are identical, and* $\lambda$ *belongs to each of them if and only if* $\mathfrak{S}(\lambda, I, x) \in L$, *or*

$$\lim_{E, E' \to \infty} \int_E \left| S_E(\lambda, I, x) - S_{E'}(\lambda, I, x) \right| dx = 0. \tag{11}$$

PROOF. (A) $\lambda \in (S, L) \rightarrow (11)$, immediately, from Lemma 1.

(B) $(11) \rightarrow \lambda \in (S, L)$.

Since $S_E(\lambda, A, x) = (2\pi)^{-n} \int_{E(u)} S_E(\lambda, I, x-u) d\Phi(E)$ it follows that (11) implies the convergence in the mean of $\{S_E(\lambda, A, x)\}$.

(C) $(11) \rightarrow \lambda \in (M, C)$.

Since $S_E(\lambda, A, x) = (2\pi)^{-n} \int_{E(u)} S_E(\lambda, I, x-u) f(u) du$, with $|f| \leqq M$ by hypotheses, the uniform convergence of the sequence $\{S_E(\lambda, A, x)\}$ follows immediately from (11).

(D) $\lambda \in (M, C) \rightarrow (11)$.

Writing

$$S_R(\lambda, A, x) = (2\pi)^{-n} \int_{E_{(u)}} S_R(\lambda, I, x - u) f(u) du,$$

we have

$$S_R(\lambda, A, 0) = (2\pi)^{-n} \int_E S_R(\lambda, I, u) f(u) du = U_f(S_R(\lambda, I, u))$$

which defines a linear function $U_f$. The sequence of values $\{U_f(S_R(\lambda, I, u))\}$ converges for every bounded $f$, that is, $\{S_R(\lambda, I, u)\}$ converges weakly. But $L$ is weakly complete,[6] whence there is an $F \in L$ such that

$$\lim_{R\to\infty} \int_E \{S_R(\lambda, I, u) - F(u)\} f(u) du = 0$$

for every bounded $f$. Consider now the two series

$$\mathfrak{S}(F, u) \simeq \sum_{(k)} C(k) e^{i(k\cdot u)}$$

and

$$\mathfrak{S}(\lambda, I, u) \simeq \sum_{(k)} \lambda(k) e^{i(k\cdot u)},$$

and note that

$$S_R(\lambda, I, u) = \sum_{(k)} \Psi\left(\frac{|K|}{R}\right) \lambda(k) e^{i(k\cdot u)}.$$

Let

$$U_N(g) = \int_E g(u) e^{-i(N\cdot u)} du,$$

which is linear on $L$. Now

$$\lim_{R\to\infty} U_N(S_R(\lambda, I, u)) = \lim_{R\to\infty} \Psi\left(\frac{|N|}{R}\right)\lambda(N) = \lambda(N)$$
$$= U_N(F) = C(N)$$

whence

$$S_R(\lambda, I, u) = S_R(F, u).$$

But

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[6] S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne, vol. 1, Warsaw, 1932, p. 141.

$$\lim_{R\to\infty}\int_{E}\left|S_R(F,u)-F(u)\right|du=0$$

which, in view of the preceding equality, implies (11).

THEOREM 3. *The transformation classes* $(S, M)$, $(L, C)$, *and* $(L, M)$ *are identical, and* $\lambda$ *belongs to each of them if and only if* $\mathfrak{S}(\lambda, I, x)\in M$, *or*

$$|S_R(\lambda, I, x)| \leqq M. \tag{12}$$

PROOF. (A) $\lambda\in(S, M)\rightarrow(12)$ follows immediately from Lemma 1.
(B) $(12)\rightarrow\lambda\in(S, M)$.
Since

$$S_R(\lambda, A, x)=\int_{E(n)}S_R(\lambda, I, x-u)d\Phi(E),$$

we have from (12)

$$|S_R(\lambda, A, x)| \leqq M\int_E |d\Phi(E)|, \quad \text{and} \quad \mathfrak{S}(\lambda, A, x)\in M.$$

(C) $(12)\rightarrow\lambda\in(L, C)$.
We have immediately

$$|S_R(\lambda, A, x)-S_{R'}(\lambda, A, x)| \leqq M\int_E |S_R(I, A, x)-S_{R'}(I, A, x)|dx$$

where $\mathfrak{S}(\lambda, I, x)=\mathfrak{S}(f, x)$, and $|f| \leqq M$. The integral on the right approaches 0 as $R$ and $R'$ approach infinity, whence $\mathfrak{S}(\lambda, A, x)\in C$.
(D) $\lambda\in(L, C)\rightarrow(12)$.
We write immediately

$$(2\pi)^n S_R(\lambda, A, x)=\int_{E(n)}S_R(\lambda, I, x-u)f(u)du$$

and (12) follows from a theorem of Steinhaus and Banach.[4]
(E) $\lambda\in(L, C)\rightarrow\lambda\in(L, M)$ is obvious.
(F) $\lambda\in(L, M)\rightarrow\lambda\in(L, C)$.
Since

$$S_R(\lambda, A, x)=(2\pi)^{-n}\int_{E(n)}S_R(\lambda, I, x-u)f(u)du$$

and $|S_R(\lambda, A, x)| \leqq M(f)$ by hypothesis, it follows that $S_R(\lambda, A, x)$ exists for every $f\in L$, and $\{S_R(\lambda, A, 0)\}$ is a sequence of linear functionals on $L$, whose norms are

$$\operatorname*{ess\,sup}_{u} |S_E(\lambda, I, u)| = \max_{u} |S_E(\lambda, I, u)|,$$

and for every $f \in L$

$$|S_E(\lambda, A, 0)| \leq M(f),$$

whence, by the theorem of Steinhaus and Banach[4] the norms are uniformly bounded. Hence $\lambda \in (L, M) \rightarrow (12) \rightarrow \lambda \in (L, C)$.

THEOREM 4. $\lambda$ *belongs to the transformation class* $(S, C)$ *if and only if* $\mathfrak{S}(\lambda, I, x) \in C$, *or*

$$(13) \qquad \lim_{E, E' \to \infty} |S_E(\lambda, I, x) - S_{E'}(\lambda, I, x)| = 0, \text{ uniformly in } x \in E.$$

PROOF. (A) $\lambda \in (S, C) \rightarrow (13)$ immediately from Lemma 1.

(B) $(13) \rightarrow \lambda \in (S, C)$.

Since

$$\begin{aligned} S_E(\lambda, A, x) - S_{E'}(\lambda, A, x) = (2\pi)^{-n} \int_{E(u)} S_E(\lambda, I, x - u) d\Phi(E) \\ - (2\pi)^{-n} \int_{E(u)} S_{E'}(\lambda, I, x - u) d\Phi(E), \end{aligned}$$

we may write

$$\begin{aligned} &|S_E(\lambda, A, x) - S_{E'}(\lambda, A, x)| \\ &\qquad \leq \sup_{x \in I} |S_E(\lambda, I, x) - S_{E'}(\lambda, I, x)| \int_I |d\Phi(E)| \end{aligned}$$

and the uniform convergence of $\{S_E(\lambda, A, x)\}$ follows from that of $\{S_E(\lambda, I, x)\}$.

6. **Conclusion.** Of the factor sequence transformations among the classes $S$, $L$, $M$, and $C$, all of those characterizable in terms of these classes applied to $\mathfrak{S}(\lambda, I, x)$ have been discussed. The class $(L, M)$ of transformations does not exist in a proper sense since, by Theorem 3, its range is a subset of $C$ contained in $M$. The results of the paper may be taulated as follows:

$$\mathfrak{S}(\lambda, I, x) \in S \equiv \lambda \in (S, S) \equiv (L, L) \equiv (M, M) \equiv (C, C),$$

$$\mathfrak{S}(\lambda, I, x) \in L \equiv \lambda \in (S, L) \equiv (M, C),$$

$$\mathfrak{S}(\lambda, I, x) \in M \equiv \lambda \in (S, M) \equiv (L, C),$$

$$\mathfrak{S}(\lambda, I, x) \in C \equiv \lambda \in (S, C).$$

BROWN UNIVERSITY AND THE CITADEL

# SOME PROPERTIES OF MEASURABLE FUNCTIONS

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**1. Introduction.** Throughout this paper the letter $I$ will denote some fixed closed interval and $f$ a numerically valued measurable function on $I$. It is our purpose to establish certain general properties of $f$. We point out in §4 that two theorems of Banach are almost immediate consequences of these properties. We suspect that further use can be made of our results.

**2. Some notations.** We define

$$X^\wedge = \underset{y}{E}\,[y = f(x) \text{ for some } x \in X], \qquad X \subset I,$$

$$Y^\vee = \underset{x}{E}\,[f(x) \in Y], \qquad Y \subset I^\wedge.$$

Writing $X^{\wedge\vee} = (X^\wedge)^\vee$ and $Y^{\vee\wedge} = (Y^\vee)^\wedge$ we note that the relations

$$X \subset X^{\wedge\vee}, \qquad Y = Y^{\vee\wedge},$$

$$\left(\sum_{n=1}^{\infty} X_n\right)^\wedge = \sum_{n=1}^{\infty} X_n^\wedge, \qquad \left(\sum_{n=1}^{\infty} Y_n\right)^\vee = \sum_{n=1}^{\infty} Y_n^\vee,$$

$$\left(\prod_{n=1}^{\infty} X_n\right)^\wedge \subset \prod_{n=1}^{\infty} X_n^\wedge, \qquad \left(\prod_{n=1}^{\infty} Y_n\right)^\vee = \prod_{n=1}^{\infty} Y_n^\vee,$$

$$X_1^\wedge - X_2^\wedge \subset (X_1 - X_2)^\wedge, \qquad Y_1^\vee - Y_2^\vee = (Y_1 - Y_2)^\vee,$$

$$YX^\wedge = (Y^\vee X)^\wedge,$$

hold whenever $X, X_1, X_2, \cdots$ are subsets of $I$ and $Y, Y_1, Y_2, \cdots$ are subset of $I^\wedge$.

We further define

$$\{y\} = \underset{z}{E}\,[z = y],$$

$$\mathfrak{F} = \underset{y}{E}\,[\{y\}^\vee \text{ is finite}],$$

$$\mathfrak{K} = \underset{y}{E}\,[\{y\}^\vee \text{ has at least } \aleph_0 \text{ elements}],$$

$$\mathfrak{D} = \underset{y}{E}\,[\{y\}^\vee \text{ has more than } \aleph_0 \text{ elements}].$$

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We denote the (outer) linear Lebesgue measure of a set $A$ by $|A|$.

## 3. Some theorems concerning $f$.

THEOREM 3.1. *If $A$ is a measurable subset of $I$, then there is a measurable subset $B$ of $A$ such that $B^{\wedge} = A^{\wedge}$ and $f$ is univalent on $B$.*

PROOF. Let[1] $0 = C_0 \subset C_1 \subset C_2 \subset \cdots$ be closed subsets of $A$ relative to each of which $f$ is continuous and for which

$$C = \sum_{n=1}^{\infty} C_n, \qquad |A - C| = 0.$$

For each positive integer $n$ let[2] $D_n$ be a measurable subset of $C_n$ such that $D_n^{\wedge} = C_n^{\wedge}$ and $f$ is univalent on $D_n$. Let

$$B_1 = \sum_{n=1}^{\infty} [D_n - C_{n-1}^{\wedge\vee}].$$

Noting that $C_{n-1}^{\wedge}$ and $C_n C_{n-1}^{\wedge\vee}$ are closed and also that

$$D_n - C_{n-1}^{\wedge\vee} = D_n - C_n C_{n-1}^{\wedge\vee}, \qquad n = 1, 2, 3, \cdots,$$

we see that $B_1$ is a measurable subset of $C$.

Now let $y_0 \in C^{\wedge}$. Let $k$ be the least integer in

$$\underset{n}{E}\,[y_0 \in C_n^{\wedge}].$$

Since

$$y_0 \in C_k^{\wedge} - C_{k-1}^{\wedge} = D_k^{\wedge} - C_{k-1}^{\wedge} \subset (D_k - C_{k-1}^{\wedge\vee})^{\wedge},$$

take a number $x_0$ such that $f(x_0) = y_0$ and

$$x_0 \in [D_k - C_{k-1}^{\wedge\vee}] \subset B_1.$$

Thus $y_0 \in B_1^{\wedge}$. Moreover if $x_0 \neq x \in B_1$ with $f(x) = y_0$, we could, for some integer $l > k$, successively infer the relations

$$x \in D_l - C_{l-1}^{\wedge\vee}, \qquad y_0 \in C_k^{\wedge} \subset C_{l-1}^{\wedge}, \qquad x \in C_{l-1}^{\wedge\vee},$$

the first and last of which contradict.

Consequently $C^{\wedge} \subset B_1^{\wedge}$, $f$ is univalent on $B_1$, $C^{\wedge} = B_1^{\wedge}$.

Let $\alpha = A - C^{\wedge\vee}$ and select a set $B_2 \subset \alpha$ such that $B_2^{\wedge} = \alpha^{\wedge}$ and $f$ is univalent on $B_2$. Now

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[1] See Saks, *Theory of the integral*, Warsaw, 1937, p. 72.

[2] See Saks, loc. cit. p. 282.

$$B_2 \subset \alpha \subset A - C, \qquad |B_2| = 0.$$

Defining $B = B_1 + B_2$ we see that $B$ is measurable and that

$$\begin{aligned} A^\wedge \supset B^\wedge &= B_1^\wedge + B_2^\wedge \\ &= C^\wedge + (A - C^{\wedge\vee})^\wedge \supset C^\wedge + (A^\wedge - C^\wedge) = A^\wedge. \end{aligned}$$

Lastly $f$ is univalent on $B$; for otherwise, in view of its univalence on $B_1$ and $B_2$, we could select points $x_1 \in B_1$ and $x_2 \in B_2$ with $f(x_1) = f(x_2)$ and deduce the false proposition

$$x_2 \in B_2 B_1^{\wedge\vee} \subset \alpha C^{\wedge\vee} = (A - C^{\wedge\vee})C^{\wedge\vee} = 0.$$

The proof is complete.

Lemma 3.2. *If $f$ is continuous relative to each of the closed sets $A_1 \subset A_2 \subset A_3 \subset \cdots$ with*

$$A = \sum_{j=1}^{\infty} A_j \subset I,$$

*then*

$$S_n = A \cdot \left( \underset{y}{E} \left[ A\{y\}^\vee \textit{ has at least } n \textit{ elements} \right] \right)^\vee$$

*is a Borel set for each positive integer $n$.*

Proof. Let $n$ be a positive integer.

For each positive integer $j$ let $W_j$ be that subset of Euclidean $n$-space such that $P = (P_1, P_2, \cdots, P_n)$ is in $W_j$ if and only if

$$\begin{aligned} &P_i \in A_j, && i = 1, 2, \cdots, n, \\ &P_i - P_{i-1} \geqq 1/j, && i = 2, 3, \cdots, n. \end{aligned}$$

For $j = 1, 2, \cdots 3, \cdots$ note that $W_j$ is bounded and closed and let

$$B_j = A_j \underset{x}{E} \left[ \text{there is a } P \in W_j \text{ with } f(x) = f(P_i) \text{ for } i = 1, 2, \cdots, n \right].$$

It follows that

$$S_n = \sum_{j=1}^{\infty} B_j. \tag{1}$$

Now let $j_0$ be a positive integer and $x_1, x_2, x_3, \cdots$ be points of $B_{j_0}$ such that

$$\lim_{m\to\infty} x_m = x_0 \in I.$$

Clearly $x_0 \in A_{j_0}$. There are points $P^1, P^2, P^3, \cdots$ of $W_{j_0}$ such that

$$f(x_m) = f(P_i^m), \qquad m = 1, 2, 3, \cdots ; i = 1, 2, 3, \cdots , n.$$

Let $\alpha$ be a set of integers and $P^0$ be a point of the compact set $W_{j_0}$ such that

$$\lim_{m\to\infty,\, m\in\alpha} P^m = P^0.$$

Since $f$ is continuous relative to $A_{j_0}$ we conclude

$$f(P_i^0) = \lim_{m\to\infty,\, m\in\alpha} f(P_i^m) = \lim_{m\to\infty,\, m\in\alpha} f(x_m) = f(x_0),$$

$$i = 1, 2, \cdots , n,$$

which implies $x_0 \in B_{j_0}$. Hence $B_{j_0}$ is closed.

In view of (1) the proof is complete.

THEOREM 3.3. *There is a measurable*[3] *set $C$ such that $C^{\wedge} = \mathfrak{R}$ and*

$$(\{y\}^{\vee} - C) \text{ is finite for each } y \in I^{\wedge}.$$

PROOF. Retaining the notation of the statement of 3.2 and demanding in addition that $|I-A| = 0$, we define

$$S = A \cdot \left( \underset{y}{E}\, [A\{y\}^{\vee} \text{ is infinite}] \right)^{\vee}, \tag{2}$$

$$T = (I - A) \cdot \left( \underset{y}{E}\, [(I - A)\{y\}^{\vee} \text{ is infinite}] \right)^{\vee}, \tag{3}$$

$$C = S + T.$$

The set $C$ is measurable because $S = \prod_{n=1}^{\infty} S_n$ and $|T| = 0$. The fact that $C^{\wedge} \subset \mathfrak{R}$ may be easily verified by deleting $A$ and $(I-A)$ from (2) and (3), respectively.

The assumption that $(A\{y\}^{\vee} - C)$ is infinite leads to the relations

$$A\{y\}^{\vee} \subset S \subset C, \qquad A\{y\}^{\vee} - C = 0,$$

the second of which is contradictory. Similarly the assumption that $((I-A)\{y\}^{\vee} - C)$ is infinite leads to a contradiction.

Thus $\{y\}^{\vee} - C$ is finite for each $y \in I^{\wedge}$, $C\{y\}^{\vee}$ is infinite (and nonvacuous) for $y \in \mathfrak{R}$; consequently $\mathfrak{R} \subset C^\wedge$.

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[3] Using the fact that there is a perfect set of measure zero which is decomposable into a continuum of disjoint perfect sets, it can be shown that a judicious choice of our function $f$ insures that neither $\mathfrak{R}^{\vee}$ nor $\mathfrak{Q}^{\vee}$ is measurable.

Hence $\mathfrak{R} = C^\wedge$.

THEOREM 3.4.[4] *Let* $\epsilon > 0$. *Then*

(i) *there is a measurable set* $L \subset I$ *with* $L^\wedge = \mathfrak{R}$ *and* $|L| < \epsilon$;

(ii) *there is a measurable set* $R \subset I$ *with* $R^\wedge = \mathfrak{Q}$ *and* $|R| = 0$.

PROOF. Let $C$ be a measurable set such that $C^\wedge = \mathfrak{R}$ and $\{y\}^\vee - C$ is finite for each $y \in \mathfrak{R}$.

Let $F$ be the family of measurable sets $X \subset C$ such that $X\{y\}^\vee$ is finite for each $y \in X^\wedge$. Note that $0 \in F \neq 0$ and define

$$M = \sup_{X \in F} |X|$$

and let $X_1, X_2, X_3, \cdots$ be sets in $F$ such that $\lim_{n\to\infty} |X_n| = M$. Letting

$$S_n = \sum_{j=1}^{n} X_j, \qquad n = 1, 2, 3, \cdots,$$

observe that $X_n \subset S_n \in F$. Thus $|X_n| \leq |S_n| \leq M$ for $n = 1, 2, 3, \cdots$ and consequently

$$|S| = \lim_{n\to\infty} |S_n| = M \qquad \text{where } S = \sum_{n=1}^{\infty} S_n.$$

Let $n_0$ be an integer such that $|S - S_{n_0}| < \epsilon$. Let $T$ be a measurable (3.1) subset of $C - S$ such that $T^\wedge = (C-S)^\wedge$ and $f$ is univalent on $T$.

Defining

$$L = T + (S - S_{n_0}), \quad R = T\mathfrak{Q}^\vee,$$

we verify first that $|T| = 0$. If $|T|$ were greater than 0 there would be an integer $k$ such that

$$|S_k| > M - |T|, \qquad (T + S_k) \in F, \qquad TS_k \subset TS = 0,$$
$$M \geq |T + S_k| = |T| + |S_k| > M, \qquad M > M.$$

Consequently $L$ and $R$ are measurable, $|L| < \epsilon$, $|R| = 0$.

Next since $S_{n_0}\{y\}^\vee$ is at most finite, $S\{y\}^\vee$ is at most countable and $C\{y\}^\vee$ has the same power as $\{y\}^\vee$ for $y \in \mathfrak{R}$, we see that

$$\mathfrak{R} \subset (C - S_{n_0})^\wedge \subset \mathfrak{R}, \qquad \mathfrak{Q} \subset (C - S)^\wedge = T^\wedge.$$

From these relations it follows almost immediately that $L^\wedge = \mathfrak{R}$ and $R^\wedge = \mathfrak{Q}$.

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[4] It is evident from Theorem 3.1 that Theorem 3.4 remains true if we add the requirement that $f$ be univalent on $L$ and on $R$.

4. **Some applications.** We say our function $f$ satisfies condition $S$ if and only if for each $\epsilon>0$ there is a $\delta>0$ such that

$$|X^\wedge\cdot[-1/\epsilon, 1/\epsilon]| < \epsilon \quad \text{whenever } X\subset I,\ |X|<\delta;$$

$f$ satisfies condition $N$ if and only if

$$|X^\wedge| = 0 \quad \text{whenever } X\subset I,\ |X| = 0;$$

$f$ satisfies condition $T_1$ if and only if $|\mathfrak{K}| = 0$; and $f$ satisfies condition $T_2$ if and only if $|\mathfrak{D}| = 0$.

The following two theorems of Banach[5] are essentially corollaries of 3.4.

THEOREM 4.1. *If $f$ satisfies condition $N$, then it satisfies $T_2$.*

THEOREM 4.2. *A necessary and sufficient condition that $f$ satisfy condition $S$ is that it satisfy both $N$ and $T_1$.*

Theorem 4.1 and the *necessity* in 4.2 are immediate consequences of 3.4. The *sufficiency* in 4.2 may be proved as follows:

Suppose $f$ satisfies $N$ and $T_1$ but not $S$. Then we can find a number $\epsilon>0$ and measurable subsets $A_1\supset A_2\supset A_3\supset\cdots$ of $I$ such that

$$\lim_{n\to\infty}|A_n| = 0, \qquad |A_n^\wedge|\geqq\epsilon, \qquad A_n^\wedge\subset[-1/\epsilon, 1/\epsilon],$$
$$n = 1, 2, 3, \cdots.$$

We note[6] that $A_n^\wedge$, $I^\wedge$, $\mathfrak{K}$, $\mathfrak{F}$, are measurable and define

$$A = \prod_{n=1}^{\infty} A_n.$$

Since the product of a descending sequence of nonvacuous *finite* sets is nonvacuous, we have

$$\mathfrak{F}\prod_{n=1}^{\infty}A_n^\wedge \subset \left(\prod_{n=1}^{\infty}A_n\right)^\wedge = A^\wedge.$$

Hence

$$|A^\wedge| \geqq \left|\mathfrak{F}\prod_{n=1}^{\infty}A_n^\wedge\right| = \lim_{n\to\infty}|\mathfrak{F}A_n^\wedge| = \lim_{n\to\infty}|A_n^\wedge| \geqq \epsilon$$

in contradiction to the relations

$$|A| = \lim_{n\to\infty}|A_n| = 0, \qquad |A^\wedge| = 0.$$

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[5] See Saks, loc. cit. p. 284.

[6] If $f$ satisfies $N$, then $X^\wedge$ is measurable, whenever $X$ is a measurable subset of $I$.

Theorem 4.2 is proved.

5. **Generalizations.** Suppose $E$ is a metric space and $\phi$ is a measure over[7] $E$ such that: closed sets are $\phi$ measurable in the sense of Carathéodory;[8] every $\phi$ measurable set is the sum of an $F_\sigma$ and a set of $\phi$ measure zero; $E$ is a countable sum of compact sets; $\phi(E) < \infty$.

Let $H$ be another metric space. We say a function $q$ is $\phi$ finitely valued, if it is on $E$, its range is a finite subset of $H$ and

$$\underset{x}{E}\,[q(x) = y]$$

is $\phi$ measurable for each $y \in H$. We call a function $\phi$ measurable, if it is $\phi$ almost everywhere in $E$ the limit of a sequence of $\phi$ finitely valued functions.

To generalize our results we replace $I$ by $E$, Lebesgue measure by $\phi$, $f$ by a $\phi$ measurable function $g$, the word "closed" in the statement of 3.2 by "compact." Conditions $S$, $N$, $T_1$, $T_2$ are to be interpreted in terms of such a measure $\psi$ over $H$ that closed subsets of $H$ are $\psi$ measurable and bounded sets have finite $\psi$ measure.

All our theorems remain true under these conditions with properly adjusted notation. Leaving details to the reader, we solve the only non-trivial problem arising in this extension by the following:

THEOREM 5.1. *If $g$ is continuous relative to the compact set $C \subset E$, then there is a Borel set $B \subset C$ such that $B^\wedge = C^\wedge$ and $g$ is univalent on $B$.*

PROOF. Select[9] a continuous function $h$ whose domain is a perfect set $A \subset [0, 1]$ and whose range is $C$.

For $n = 1, 2, 3, \cdots$, let $A_n$ be the set such that $t \in A_n$ if and only if $t \in A$ and the relations

$$s \in A, \quad s \leqq t - n^{-1}, \quad g[h(s)] = g[h(t)],$$

are incompatible. Note that $A_n$ is open with respect to $A$. Let

$$B_n = C \cdot \underset{x}{E}\,[x = h(t) \text{ for some } t \in A_n],$$

$$B = \prod_{n=1}^{\infty} B_n.$$

Observe that $B$ is an $F_{\sigma\delta}$.

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[7] We say $\Lambda$ is over $E$ if and only if $\Lambda$ is a function whose domain is the set of subsets of $E$.

[8] See H. Hahn, *Theorie der reellen Funktionen*, vol. 1, Berlin, 1921, p. 424.

[9] See W. Sierpinski, *Introduction to general topology*, Toronto, 1934, p. 166.

Now choose $y_0 \in C^\wedge$. Taking

$$t_0 = \inf A \cdot \underset{t}{E}\,[g[h(t)] = y_0], \qquad x_0 = h(t_0),$$

we easily check that

$$t_0 \in \prod_{n=1}^{\infty} A_n, \qquad x_0 \in B, \qquad g(x_0) = y_0.$$

If $x_0 \neq x_1 \in B$ with $g(x_1) = y_0$, we infer that

$$T = A \cdot \underset{t}{E}\,[h(t) = x_1]$$

is a closed set lying entirely to the right of $t_0$ with distance $d > 0$ from $t_0$; hence $n > d^{-1}$ implies $T \subset A - A_n$ which in turn implies $x_1 \in E - B_n \subset E - B$ contrary to the assumption $x_1 \in B$.

Theorem 5.1 is proved.

Now we replace the condition that $\phi(E) < \infty$ by the hypothesis:

$$E = \sum_{n=1}^{\infty} E_n, \qquad E_n E_m = 0 \quad \text{for} \quad n \neq m,$$

$$E_n \text{ is } \phi \text{ measurable with } \phi(E_n) < \infty.$$

Obviously Theorem 3.4 part (i) and Theorem 4.2 do not[10] hold under these new conditions. However 3.1, 3.2, 3.3, 3.4 part (ii), 4.1 are still valid. To prove this it is sufficient to know of the existence of a measure $\Phi$ over $E$ such that: $\Phi(E) < \infty$; a set is $\Phi$ measurable if and only if it is $\phi$ measurable; a set has $\Phi$ measure zero if and only if it has $\phi$ measure zero.

Let $\lambda_1, \lambda_2, \lambda_3, \cdots$ be *positive* numbers such that

$$\sum_{n=1}^{\infty} \lambda_n \phi(E_n) < \infty$$

and define $\Phi$ over $E$ by the relation:

$$\Phi(X) = \sum_{n=1}^{\infty} \lambda_n \phi(X E_n) \qquad \text{for } X \subset E.$$

The reader will find no difficulty in checking that $\Phi$ is a measure with the required properties.

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[10] Counterexample: $g(x) = \sin x$, $-\infty < x < +\infty$.

# SOME THEOREMS ON THE EULER $\phi$-FUNCTION

N. G. GUNDERSON

The Euler $\phi$-function, $\phi(m)$, denotes the number of positive integers not greater than $m$ which are relatively prime to $m$.[1] It was noted by U. Scarpis[2] that $n \mid \phi(p^n - 1)$. Generalizations of this result are obtained in Theorems 9 and 10.

The first five theorems are either well known or self-evident.[3]

THEOREM 1. *If $p_1, \cdots, p_k$ are the distinct prime factors of $m$, then*

$$\phi(m) = m(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)/p_1 p_2 \cdots p_k.$$

THEOREM 2. *If $a_1, \cdots, a_k$ are relatively prime in pairs, then*

$$\phi(a_1 \cdots a_k) = \phi(a_1) \cdot \phi(a_2) \cdots \phi(a_k).$$

THEOREM 3. *If $w$ is the product of the distinct prime factors common to $m$ and $n$, then*

$$\phi(mn) = w \cdot \phi(m) \cdot \phi(n)/\phi(w).$$

THEOREM 4. *If $a \mid b$, then $\phi(a) \mid \phi(b)$.*

THEOREM 5. *If $q \mid a$ and $q \equiv 1 \pmod{p^\alpha}$, then $p^\alpha \mid \phi(a)$.*

THEOREM 6. *If $p$ is an odd prime, $a \neq b$, and $\alpha \geqq 1$, then*

$$p^{2\alpha-1} \mid \phi(a^{p^\alpha} + b^{p^\alpha}).$$

The proof is by induction on $\alpha$. We assume $a > b$. In the notation of Birkhoff and Vandiver,[4] $a^p + b^p = V_{2p}/V_p$. By their Theorems V and I, there is a prime divisor $q$ of $a^p + b^p$ such that $q \equiv 1 \pmod{p}$ unless $p = 3$, $a = 2$, $b = 1$. Then by Theorem 5, $p \mid \phi(a^p + b^p)$, and in the exceptional case, $3 \mid \phi(2^3 + 1^3)$. Thus the theorem holds for $\alpha = 1$, starting the induction, so we assume it for all positive integers less than $\alpha$. We adopt the notation $C = AB$, where

$$C = a^{p^\alpha} + b^{p^\alpha}, \qquad P = p^{\alpha-1}, \qquad A = a^p + b^p,$$
$$B = a^{(p-1)P} - a^{(p-2)P} \cdot b^P + \cdots - a^P \cdot b^{(p-2)P} + b^{(p-1)P}.$$

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[1] In this discussion all letters represent positive integers. In particular, $p$ and $q$ represent primes.

[2] Period. Mat. vol. 29 (1913) p. 138.

[3] See, for example, L. E. Dickson, *History of the theory of numbers*, vol. 1, chap. 5.

[4] Ann. of Math. (2) vol. 5 (1903) pp. 173–180.

*Case* 1. $(a, b)=1$.

Again using the notation of Birkhoff and Vandiver, $B=C/A =(V_{2pP}/V_{pP})(V_P/V_{2P})$, so we see by their Theorems V and I that there is a prime divisor $q$ of $B$ such that $q\equiv 1 \pmod{p^\alpha}$. Hence by Theorem 5 we have that $p^\alpha | \phi(B)$. By the hypothesis of induction we have $p^{2(\alpha-1)-1} | \phi(A)$.

Now, if $(A, B)\neq 1$, let $r$ be a common prime factor of $A$ and $B$. Then $a^P\equiv -b^P \pmod{r}$, so that $B\equiv p\cdot a^{(p-1)P}\equiv 0 \pmod{r}$. If $r|a$, then $r|b$, contrary to $(a, b)=1$, so $r=p$. Then by Theorem 3,

$$\phi(C) = p\cdot\phi(A)\cdot\phi(B)/(p-1),$$

so we have that $p\cdot p^{2(\alpha-1)-1}\cdot p^\alpha|\phi(C)$. But since $\alpha>1$, $3\alpha-2>2\alpha-1$, so $p^{2\alpha-1}|\phi(C)$.

If $(A,B)=1$, then $\phi(C)=\phi(A)\cdot\phi(B)$, and $p^{2(\alpha-1)-1}\cdot p^\alpha|\phi(C)$. Again, since $\alpha>1$, $3\alpha-3\geqq 2\alpha-1$, and $p^{2\alpha-1}|\phi(C)$.

*Case* 2. $(a, b)\neq 1$.

Let $(a, b)=c$, $a=ca_1$, $b=cb_1$, $(a_1, b_1)=1$. Further, since $a>b$, $a_1$ and $b_1$ are not both 1, and so $a_1>b_1$. By Case 1, $p^{2\alpha-1}|\phi(a_1^{p^\alpha}+b_1^{p^\alpha})$ so by Theorem 4, $p^{2\alpha-1}|\phi(a^{p^\alpha}+b^{p^\alpha})$.

THEOREM 7. *If* $a\neq b$, *then* $2^{\alpha+1}|\phi(a^{2^\alpha}+b^{2^\alpha})$.

We note that $a^{2^\alpha}+b^{2^\alpha}=2^\beta$ would imply $a=b=2$, so $a^{2^\alpha}+b^{2^\alpha}$ has an odd factor, say $q$. For $(a, b)=1$, Euler[5] has shown that $q\equiv 1 \pmod{2^{\alpha+1}}$, so by Theorem 5, $2^{\alpha+1}|\phi(a^{2^\alpha}+b^{2^\alpha})$. For $(a, b)\neq 1$ we proceed as in Case 2 of Theorem 6.

THEOREM 8. *If* $a>b$, *then* $p^{2\alpha-1}|\phi(a^{p^\alpha}-b^{p^\alpha})$.

The proof of this theorem parallels that of Theorem 6.

THEOREM 9. *If* $a>b$, *and* $m$ *is the product of the distinct prime factors of* $n$, *then* $(n^2/m)|\phi(a^n-b^n)$.

Let $n=P_1P_2\cdots P_k$ where $P_i=p_i^{\alpha_i}$. Then $a^n-b^n=(a^{n_i})^{P_i}-(b^{n_i})^{P_i}$ where $n_i=P_1\cdots P_{i-1}P_{i+1}\cdots P_k$. Then by Theorem 8, $p_i^{2\alpha_i-1}|\phi(a^n-b^n)$, and the theorem follows immediately.

THEOREM 10. *If* $a\neq b$, *then* $n|\phi(a^n+b^n)$.

This proof parallels that of Theorem 9.

Theorems 9 and 10 can be combined in various ways, for example, we have this theorem:

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[5] Commentationes Arithmeticae Collectae, vol. 1, p. 55. See also Amer. Math. Monthly vol. 10 (1903) p. 171, misprint of $2m$ for $2^m$.

THEOREM 11. *If* $(a, b)=1$, *and* $m$ *is the product of the distinct prime factors of* $n$, *then*

$$(n^3/m) \mid \phi(a^{2n}-b^{2n}).$$

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# APPLICATIONS OF TRANSITIVITIES OF BETWEENNESS IN LATTICE THEORY[1]

M. F. SMILEY AND W. R. TRANSUE

**Introduction.** This paper solves three characterization problems for lattices[2] [1]. Problem I is to characterize those metric spaces [2] into which lattice operations which are consistent with the given metric [1, p. 41] may be introduced. Problem II is to characterize those members of a rather general class of abstract systems which are modular lattices, while Problem III consists in the characterization of lattices in an even larger class of abstract systems.

Problem I has already been solved by V. Glivenko [3]. He showed that the property: "Among those elements metrically between [4, p. 76; 2] two elements $a$ and $b$, the element $a \cup b$ is farthest from $O$," and its dual characterize those metric spaces which are also metric lattices with the same metric and least element $O$. Our approach to Problem I is through the existence of certain metric singularities [2, p. 47] in every metric lattice. Our solution also involves certain five point transitivities [5, Part I] of metric betweenness. The abstract system involved in Problem II (Problem III) is a wide generalization of the concept of a metric space—so general, in fact, that it also includes the concept of a modular lattice (lattice). We find it not difficult to extend the ideas essential to our solution of Problem I to give analogous solutions of Problems[3] II and III. Briefly, our results consist in characterizing the three important systems: metric

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[1] This paper is an expansion of our note *Metric lattices as singular metric spaces* which was presented to the Society on December 29, 1941, and which was to appear in Bull. Amer. Math. Soc.

[2] Numbers in square brackets refer to the list of references at the end of the paper.

[3] It would be interesting to characterize metric spaces among our general systems. This problem appears difficult to us, and we make no attempt to solve it.

lattices, modular lattices, and lattices within three increasingly general abstract systems.

## 1. A characterization of metric lattices.

In this section we shall give our solution of Problem I. The metric singularities which we shall encounter are *pseudo-linear quadruples* [2, p. 48]. Such a quadruple is one which cannot be imbedded in the Euclidean line, but is such that each of the four triples chosen from it can be so imbedded. In metric lattices such configurations abound. Every pair of elements which are not comparable together with their meet and join form such a quadruple. But the existence of "sufficiently many" pseudo-linear quadruples in a metric space $(M, \delta)$ is not enough to ensure that lattice operations can be introduced in such a way that $(M, \delta)$ becomes a metric lattice $(M, \delta, <)$. However, the additional assumption of a weak form of either of two five point transitivities for the metric betweenness of $(M, \delta)$ suffices.

Before we proceed to prove these statements, let us agree on the following matters of notation. The symbols $(M, \delta)$ represent a metric space $M$ with metric $\delta$. For elements $a, b, c \in M$ we use the notation $abc$ to indicate that $b$ is metrically between $a$ and $c$ [4, p. 75; 5, Part II], that is, that $\delta(a, c) = \delta(a, b) + \delta(b, c)$. This relation has the fundamental properties:[4]

$$(\alpha) \qquad abc \leftrightarrow cba.$$

$$(\beta) \qquad abc \cdot acb \leftrightarrow b = c.$$

We shall frequently refer to the transitivities:

$$(t_1) \qquad abc \cdot adb \rightarrow dbc.$$

$$(t_2) \qquad abc \cdot adb \rightarrow adc.$$

which are valid for the metric betweenness of $M$ [4, p. 76; 5, Part II]. If $(M, \delta)$ is also a metric lattice $(M, \delta, <)$, then the relation $abc$ is equivalent [3] to the equations

$$(a \cap b) \cup (b \cap c) = b = (a \cup b) \cap (b \cup c). \tag{1.1}$$

These equations define, in fact, a relation of betweenness in arbitrary lattices [5, Part II]. We shall also use the notation $abc$ to indicate that the equations (1.1) are valid, even when no metric enters the discussion. In this paper we shall need the fact that this relation of lattice betweenness satisfies ($\alpha$), ($\beta$), ($t_1$), and

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[4] The second of these statements is intended to be read as, "For each three elements $a, b, c \in M$, the relations $abc$ and $acb$ hold if and only if $b = c$." This convenient notation was used with profit in [5]. We shall attempt to preserve as many of the other notations introduced in [5] as possible.

$$abc \rightarrow a \cap c \leqq b \leqq a \cup c. \tag{1.2}$$

Further properties of this relation are given in [5, Part II].

With reference to each point $p \in M$ of a metric space $(M, \delta)$ we may define a partial ordering of $M$ as follows [3].

*If $a, b \in M$, then $a <_p b$ in case $pab$ and $a \neq b$.* The transitivity of the relation $<_p$ follows from the transitivity $t_1$ of metric betweenness. The antisymmetry of $\leqq_p$ is apparent. We shall be interested in the following conditions [5, Part I].

$$(T_5^p) \qquad pbc \cdot pdc \cdot bxd \rightarrow pxc.$$

$$(T_7^p) \qquad pbc \cdot pbd \cdot cxd \rightarrow pbx.$$

REMARK 1. Condition ($T_5^p$) states that an upper bound for the set $[b, d]$ is greater than every element between $b$ and $d$, while condition ($T_7^p$) states that a lower bound for the set $[c, d]$ is smaller than every element between $c$ and $d$.

We shall also need the following condition.

($A^p$) *If $a, b \in M$ and if $a$ and $b$ are not comparable by the relation $\leqq_p$, then there are elements $a \cup b$, $a \cap b \in M$ for which we have $(a \cup b)a(a \cap b)$, $(a \cup b)b(a \cap b)$, $a(a \cup b)b$, $a(a \cap b)b$, and $p(a \cap b)(a \cup b)$.*

We may now state the following theorem.

THEOREM 1. *Let $(M, \delta)$ be a metric space. If $O \in M$, then $(M, \delta, <_0)$ is a metric lattice with least element $O$ if and only if $(M, \delta)$ satisfies ($A^0$) and ($T_5^0$) or ($A^0$) and ($T_7^0$).*

PROOF. The necessity of the condition ($A^0$) is obvious. But both ($T_5^p$) and ($T_7^p$) also hold in every lattice if $p$ is a distributive element. We prove this in the following lemma which is a generalization of Theorem 9.4 of [5].

LEMMA 1. *If $L$ is a lattice and $p$ is a distributive element of $L$, then the conditions ($T_5^p$) and ($T_7^p$) are true.*

PROOF. We prove ($T_5^p$) first. Let $p \in L$ be a distributive element of a lattice $L$, and consider elements $b, c, d, x \in L$ satisfying $pbc$, $pdc$, and $bxd$. From (1.2) we obtain $b \leqq p \cup c$, $d \leqq p \cup c$, and $x \leqq b \cup d$. Hence $x \leqq p \cup c$. Dually, $x \geqq p \cap c$. Since $p$ is distributive, it follows that $(p \cap x) \cup (x \cap c) = x \cap (p \cup c) = x$, and dually. Thus ($T_5^p$) holds. To prove ($T_7^p$), let $p \in L$ be a distributive element and consider elements $b, c, d, x \in L$ satisfying $pbc$, $pbd$, and $cxd$. As before, we have $b \leqq p \cup c$,

$b \leqq p \cup d$, and $x \geqq c \cap d$. Consequently, since $p$ is distributive, $p \cup x \geqq p \cup (c \cap d) = (p \cup c) \cap (p \cup d) \geqq b$. Dually, $p \cap x \leqq b$. Another application of the distributivity of $p$ yields $(p \cap b) \cup (b \cap x) = b \cap (p \cup x) = b$. By duality we obtain the relation $pbx$. Thus $(T_7^p)$ also holds and the proof of the lemma is complete.

The necessity of the conditions $(T_8^O)$ and $(T_7^O)$ now follows from Lemma 1 and the fact that the least element of a lattice is a distributive element.

We pass to the proof of the sufficiency of our conditions. Using only $(A^O)$ we see that $a \cup b$ and $a \cap b$ are upper and lower bounds, respectively, for the set $[a, b]$. This follows on applying the transitivities $t_1$ and $t_2$ to the relations $(a \cup b)a(a \cap b)$, $(a \cup b)(a \cap b)O$, and to the relations $(a \cup b)b(a \cap b)$, $(a \cup b)(a \cap b)O$. The force of the conditions $(T_8^O)$ and $(T_7^O)$ is that each ensures that $a \cup b$ and $a \cap b$ are *unique* least upper and greatest lower bounds, respectively. We shall give the details of the proof only in the case in which $(A^O)$ and $(T_8^O)$ are assumed. The other set of conditions may be handled by almost identical arguments. Hence suppose that the conditions $(A^O)$ and $(T_8^O)$ are valid. We prove first that $a \cup b$ is the least upper bound of the set $[a, b]$ if $a$ and $b$ are not comparable. For, if $z \geqq a, b$ by definition we have $Oaz$ and $Obz$. Condition $(A^O)$ yields $a(a \cup b)b$, and condition $(T_8^O)$ then gives $O(a \cup b)z$, that is, $z \geqq a \cup b$. The treatment of $a \cap b$ is less trivial. Consider an element $w \in M$ for which $w \leqq a, b$; that is, for which $Owa$ and $Owb$. We show first that $w > a \cap b$ is impossible as follows. If it were true we should have $O(a \cap b)w$, and this with $Owa$ and $Owb$ would yield, via $t_1$, the relations $(a \cap b)wa$ and $(a \cap b)wb$. Combining $aw(a \cap b)$ with $a(a \cap b)b$ we would obtain, again by $t_1$, the relation $w(a \cap b)b$. From this and the relation $(a \cap b)wb$ we would get $(a \cap b)w(a \cap b)$. But this last relation implies that $w = a \cap b$, which is contrary to our assumption. Thus the relation $w > a \cap b$ cannot hold. To show that $w \leqq a \cap b$ we again use indirect proof. If this relation fails to hold, then, since $w \not> a \cap b$, the elements $w$ and $a \cap b$ would not be comparable. The condition $(A^O)$ then implies the existence of elements $w \cup (a \cap b)$ and $w \cap (a \cap b)$ satisfying with $w$ and $a \cap b$ the relations listed in $(A^O)$. The application of the condition $(T_8^O)$ to the relations $O(a \cap b)a$, $Owa$, and $(a \cap b)((a \cap b) \cup w)w$ gives the relation $O((a \cap b) \cup w)a$. Likewise we obtain $O((a \cap b) \cup w)b$. Applying the transitivity $t_1$ to $((a \cap b) \cup w)(a \cap b)O$ and $a((a \cap b) \cup w)O$ yields $a((a \cap b) \cup w)(a \cap b)$. Similarly, $b((a \cap b) \cup w)(a \cap b)$. Again, the relations $b((a \cap b) \cup w)(a \cap b)$ and $b(a \cap b)a$ would give, by $t_1$, the relation $a(a \cap b)((a \cap b) \cup w)$. A final application of the transitivity $t_1$ to the relations $a(a \cap b)((a \cap b) \cup w)$ and $a((a \cap b) \cup w)(a \cap b)$ would produce the relation $(a\cap b)((a\cap b)\cup w)(a\cap b)$. But this implies that $a\cap b=(a\cap b)\cup w$, contrary to our assumption that $w\leqq a\cap b$ fails to hold. We conclude that $w\leqq a\cap b$ and that $a\cap b$ is the unique greatest lower bound for the set $[a, b]$. We have now shown that the relation $<_O$ has the lattice property. It is an easy consequence of the condition $(A^O)$ that $\delta(O, a)$ is a sharply positive modular functional and that

$$\delta(a, b) = \delta(a \cup b, O) - \delta(a \cap b, O).$$

This completes the proof that $(M, \delta, <_O)$ is a metric lattice.

REMARK 2. To show that one of the conditions $(T_5^O)$ or $(T_7^O)$ is actually needed in Theorem 1, we exhibit a metric space $(M, \delta)$ which satisfies the condition $(A^O)$ but for which the partial orderings $<_q$ fail to have the lattice property for every $q\in M$.

Let $M$ consist of the six points $O$, $a$, $b$, $c$, $d$, $I$, with the distances of distinct points (we omit $\delta$) as follows: $Ia=Ib=Oc=Od=ab=cd=2$, $IO=3$, all other distances between distinct points equal to one. The condition $(A^O)$ is valid for $(M, \delta)$ because only the pairs $a$, $b$ and $c$, $d$ are not comparable by the relation $<_O$ and for these pairs the choices $c\cup d=I$, $c\cap d=a$ or $b$, $a\cap b=O$, and $a\cup b=c$ or $d$ are effective in the condition $(A^O)$. Thus $<_O$ is not a lattice ordering. By the symmetry of $M$ it will suffice to show that $<_a$ is not a lattice ordering to prove our assertion completely. In $(M, \delta, <_a)$ we have, besides the obvious fact that $a$ is the least element, only the relations $O<b$, $c<b$, $d<b$, $c<I$, and $d<I$. This is obviously not a lattice ordering of $M$.

2. **A characterization of modular lattices.** We turn now to a consideration of Problem II. The abstract systems which we shall consider consist of a class $K$ of elements $a, b, c, \cdots$ together with a triadic relation $R$ defined on $K$ and satisfying $(\alpha)$, $(\beta)$, $(t_1)$, and $(t_2)$. We shall find conditions which are necessary and sufficient that such a system be a modular lattice whose lattice betweenness is identical with the relation $R$.

For an arbitrary triadic relation $R$ defined on a class $K$ of elements $a, b, c, \cdots$ we make the following definition.

(2.1) *If $a, b\in K$ then $a<_p b$ in case $(p, a, b)\ R$ and $a\neq b$.*

Using this notation, our solution of Problem II takes the following form.

THEOREM 2.1. *If a triadic relation $R$ defined on a class $K$ satisfies $(\alpha)$, $(\beta)$, $(t_1)$, and $(t_2)$, then $(K, \leqq_O)$ is a modular lattice with least element $O$ if and only if $(K, R)$ satisfies $(A^O)$ and $(T_5^O)$ or $(A^O)$ and $(T_7^O)$. When either of these alternatives holds, the relation $R$ is identical with the lattice betweenness of $K$.*

PROOF. We note first that if $(A^O)$ and $(T_5^O)$ $((A^O)$ and $(T_7^O))$ hold for $R$, the proof of Theorem 1 assures us that $(K, \leqq_0)$ is a lattice. Notice also that $a \leqq_0 b \leqq_0 c$ implies that $(a, b, c)R$ holds by the definition of $\leqq_0$ and $t_1$. The truth of the second sentence of the theorem is then evident from the following theorem [5, Theorem 10.1].

THEOREM 2.2. *If $L$ is a lattice and $R$ is a triadic relation defined on $L$, then $R$ is the relation of lattice betweenness of $L$ if and only if the following conditions hold.*

(i) *The relation $R$ satisfies $(\alpha)$, $(\beta)$, and $(t_1)$.*

(ii) *If $a \leqq b \leqq c$, then we have $(a, b, c)R$.*

(iii) *If the relation $abc$ holds, then in the sublattice generated by $a$, $b$, $c$, the transitivity $t_2$ holds for $R$.*

That the modular law holds in $(K, \leqq_0)$ is now implied by $t_2$ [5, Theorem 9.1]. This completes the proof that $(A^O)$ and $(T_5^O)$ $((A^O)$ and $(T_7^O))$ are sufficient. Their necessity follows from Theorem 2.2 and Lemma 1. The proof of Theorem 2.1 is now complete.

3. **A characterization of lattices.** In this final section we shall present our solution of Problem III. The abstract systems which we shall consider consist of a class $K$ of elements $a, b, c, \cdots$ together with a triadic relation $R$ defined on $K$ and satisfying $(\alpha)$, $(\beta)$, and $(t_1)$. We shall find sufficient conditions that such a system be a lattice under an ordering given by (2.1). In the presence of these conditions, we shall find necessary and sufficient conditions that the postulated relation $R$ coincide with the lattice betweenness of $K$. The first of these results is given in the following theorem.

THEOREM 3.1. *If a triadic relation $R$ defined on a class $K$ satisfies $(\alpha)$, $(\beta)$, and $(t_1)$, $(A^O)$ and $(T_5^O)$, then the relation $\leqq_0$ is a partial ordering of $K$ with the lattice property.*

PROOF. The antisymmetry of $\leqq_0$ is a consequence of $(\beta)$, while the transitive law for $\leqq_0$ may be easily obtained if we notice that identifying $d$ and $O$ in $(T_5^O)$ yields the condition

$$(t_2^O) \qquad (O, b, c)R\ (O, x, b)R \rightarrow (O, x, c)R.$$

We may also note that identification of $c$ and $d$ in $(T_5^O)$ yields the condition

$$(3.1) \qquad (O, b, c)R \cdot (b, x, c)R \rightarrow (O, x, c)R.$$

This last condition may be used to replace $t_2$ in the proof given in Theorem 1 of the sufficiency of the conditions $(A^O)$ and $(T_5^O)$. A careful examination of our proof will reveal that $t_1$ was used only to show that the relations $(O,a,a\cup b)R$ and $(O,b,a\cup b)R$ follow from $(O,a\cap b,a\cup b)R$, $(a\cap b,a,a\cup b)R$ and $(a\cap b,b,a\cup b)R$. But it is easy to see that these conclusions follow from the same hypotheses under condition (3.1) as well as under $t_1$. We can then follow the proof of Theorem 1 from this point until the conclusion that $(K, \leqq_0)$ is a lattice has been reached. The proof of Theorem 3.1 is complete.

REMARK 3. That the sufficient conditions of Theorem 3.1 are not necessary conditions can be seen by the following example. Let the class $K$ consist of the five points $O$, $a$, $b$, $c$, $I$, and let the relation $R$ hold for the triples $OaI$, $ObI$, $OcI$, $Obc$, and $bcI$, as well as for those then required by ($\alpha$) and ($\beta$). It is easy to verify that $t_1$ holds in this example while $(A^0)$ fails. But $R$ gives rise through $\leqq_0$ to the simplest non-modular lattice.

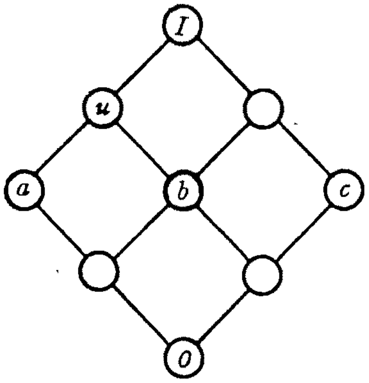

FIG. 1

REMARK 4. Theorem 3.1 leaves us in doubt as to the question of the identity of the triadic relation $R$ and lattice betweenness in $K$. The following example shows that still further assumptions must be made if we are to obtain this conclusion. Let $K$ consist of the elements of the lattice shown in Figure 1. Define $R$ to be lattice betweenness except that the relation $(a, b, c)R$ is rejected. That $R$ satisfies the conditions ($\alpha$) and ($\beta$) is clear since they hold for lattice betweenness and their conclusions do not involve three distinct elements. To verify that $t_1$ holds for $R$ we must make sure that we cannot arrange its hypotheses so as to obtain $(a, b, c)R$ in the conclusion. To obtain $(a, b, c)R$ from $t_1$ would require hypotheses of the form $(d, b, c)R$, $(d, a, b)R$ or of the form $(d, b, a)R$, $(d, c, b)R$. But these sets cannot hold in our example, since if we have $(d, a, b)R$ and $(d, b, c)R$, then $d\neq a$, and $d\cap b\leqq a\leqq d\cup b$, by (1.2). It follows that $d\cup b=u$ or $I$, contrary to $d\cap b\leqq a$. The other set of hypotheses may be treated similarly by interchanging $a$ and $c$. It is also easy to check that $R$ satisfies $(A^O)$. It remains only to remark that the conclusion of $(T_5^O)$ cannot be $(a, b, c)R$ (since $O$ must appear in the conclusion of $(T_5^O)$) to assure ourselves that $R$ satisfies all of the conditions of Theorem 4. Thus $R$ generates through $\leqq_0$ the lattice of Figure 1 but $R$ is not the relation of lattice betweenness of this lattice.

To overcome this difficulty we propose the following strengthened condition.

(3.2) *If* $(p\cap x)\cup(x\cap c)=x\cap(p\cup c)$ *and dually, then*

$$(p, b, c)R\cdot(p, d, c)R\ (b, x, d)R\rightarrow(p, x, c)R.$$

The proof of Lemma 1 shows that (3.2) holds for lattice betweenness in arbitrary lattices.

We conclude with the following theorem.

THEOREM 3.2. *If a triadic relation $R$ defined on a class $K$ satisfies* $(\alpha)$, $(\beta)$, $(t_1)$, $(T_5^O)$, *and* $(A^O)$, *then $R$ is the lattice betweenness of the lattice* $(K, \leqq_0)$ *if and only if $R$ satisfies* (3.2).

PROOF. The remark preceding the statement of the theorem disposes of the necessity of (3.2). To establish the sufficiency we rely on the proof of Theorem 10.1 of [5] to assure us that the relation $(a, b, c)R$ implies $abc$ (lattice betweenness). It remains, then, to prove that the relation $abc$ implies the relation $(a, b, c)R$. It was proved in [5] that when $abc$ holds the sublattice generated by $a$, $b$, $c$ is distributive. Hence the transitivity $(T_5)$ (see [5]) is available to us *in this sublattice* by virtue of (3.2). Now notice that since $a\cup c\geqq_0 b\geqq_0 a\cap c$ (by (1.2)), we have $a\cup b\cup c=a\cup c$ and dually. The condition $(A^O)$ then gives the relations $(a, a\cup b\cup c, c)R$ and $(a, a\cap b\cap c, c)R$. Clearly $a\cup b\cup c\geqq_0 b\geqq_0 a\cap b\cap c$; from which we obtain $(a\cup b\cup c, b, a\cap b\cap c)R$. The transitivity $(T_5^O)$ then yields $(a, b, c)R$ as desired and the proof is complete.

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# ON A THEOREM OF NEWSOM

H. K. HUGHES

1. **Extension of the theorem.** In 1938, Newsom[1] published a paper containing a theorem regarding the behavior for large values of $|z|$ of the function

$$(1)\qquad f(z) = \sum_{n=0}^{\infty} g(n) z^n,$$

radius of convergence equal to $\infty$. It is assumed that the function $g(w)$, where $w = x + iy$, satisfies the following two conditions:

(a) it is single-valued and analytic in the finite $w$-plane;

(b) it is such that for all values of $x$ and $y$, one may write

$$(2)\qquad |g(x+iy)| < Ke^{\pi|y|},$$

where $K$ is a positive constant and $k$ is a positive integer. Under these conditions, according to the theorem, $f(z)$ may be expressed in the form

$$(3)\qquad f(z) = \int_{-l-1/2}^{\infty} g(x)[\pm z]^x \frac{\sin k\pi x}{\sin \pi x} dx + \sum_{m=0}^{l} \frac{g(-m)}{z^m} + \xi(z, l),$$

where $l$ is any positive integer, where the symbol $[\pm z]^x$ means $z^x$ or $(-z)^x$ according as $k$ is odd or even, respectively, and where if $|\arg[\pm z]| < \pi$, we have $\lim_{|z|\to\infty} z^l \xi(z, l) = 0$ for every value of $l$.

In the present paper we shall consider the situation when conditions (a) and (b) are made somewhat less restrictive. The theorem which we wish to prove is as follows:

THEOREM. *Let the coefficient* $g(n)$ *in* (1) *satisfy condition* (a) *except for a singularity at the point* $w = w_1$, *which is not a negative integer; and let inequality* (2) *be satisfied for all values of* $|x|$ *and* $|y|$ *sufficiently large. Then* (3) *continues to hold provided one subtracts from the right member the loop integral*

$$(4)\qquad \frac{1}{2i} \int_C \frac{g(w) z^w}{e^{k\pi w} \sin \pi w} dw$$

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[1] *On the character of certain entire functions in distant portions of the plane*, Amer. J. Math. vol. 60 (1938) pp. 561–572.

*where the loop $C$ surrounds the point $w_1$, and extends to infinity in any convenient direction lying in either the third or the fourth quadrant.*

PROOF. In a concluding remark in his paper, Newsom infers that, under the conditions which we have postulated, the theorem continues to hold provided one subtracts from the right member of (3) a suitable loop integral, which upon examination of the analysis, is seen to be

$$I = \frac{k!\pi^k}{2i}\int_C \frac{P(w, z)}{(\sin \pi w)^{k+1}}\,dw, \tag{5}$$

where[2]

$$\begin{aligned} P(w, z) &= \sum_{j=1}^{k} b_j e^{r_j \pi i w}\int_0^w e^{-r_j \pi i w} g(w)[\pm z]^w dw, \\ r &= k + 1 - 2j, \qquad\qquad 1 \leqq j \leqq k, \\ b_j &= \frac{(-1)^{j-1}(2\pi i)^{1-k}}{(k-1)!} C_{j-1}^{k-1}, \qquad C_{j-1}^{k-1} = 1 \text{ when } j = 1. \end{aligned}$$

We shall show that the integral $I$ in (5) is equivalent to the integral (4). Evidently we may write

$$P(w, z) = \int_0^w \left\{ \sum_{j=1}^{k} b_j e^{r_j \pi i (w-t)} \right\} g(t)[\pm z]^t dt.$$

However, the sum appearing in the above integrand is readily shown to be equal to

$$\frac{1}{\pi^{k-1}(k-1)!}[\sin \pi(w - t)]^{k-1},$$

and this last expression can be written in the form

$$\frac{1}{\pi^{k-1}(k-1)!}\sum_{j=1}^{k}(-1)^{j-1}C_{j-1}^{k-1}(\sin \pi w)^{k-j}(\cos \pi w)^{j-1}(\cos \pi t)^{k-j}(\sin \pi t)^{j-1}.$$

Introducing into (5) the resulting new form for $P(w, z)$, and making simple reductions, we obtain the equation

$$I = \frac{k\pi}{2i}\int_C \left\{ \sum_{j=1}^{k} C_{j-1}^{k-1}(-\cot \pi w)^{j-1}\csc^2 \pi w\, \phi_j(w, z) \right\} dw, \tag{6}$$

where

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[2] The form of $b_j$ as given here may be seen to be equivalent to that given by Newsom.

$$\phi_j(w, z) = \int_0^w (\cos \pi t)^{k-j}(\sin \pi t)^{j-1} g(t) [\pm z]^t dt.$$

We may simplify (6) by an integration by parts. Consider first the general term of the sum appearing in the integrand. If we let $u = \phi_j(w, z)$, $dv = \pi(-\cot \pi w)^{j-1} \csc^2 \pi w \, dw$, then, taking account of the constant factors, we obtain as an integral of the general term the expression

$$\frac{k}{2ij} C_{j-1}^{k-1} \left\{ (-\cot \pi w)^j \phi_j(w, z) \right.$$
$$\left. - \int (-\cot \pi w)^j (\cos \pi w)^{k-j} (\sin \pi w)^{j-1} g(w) [\pm z]^w dw \right\}.$$

Moreover the expression $\phi_j(w, z)$ and the indefinite integral here appearing are both such that they can be evaluated along the loop $C$. Simplifying the above form and summing with respect to $j$, we have, as an integral of the entire integrand in (6), the sum

$$\frac{k}{2i} \sum_{j=1}^{k} \frac{(-1)^j}{j} C_{j-1}^{k-1} \left\{ (\cot \pi w)^j \phi_j(w, z) \right.$$
$$\left. - \int (\cos \pi w)^k \frac{g(w) [\pm z]^w}{\sin \pi w} dw \right\}. \tag{7}$$

If we now agree to let $C$ extend to infinity in a direction of either of the third or fourth quadrants, then at the infinite extremities, the function $\cot \pi w$ has the value $i$. Consequently, upon evaluating (7) along $C$ and combining properly the definite integrals which arise, we arrive at the result

$$I = \frac{k}{2i} \int_C \left\{ \sum_{j=1}^{k} \frac{(-1)^j}{j} C_{j-1}^{k-1} (a^j b^{k-j} - b^k) \right\} \frac{g(w) [\pm z]^w}{\sin \pi w} dw, \tag{8}$$

where $a$ and $b$ denote $i \sin \pi w$ and $\cos \pi w$, respectively. The sum appearing in the integrand of (8) is easily shown to be equal to $k^{-1}(b-a)^k$, or $k^{-1}e^{-k\pi w}$. Substituting this quantity for the sum, we have the integral (4).

The above theorem assumes only one singular point of $g(w)$, but the extension to the case when a finite number of singularities occurs is obvious. Moreover, if a singularity is polar in character, then the corresponding loop integral reduces to the residue of the function

$$\frac{1}{2i} \frac{g(w) [\pm z]^w}{e^{k\pi w} \sin \pi w}$$

at the pole. Furthermore, if a singularity is a negative integer, say $-n$, then the corresponding term in $z^{-n}$ in (3) is to be suppressed and in its place is to be substituted the proper loop integral.

2. **Asymptotic developments.** The theorem of Newsom, together with the extension just established, finds application in the determination of asymptotic developments for large values of $|z|$ of such integral functions as are defined by (1). It is evident that in order to determine completely the asymptotic development of a given function $f(z)$, one must first find such developments for the integral appearing in the right member of (3), and each loop integral (4) that may occur. The first of these questions is a problem about which we shall not concern ourselves here, but we shall discuss briefly a method by which the second question can often be answered.

If the singularity $w_1$, of $g(w)$, is algebraic in character, then $g(w)$ can be written in the form $g(w)=(w-w_1)^\theta\Phi(w)$, where $\Phi(w)$ is analytic at $w=w_1$, $\phi(w_1)\neq 0$, and where $\theta$ is a real constant not equal to zero or a positive integer. One of the transformations $w'=\pm(w-w_1)$ will then transform (4) into an integral of the form

$$(9)\qquad I(z,\beta)=\frac{i}{2\pi}\int_{C'}F(w)(-w)^{\beta-1}(-[\pm z])^{-w}dw$$

where $\beta$ is a constant, and where $C'$ is a loop about the origin $w=0$ (the primes having been dropped), and extending to infinity in a direction lying in either the first or the fourth quadrant. Moreover, $F(w)$ is analytic at the origin, and possesses a convergent series development of the form

$$F(w)=\sum_{n=0}^{\infty}c_n(-w)^n,$$

radius of convergence greater than 0. According to a theorem due to Barnes,[3] if $F(w)$ is bounded in the distant right half $w$-plane, and if the expressions $(-w)^{\beta-1}$ and $(-[\pm z])^{-w}$ are made precise by suitable definitions, then the loop integral $I(z, \beta)$ in (10) is developable asymptotically in the form

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[3] For a full statement and proof of the theorem of Barnes, see Ford, ***Asymptotic developments of functions defined by Maclaurin series***, Michigan Science Series vol. 11 (1936) p. 16. This book also contains a proof of the theorem of Newsom for the special case in which $k=1$, together with the extension and numerous applications. For proof of the theorem, see chap. 4; for applications, see chaps. 6 and 7. For further applications of the theorems of Ford and Newsom, and of the theorem in the present paper, see C. G. Fry and H. K. Hughes, *Asymptotic developments of certain integral functions*, Duke Math. J. vol. 9 (1942) pp. 791–802.

$$I(z, \beta) \sim \sum_{n=0}^{\infty} \frac{c_n}{[\log(-[\pm z])]^{\beta+n}\Gamma(1-\beta-n)}.$$

It thus appears that the presence of an algebraic singularity of $g(w)$ presents no serious difficulty.

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# ON SOME FORMULAS INVOLVING THE DIVISOR FUNCTION

HERBERT S. ZUCKERMAN

Viggo Brun[1] has proved the formulas

$$T_1(n) - T_2(n) + T_3(n) - \cdots = -\mu(n), \qquad n > 1, \tag{1}$$

$$h(n) = T_1(n) - (1/2)T_2(n) + (1/3)T_3(n) - \cdots$$
$$= \begin{cases} 0 \text{ if } n \text{ is not a prime power,} \\ 1/t \text{ if } n = p^t,\ p \text{ a prime;} \end{cases} \tag{2}$$

where $T_l(n)$ is the number of ways that $n$ can be expressed as a product of $l$ factors, each greater than 1. He obtains them as special cases of combinatorial theorems. Pavel Kuhn[2] has also given proofs but it seems that no one has attempted to give elementary number theory proofs of these formulas. It is the purpose of this note to give such proofs and to point out a few other formulas similar to (1) and (2).

All the formulas which we shall prove can be proved very concisely by using the generating function

$$\sum_{n=1}^{\infty} T_l(n)n^{-s} = \{\zeta(s) - 1\}^l,$$

and some simple properties of the zeta-function.[3] Our number theory

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[1] Netto, *Lehrbuch der Combinatorik*, 2d edition, 1927, chap. 14, especially pp. 276–277.

[2] *Det Kongelige Norske Videnskabers Selskab*, Forhandlinger, 1939.

[3] Interchanging the order of summation we have $\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{l-1}T_l(n)n^{-s} = \sum_{l=1}^{\infty}(-1)^{l-1}\{\zeta(s)-1\}^l = -\zeta(s)^{-1} = -\sum_{n=1}^{\infty}\mu(n)n^{-s}$, and (1) is obtained by comparing coefficients of $n^{-s}$ in the two members. Similarly, (2) follows from $\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{l-1}l^{-1}T_l(n)n^{-s} = \sum_{l=1}^{\infty}(-1)^{l-1}l^{-1}\{\zeta(s)-1\}^l = \log\zeta(s) = \sum_p \log(1-p^{-s})^{-1} = \sum_p\sum_{t=1}^{\infty}t^{-1}p^{-ts}$.

proofs are the algebraic translations of these proofs.

In order to avoid certain special cases it will be convenient to define

$$T_0(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{3}$$

The function $T_1(n)$ has already been defined as the number of ways $n$ can be expressed as a single factor greater than 1, so we have

$$T_1(n) = \begin{cases} 0 & \text{if } n = 1, \\ 1 & \text{if } n > 1. \end{cases} \tag{4}$$

In writing an expression such as the left member of (1) it will be convenient to write it as $\sum_{l=1}^{\infty}(-1)^{l-1}T_l(n)$ but it should be noted that all the terms in the series are zero after a certain point. This will be true of all the series which we write in this way, so we are really dealing only with finite sums.

To prove (1) we first note that from the definition of $T_l(n)$ we have

$$T_{l+1}(n) = \sum_{d\mid n,\, d\neq n} T_l(d) = \sum_{d\mid n} T_l(d) - T_l(n), \tag{5}$$

for $l \geqq 0$. We let $f(n) = \sum_{l=0}^{\infty}(-1)^l T_l(n)$ and have, by (5),

$$\sum_{d\mid n} f(d) = \sum_{l=0}^{\infty}(-1)^l \sum_{d\mid n} T_l(d) = \sum_{l=0}^{\infty}(-1)^l\{T_l(n) + T_{l+1}(n)\} = T_0(n).$$

Then by the Möbius inversion formula we have

$$f(n) = \sum_{l=0}^{\infty}(-1)^l T_l(n) = \sum_{d\mid n} T_0(n/d)\mu(d) = \mu(n), \tag{6}$$

where we have used (3). This is equivalent to (1).

This proof has an obvious extension. Putting $g(n) = \sum_{l=1}^{\infty}(-1)^l \cdot (l+1)T_l(n)$ we have, using (5),

$$\begin{aligned} \sum_{d\mid n} g(d) &= \sum_{l=0}^{\infty}(-1)^l(l+1)\sum_{d\mid n} T_l(d) \\ &= \sum_{l=0}^{\infty}(-1)^l(l+1)\{T_l(n) + T_{l+1}(n)\} \\ &= \sum_{l=0}^{\infty}(-1)^l T_l(n) = f(n) = \mu(n). \end{aligned}$$

Inverting this we obtain

$$g(n) = \sum_{l=0}^{\infty} (-1)^l (l+1) T_l(n) = \sum_{d \mid n} \mu(d)\mu(n/d)$$

$$(7) \qquad = \begin{cases} 1 \text{ if } n = 1, \\ (-1)^s 2^s \text{ if } n = p_1 \cdots p_s q_1^2 \cdots q_t^2, \\ 0 \text{ if } n \text{ is divisible by a cube.} \end{cases}$$

In the same way we can prove

$$\sum_{l=0}^{\infty} (-1)^l \frac{(l+1)(l+2)}{2} T_l(n) = \sum_{d\delta \mid n} \mu(d)\mu(\delta)\mu(n/d\delta)$$

$$(8) \qquad = \begin{cases} 1 \text{ if } n = 1, \\ (-1)^{s+u} 3^{s+t} \text{ if } n = p_1 \cdots p_s q_1^2 \cdots q_t^2 r_1^3 \cdots r_u^3, \\ 0 \text{ if } n \text{ is divisible by a fourth power.} \end{cases}$$

To prove (2) we need another expression for $T_l(n)$. From the definitions we have

$$(9) \qquad T_l(n) = \sum_{r_1 \cdots r_l = n;\, r_i > 1} 1 = \sum_{r_1 \cdots r_l = n} T_1(r_1) \cdots T_1(r_l),$$

for $l \geq 1$. Multiplying this by $\log n$, rearranging, and using (9), we have

$$\begin{aligned} T_l(n) \log n &= \sum_{r_1 \cdots r_l = n} T_1(r_1) \cdots T_1(r_l)(\log r_1 + \cdots + \log r_l) \\ &= l \sum_{r_1 \cdots r_l = n} T_1(r_1) \cdots T_1(r_l) \log r_l \\ &= l \sum_{r_l \mid n} T_1(r_l) \log r_l \sum_{r_1 \cdots r_{l-1} = n/r_l} T_1(r_1) \cdots T_1(r_{l-1}) \\ &= l \sum_{r_l \mid n} T_1(r_l) \log r_l \cdot T_{l-1}(n/r_l) \\ &= l \sum_{d \mid n} T_{l-1}(d) T_1(n/d) \log (n/d) \\ &= l \log n \sum_{d \mid n} T_{l-1}(d) T_1(n/d) - l \sum_{d \mid n} T_{l-1}(d) T_1(n/d) \log d. \end{aligned}$$

Reducing this by (5) and (4) we have

$$T_l(n) \log n = l T_l(n) \log n - l \sum_{d \mid n} T_{l-1}(d) T_1(n/d) \log d,$$

which we can write in the form

$$(10) \qquad (1/(l-1)) \sum_{d \mid n} T_{l-1}(d) T_1(n/d) \log d = (1/l) T_l(n) \log n.$$

If we now let

$$h(n) = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} T_l(n), \tag{11}$$

we have, by (4) and (10),

$$\begin{aligned}\sum_{d|n} h(d) \log d &= \sum_{l=1}^{\infty} ((-1)^{l-1}/l) \left\{ \sum_{d|n} T_l(d) T_1(n/d) \log d + T_l(n) \log n \right\} \\ &= \sum_{l=1}^{\infty} (-1)^{l-1} \{ (1/l) T_l(n) \log n + (1/(l+1)) T_{l+1}(n) \log n \} \\ &= T_1(n) \log n.\end{aligned}$$

Inverting this we obtain

$$\begin{aligned}h(n) \log n &= \sum_{d|n} T_1(n/d) \mu(d) \log n/d \\ &= \log n \sum_{d|n} T_1(n/d)\mu(d) - \sum_{d|n} T_1(n/d)\mu(d) \log d \\ &= \log n \left\{ \sum_{d|n} \mu(d) - \mu(n) \right\} \\ &\qquad - \left\{ \sum_{d|n} \mu(d) \log d - \mu(n) \log n \right\} \\ &= - \sum_{d|n} \mu(d) \log d,\end{aligned}$$

where we have used (4) and the fact that $\sum_{d|n}\mu(d) = 0$ for $n > 1$. Now it is easily verified that $-\sum_{d|n}\mu(d) \log d$ takes the value $\log p$ if $n = p^t$ and the value 0 if $n$ is not a prime power.[4] Hence we have

$$h(n) = \begin{cases} 1/t \text{ if } n = p^t, \\ 0 \text{ if } n \text{ is not a prime power,} \end{cases} \tag{12}$$

which is formula (2).

We can combine (6) and (12) to obtain another formula. We first note that, from (9), we have

$$\begin{aligned}T_l(n) &= \sum_{d|n} \left\{ \sum_{r_1 \cdots r_\mu = d} T_1(r_1) \cdots T_1(r_\mu) \sum_{r_{\mu+1} \cdots r_l = n/d} T_1(r_{\mu+1}) \cdots T_1(r_l) \right\} \\ &= \sum_{d|n} T_\mu(d) T_{l-\mu}(n/d), \qquad 0 \leqq \mu \leqq l.\end{aligned} \tag{13}$$

---

[4] If $n = p^r m$ and $p$ does not divide $m$ we have $-\sum_{d|n}\mu(d) \log d = -\sum_{\delta|m}\sum_{\gamma|p^r} \mu(\delta\gamma) \log (\delta\gamma) = -\sum_{\delta|m}\{\mu(\delta) \log \delta + \mu(\delta p) \log (\delta p)\} = -\sum_{\delta|m}\{\mu(\delta) \log \delta - \mu(\delta) \log (\delta p)\} = \log p \sum_{\delta|m}\mu(\delta)$ and this has the value 0 if $m > 1$, $\log p$ if $m = 1$.

We let

$$
\begin{aligned}
j(n) &= \sum_{l=1}^{\infty}\sum_{\mu=1}^{l}(-1)^{l-1}(1/\mu)T_l(n) \\
&= T_1(n) - (3/2)T_2(n) + (11/6)T_3(n) - (25/12)T_4(n) \\
&\quad + (137/60)T_5(n) - \cdots ,
\end{aligned} \tag{14}
$$

and have, by (13),

$$
\begin{aligned}
j(n) &= \sum_{l=1}^{\infty}\sum_{\mu=1}^{l}(-1)^{l-1}(1/\mu)\sum_{d|n}T_\mu(d)T_{l-\mu}(n/d) \\
&= \sum_{d|n}\sum_{l=1}^{\infty}\sum_{\mu+\nu=l;\,\mu\geq 1}(-1)^{\mu-1}(1/\mu)T_\mu(d)(-1)^\nu T_\nu(n/d) \\
&= \sum_{d|n}\left\{\sum_{\mu=1}^{\infty}(-1)^{\mu-1}(1/\mu)T_\mu(d)\sum_{\nu=0}^{\infty}(-1)^\nu T_\nu(n/d)\right\} \\
&= \sum_{d|n}h(d)f(n/d),
\end{aligned}
$$

where we have used (11) and (6). Then by (6) and (12) we have

$$
\begin{aligned}
j(n) &= \sum_{p^t|n}(1/t)\mu(n/p^t) \\
&= \begin{cases} 0 \text{ if } n \text{ is divisible by the squares of two distinct primes,} \\ (-1)^{s+1}/t(t-1) \text{ if } n = p^t q_1\cdots q_s,\ t>1, \\ (-1)^{s-1}s \text{ if } n = q_1\cdots q_s. \end{cases}
\end{aligned} \tag{15}
$$

In quite a similar way we can prove

$$
\begin{aligned}
k(n) &= \sum_{l=2}^{\infty}(-1)^l\sum_{\nu+\mu=l;\,\nu,\mu\geq 1}(1/\nu\mu)T_l(n) \\
&= T_2(n) - T_3(n) + (11/2)T_4(n) - (5/6)T_5(n) \\
&\quad + (137/180)T_6(n) - \cdots \\
&= \begin{cases} 0 \text{ if } n \text{ has more than two distinct prime factors,} \\ 2/tu \text{ if } n = p^t q^u, \\ \sum_{\nu=1}^{t-1}(1/\nu(t-\nu)) \text{ if } n = p^t. \end{cases}
\end{aligned} \tag{16}
$$

Brun notes that $\sum_{n=1}^{m}T_1(n)=\{m\}$, $\sum_{n=1}^{m}T_2(n)=\sum_{\mu=2}^{m}\{m/\mu\}$, $\sum_{n=1}^{m}T_3(n)=\sum_{\mu,\nu=2}^{m}\{m/\mu\nu\}$, and so on, where $\{x\}$ denotes the number of integers not less than 2 and not greater than $x$. Summing (2) he then has

$$H(m) = \sum_{n=1}^{m} h(n) = \{m\} - (1/2)\sum_{\mu=2}^{\infty}\{m/\mu\} + (1/3)\sum_{\mu,\nu=2}^{\infty}\{m/\mu\nu\} - \cdots . \tag{17}$$

This formula can be used to determine $h(n) = H(n) - H(n-1)$ and hence, by (2), it gives a method (although impractical) of determining whether $n$ is a prime power. A similar thing can be done with each of our formulas. For example from (14) we have

$$J(m) = \sum_{n=1}^{m} j(n) = \{m\} - (3/2)\sum_{\mu=2}^{\infty}\{m/\mu\} + (11/6)\sum_{\mu,\nu=2}^{\infty}\{m/\mu\nu\} - \cdots .$$

Then $j(n) = J(n) - J(n-1)$, with the aid of (15), determines the number of prime factors of $n$ if $n$ is not divisible by the squares of more than one prime. As a numerical example consider $n = 6$. We have

$$J(6) = \{6\} - (3/2)[\{3\} + \{2\}] = 1/2,$$

$$J(5) = \{5\} - (3/2)\{5/2\} = 5/2,$$

$$j(6) = (1/2) - (5/2) = -2,$$

and hence 6 has two distinct prime factors, each entering to the first power.

Brun also points out that from (2) we have

$$H(m) = \sum_{n=1}^{m} h(n) = \sum_{t=1}^{\infty}(1/t)\pi(m^{1/t}).$$

The other formulas do not lead to such simple results but (16) gives

$$K(m) = \sum_{n=1}^{m} k(n) = \sum_{\mu=1}^{\infty}\sum_{p^{\nu}}(1/\mu\nu)\pi((m/p^{\nu})^{1/\mu}).$$

If (10) is put into exponential form it can be stated in a more interesting way. From (10) we have

$$[n^{T_l(n)}]^{1/l} = \left[\prod_{d\mid n;\, d\neq n} d^{T_{l-1}(d)}\right]^{1/(l-1)}.$$

The left member can be obtained by writing $n$ in all possible ways as a product of $l$ factors greater than 1, multiplying them together, and taking the $l$th root. The right member can be obtained by writing all proper divisors of $n$ in all possible ways as products of $l-1$ factors greater than 1, multiplying them all together, and taking the $(l-1)$th root. Our result is that these two numbers are equal. Thus for $n=24$, $l=3$ we have

$$[(2\cdot 2\cdot 6)(2\cdot 6\cdot 2)(6\cdot 2\cdot 2)(2\cdot 3\cdot 4)(2\cdot 4\cdot 3)(3\cdot 2\cdot 4)(3\cdot 4\cdot 2)(4\cdot 2\cdot 3)(4\cdot 3\cdot 2)]^{1/3} = 2^9\cdot 3^3,$$

$$[(2\cdot 2)\times(2\cdot 3)(3\cdot 2)\times(2\cdot 4)(4\cdot 2)\times(4\cdot 3)(3\cdot 4)\times(6\cdot 2)(2\cdot 6)]^{1/2}=2^9\cdot 3^3.$$

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# ON THE FOURIER DEVELOPMENTS OF A CERTAIN CLASS OF THETA QUOTIENTS

M. A. BASOCO

1. **Introduction.** In this paper we shall be concerned with the functions $\phi_\alpha^k(z)$ defined by the relation

$$(1) \qquad \phi_\alpha^k(z) = \left\{\frac{d}{dz}\log\vartheta_\alpha(z, q)\right\}^k = \left\{\frac{\vartheta_\alpha'(z, q)}{\vartheta_\alpha(z, q)}\right\}^k, \qquad \alpha = 0, 1, 2, 3,$$

where $\vartheta_\alpha(z, q)$ is a Jacobi theta function and $k$ is a positive integer. In the first place, we shall derive the Fourier developments which represent these functions in a certain strip of the complex plane; it will be seen that the Fourier coefficients of $\phi_\alpha^k(z)$ depend on those of $\phi_\alpha^s(z)$, $s=1, 2, 3, \cdots, k-1$, through a recurrence relation of order $k$. Secondly, these developments, in conjunction with certain obvious identities, yield, through the method of paraphrase, some general arithmetical formulae of a type first given by Liouville.[1] Indeed, we recover, in a simple manner, some results given without proof by Liouville, which were later proved by Bell[2] through the use of somewhat complex identities involving a certain set of doubly periodic functions of the second kind. One of these results has recently been proved in a strictly elementary, but very ingenious way, by Uspensky.[3] Finally, we indicate some applications of these formulae to the derivation of a certain type of arithmetic and algebraic identities.

2. **The functions $\phi_\alpha^k(z)$.** It should be pointed out that the case $k=1$, is implicit in §§47 and 48 of Jacobi's *Fundamenta nova*.[4] Likewise, the case $k=2$, has been obtained by G. D. Nichols[5] through the use of certain results due to the present writer.[6] The following is a direct derivation of the necessary procedure for the general case; it depends on a straightforward application of contour integration and the theory of residues. It is convenient to treat the two functions $\phi_0^k(z)$ and

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[1] J. Math. Pures Appl. (2) vol. 3 (1858) et seq. See, for example, vol. 3 p. 247 (H).

[2] Trans. Amer. Math. Soc. vol. 22 (1921) p. 215 formula (xiv').

[3] J. V. Uspensky and M. A. Heaslet, *Elementary number theory*, New York, 1939. See chap. 13 p. 462 formula (R).

[4] Jacobi, *Gesammelte Werke*, vol. 1 p. 187.

[5] G. D. Nichols, Tôhoku Math. J. vol. 40 (1935) pp. 252–258.

[6] M. A. Basoco, Bull. Amer. Math. Soc. vol. 38 (1932) pp. 560–568.

$\phi_1^k(z)$ separately; from the results for these functions we may obtain the corresponding ones for $\alpha = 2, 3$ by merely replacing $z$ by $z+\pi/2$.

It follows from the properties of the theta functions[7] and Fourier's theorem for analytic functions that

$$(2) \qquad \phi_0^k(z) = \sum_{n=-\infty}^{\infty} A_n^{(k)} e^{-2niz}, \qquad -\Im(\pi\tau/2) < \Im(z) < \Im(\pi\tau/2),$$

where,

$$(3) \qquad A_n^{(k)} = (1/\pi)\int_{-\pi/2}^{\pi/2} \phi_0^k(z) e^{2niz}\,dz.$$

To evaluate this integral, consider the contour integral $\int_C \phi_0^k(z)e^{2niz}dz$, the contour $C$ being the boundary of the parallelogram with vertices at $z = \pm\pi/2$ and $z = \pm\pi/2+\pi\tau/2$. Cauchy's theorem applied to this integral leads without difficulty to the recurrence relation

$$(4) \qquad A_n^{(k)} - q^{2n}\sum_{j=0}^{k-1} C_{k,j}(-2i)^j A_n^{(k-j)} = 2iR, \qquad q = \exp \pi i\tau,$$

where $C_{k,j}$ is the binomial coefficient $k!/j!(k-j)!$ and $R$ is the residue of the integrand at the pole of order $k$ with affix $z = \pi\tau/2$.

We thus obtain, on placing $k = 1, 2, 3$ and computing the necessary residues, the following expressions for the coefficients in the expansions of $\phi_0^k(z)$, for the values of $k$ indicated:

$$\text{(a)} \quad A_0^{(1)} = 0;\ A_n^{(1)} = -A_{-n}^{(1)} = \frac{2iq^n}{1-q^{2n}}, \qquad n \neq 0,$$

$$\text{(b)} \quad A_n^{(2)} = A_{-n}^{(2)} = \frac{8q^2}{(1-q^{2n})^2} - \frac{4(n-1)}{1-q^{2n}},\ n \neq 0;\ A_0 = -2\sum_{n=1}^{\infty} A_n^{(2)}$$

$$\text{(c)} \quad A_0^{(3)} = 0;\ A_n^{(3)} = -A_{-n}^{(3)} = \frac{\vartheta_1'''}{\vartheta_1'}\cdot\frac{2iq^n}{1-q^{2n}}$$
$$- \frac{2iq^n}{(1-q^{2n})^3}\left\{(2n^2+6n+3)q^{4n}\right.$$
$$\left. + (18-4n^2)q^{2n} + (2n^2-6n+3)\right\}.$$

With respect to the function $\phi_1^k(z)$ the process used in the preceding must be modified slightly on account of the presence of a singularity (pole of order $k$) at the origin. Define the function $\psi_1^{(k)}(z)$ to be of the form

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[7] See Whittaker and Watson, *Modern analysis*, chap. 21.

(5) $$\psi_1^{(k)}(z) \equiv \phi_1^k(z) - T^{(k)}(z),$$

where $T^{(k)}(z)$ is a suitable function of sin $z$ and cos $z$, having the period $\pi$ and such that the principal part of its Laurent expansion about the origin coincides with that of $\phi_1^k(z)$. The function $\psi_1^{(k)}(z)$ is, therefore, analytic along the real axis. It is found that, for the cases considered (that is, for $k=1, 2, 3$), the functions $T^{(k)}(z)$ are of the form:

$T^{(1)}(z) = \cot z$; $T^{(2)}(z) = \cot^2 z$; $T^{(3)}(z) = (1+\vartheta_1'''/\vartheta_1')\cot z + \cot^3 z$.

We may now write

$$\psi_1^{(k)}(z) = \sum_{n=1}^{\infty} B_n^{(k)}(e^{2niz} - e^{-2niz}), \text{ if } k \text{ is odd},$$

$$\psi_1^{(k)}(z) = B_0^{(k)} + \sum_{n=1}^{\infty} B_n^{(k)}(e^{2niz} + e^{-2niz}), \text{ if } k \text{ is even}.$$

These expansions are valid in the strip defined by the inequalities $-\Im(\pi\tau) < \Im(z) < \Im(\pi\tau)$. In either case, we have,

(6) $$B_n^{(k)} = (1/\pi)\int_{-\pi/2}^{\pi/2} \psi_1^{(k)}(z)e^{-2niz}\,dz, \qquad n \neq 0.$$

The value of $B_0^{(k)}$ can be obtained from the relation

(7) $$B_0^{(k)} = -2\sum_{n=1}^{\infty} B_n^{(k)} + \lim_{z\to 0}\psi_1^{(k)}(z).$$

To calculate the integral in (6), consider the contour integral $\int_C \psi_1^{(k)}(z)e^{-2niz}dz$ where $C$ is the boundary of the parallelogram with vertices at $z=\pm\pi/2$ and $z=\pm\pi/2-(2s+1)\pi\tau/2$, where $s$ is an arbitrary positive integer. This contour contains precisely $s$ poles of the function $\psi_1^{(k)}(z)$, these being located at the points with affixes $z=-r\pi\tau$, $r=1, 2, 3, \cdots, s$. On account of the properties of the functions involved we may pass to the limit as $s$ tends to infinity, thus obtaining

$$B_n^{(k)} = -2i\sum_{s=1}^{\infty} R_s$$

where $R_s$ is the residue of $\psi^{(k)}(z)e^{-2niz}$ at the poles of order $k$, $z=-s\pi\tau$.

It is found in this manner that

(d) $$B_0^{(1)} = 0;\ B_n^{(1)} = \frac{-2iq^{2n}}{1-q^{2n}}, \qquad n \neq 0.$$

$$\text{(e)}\quad B_0^{(2)} = 8\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}};\ B_n^{(2)} = \frac{8q^{2n}}{(1-q^{2n})^2} - \frac{4nq^{2n}}{1-q^{2n}}, \qquad n \neq 0.$$

$$\text{(f)}\quad B_0^{(3)} = 0;\ B_n^{(3)} = \frac{\vartheta_1'''}{\vartheta_1'}\cdot\frac{4q^{2n}}{1-q^{2n}}$$

$$- \frac{8q^{2n}}{(1-q^{2n})^3}\left\{n^2q^{4n} - (2n^2-6n-6)q^{2n} + (n^2-6n+6)\right\}, \qquad n \neq 0.$$

On substituting the values of the coefficients given in (a), $\cdots$, (f), into (2) and (5) and transforming the resulting series into an arithmetical form, we find the following:

$$\text{(I)}\quad \phi_0(z) = 4\sum_{(n)} q^n\left\{\sum \sin 2tz\right\}, \qquad n = 1, 2, 3, 4, \cdots,$$

$$\text{(II)}\quad \phi_1(z) = \cot z + 4\sum_{(n)} q^{2n}\left\{\sum \sin 2dz\right\},$$

$$\text{(III)}\quad \phi_2(z) = -\tan z + 4\sum_{(n)} q^{2n}\left\{\sum (-1)^d \sin 2dz\right\},$$

$$\text{(IV)}\quad \phi_3(z) = 4\sum_{(n)} (-1)^n q^n\left\{\sum \sin 2tz\right\},$$

$$\text{(V)}\quad \phi_0^2(z) = 8\sum_{(n)} q^n\sigma(n) + 8\sum_{(n)} q^n\left\{\sum (\tau - t)\cos 2tz\right\},$$

$$\text{(VI)}\quad \phi_1^2(z) = \cot^2 z + 8\sum_{(n)} q^{2n}\zeta_1(n) + 8\sum_{(n)} q^{2n}\left\{\sum (2\delta - d)\cos 2dz\right\},$$

$$\text{(VII)}\quad \begin{aligned}\phi_2^2(z) &= \tan^2 z + 8\sum_{(n)} q^{2n}\zeta_1(n)\\ &\quad - 8\sum_{(n)} q^{2n}\left\{\sum (-1)^d(2\delta - d)\cos 2dz\right\},\end{aligned}$$

$$\text{(VIII)}\quad \phi_3^2(z) = 8\sum_{(n)} (-1)^n q^n\sigma(n) + 8\sum_{(n)} (-1)^n q^n\left\{\sum (\tau - t)\cos 2tz\right\},$$

$$\text{(IX)}\quad \phi_0^3(z) = (\vartheta_1'''/\vartheta_1')\phi_0(z) - 4\sum_{(n)} q^n\left\{\sum (3\tau^2 - 6t\tau + 2t^2)\sin 2tz\right\},$$

$$\text{(X)}\quad \begin{aligned}\phi_1^3(z) &= \cot z + \cot^3 z + (\vartheta_1'''/\vartheta_1')\phi_1(z)\\ &\quad - 8\sum_{(n)} q^{2n}\left\{\sum (d^2 - 6d\delta + 6\delta^2)\sin 2dz\right\},\end{aligned}$$

$$\text{(XI)}\quad \begin{aligned}\phi_2^3(z) &= -\tan z - \tan^3 z + (\vartheta_1'''/\vartheta_1')\phi_2(z)\\ &\quad - 8\sum_{(n)} q^{2n}\left\{\sum (-1)^d(d^2 - 6d\delta + 6\delta^2)\sin 2dz\right\},\end{aligned}$$

$$\begin{aligned}\phi_3^2(z) &= (\vartheta_1'{}'/\vartheta_1')\phi_3(z)\\ &\quad - 4\sum_{(n)}(-1)^n q^n\left\{\sum(3\tau^2 - 6t\tau + 2t^2)\sin 2tz\right\},\end{aligned} \tag{XII}$$

$$(\vartheta_1'''/\vartheta_1') = -1 + 24\sum_{(n)} q^{2n}\zeta_1(n).$$

In these expansions, the notation is as usual: The inner sigma refers to the divisors $d$, $\delta$, $t$, $\tau$ of the positive integer $n$, $\tau$ being odd, and

$$\sigma(n) = \psi_1(n) - \zeta_1'(n)$$

where,

$\psi_1(n)$ = sum of the divisors of $n$ whose conjugates are odd,
$\zeta_1'(n)$ = sum of the odd divisors of $n$, and
$\zeta_1(n)$ = sum of all the divisors of $n$.

3. **Paraphrases.** The obvious identities

$$\phi_a(z)\cdot\phi_a(z) = \phi_a^2(z), \qquad a = 0, 1, 2, 3, \tag{8}$$

in conjunction with the expansions (I) to (VIII) of the preceding section yield, through the method of paraphrase,[8] the following arithmetical theorems:

THEOREM ($\alpha$). *Let $F(x)$ be an even, single-valued function which is well defined for all integral values of the argument including zero, but is otherwise arbitrary; let $n$ be any positive integer. Then,*

$$\sum_{(k)}\{F(t' - t'') - F(t' + t'')\} = \sum_{(l)}(\tau - t)\{F(t) - F(0)\},$$

*where* ($k$) *and* ($l$) *refer to the following partitions of $n$ in positive integers:*

$$(k) \quad n = t'\tau' + t''\tau'', \qquad (l) \quad n = t\tau, \qquad \tau, \tau', \tau'' \textit{ odd}.$$

This is the arithmetical equivalent of (8) with $a = 0$. It is of interest to note that a strictly elementary proof of this theorem has recently been given by Uspensky.[9]

THEOREM ($\beta$). *Let $F(x)$ and the integer $n$ be as in Theorem* ($\alpha$). *Then,*

$$\begin{aligned}\sum_{(i)}\{F(d' - d'') - F(d' + d'')\}\\ = \{\zeta_1(n) - \zeta_0(n)\}F(0) + \sum_{(j)}(2\delta - d - 1)F(d)\\ - 2\sum_{(f)}\{F(1) + F(2) + F(3) + \cdots + F(d - 1)\}\end{aligned}$$

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[8] E. T. Bell, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 1–30; 198–219.

*where (i), (j) refer to the following partitions of $n$ in positive integers:*

$$(i)\quad n = d'\delta' + d''\delta'', \qquad (j)\quad n = d\delta.$$

*$\zeta_0(n)$ represents the number of positive integral divisors of $n$, while $\zeta_1(n)$ represents their sum.*

This theorem was first given by Liouville,[9] who stated it without giving a proof; it is conjectured that he obtained this as well as other similar results through the aid of the theory of elliptic functions. At any rate, this theorem was later rediscovered by Bell (loc. cit.) as a special case of a more general result derived from one of the addition theorems of the theta functions and involving a certain set of doubly periodic functions of the second kind. We have obtained this result as the equivalent of the identity (8) with $a=1$.

THEOREM ($\gamma$). *Let $F(x)$, $n$, and the partitions (i) and (j) be as in the preceding theorem. Then,*

$$\sum_{(i)} (-1)^{d'+d''}\{F(d'-d'') - F(d'+d'')\}$$
$$= \{\zeta_1(n) - \zeta_0(n)\}F(0) - \sum_{(j)} (-1)^d(2\delta - d + 1)F(d)$$
$$+ 2\sum_{(j)} \{F(1) - F(2) + F(3) - \cdots + (-1)^d F(d-1)\}.$$

This theorem is the arithmetical equivalent of identity (8) with $a=2$. It is, however, not essentially distinct from Theorem ($\beta$); for either theorem may be obtained from the other by replacing $F(x)$ by $(-1)^x F(x)$. Finally it should be noted that the case $a=3$ merely yields the result stated in Theorem ($\alpha$).

As might be expected, those theorems analogous to the preceding which may be deduced from identities involving the functions $\phi_a^3(x)$ are considerably more complicated; it will perhaps suffice, in the present instance, to state merely the following two results by way of samples.

THEOREM ($\delta$). *Let $G(x)$ be an odd, single-valued function, which is well defined for all integral values of the argument, but is otherwise arbitrary; let $n$ be any positive integer. Then,*

$$4\sum_{(i)} \{G(t_1 - t_2 + t_3) + G(t_1 + t_2 - t_3) + G(t_2 + t_3 - t_1) - G(t_1 + t_2 + t_3)\}$$
$$= 24\sum_{(j)} \{\zeta_1(n_1)\sum G(t_2)\} - \sum_{(h)} (3\tau^2 - 6t\tau + 2t^2 + 1)G(t),$$

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[9] Liouville, loc. cit.

*where* (*i*), (*j*) *and* (*k*) *refer to the following partitions of* $n$ *in positive integers*:

$$(i)\quad n = t_1\tau_1 + t_2\tau_2 + t_3\tau_3 \qquad \tau_1,\ \tau_2,\ \tau_3 \text{ all odd},$$

$$(j)\quad n = 2n_1 + n_2, \qquad n_2 = t_2\tau_2, \qquad \tau_2 \text{ odd},$$

$$(k)\quad n = t\tau, \qquad \tau \text{ odd}.$$

This theorem is the arithmetical equivalent of the otherwise apparently trivial identity:

$$\phi_0(z)\cdot\phi_0(z)\cdot\phi_0(z) \equiv \phi_0^3(z).$$

THEOREM ($\epsilon$). *Let* $F(x)$, $n$, *and the partitions* (*i*) *and* (*j*) *be as in Theorem* ($\beta$). *Then*,

$$\begin{aligned}6\sum_{(i)} d'(d'' - 2\delta'')\{F(d' + d'') + F(d' - d'')\} \\ = 6(n\zeta_0(n) - \zeta_2(n))F(0) + 12\sum_{j}(\delta - d)\left\{\sum_{r=1}^{d}\sum_{s=1}^{r-1}F(s)\right\} \\ + \sum_{(j)}(d^3 - 6d^2\delta + 6d\delta^2 - d)F(d).\end{aligned}$$

This theorem results on paraphrasing the identity:

$$\frac{d}{dz}\phi_1^3(z) \equiv 3\phi_1^2(z)\frac{d}{dz}\phi_1(z).$$

4. **An application.** If in Theorem ($\alpha$) we place $F(x) = x^2$, the following may be deduced

$$(9)\qquad 4\sum_{k=1}^{n-1}\psi_1(k)\psi_1(n - k) = \psi_3(n) - n\psi_1(n),$$

where $\psi_r(n)$ = sum of the $r$th powers of the divisors of $n$ whose conjugates are odd. In particular, if $n$ is a prime $p$, we obtain

$$4\sum_{k=1}^{p-1}\psi_1(k)\psi_1(p - k) = (p - 1)(p^2 - 1).$$

It also follows from (9) and the easily verified relationships:

$$\sum_{n=1}^{\infty}\frac{nx^n}{1 - x^{2n}} = \sum_{n=1}^{\infty}\psi_1(n)x^n, \quad \sum_{n=1}^{\infty}\frac{n^3x^n}{1 - x^{2n}} = \sum_{n=1}^{\infty}\psi_3(n)x^n,$$

$$\sum_{n=1}^{\infty}\frac{n^2(1 + x^{2n})x^n}{(1 - x^{2n})^2} = \sum_{n=1}^{\infty}n\psi_1(n)x^n,$$

that,

$$4\left\{\sum_{n=1}^{\infty}\frac{nx^n}{1-x^{2n}}\right\}^2=\sum_{n=1}^{\infty}\frac{n^3x^n}{1-x^{2n}}-\sum_{n=1}^{\infty}\frac{n^2x^n}{(1-x^{2n})^2}-\sum_{n=1}^{\infty}\frac{n^2x^{3n}}{(1-x^{2n})^2}. \tag{10}$$

Again, if in Theorem ($\beta$) we place $F(x)=x^2$, we deduce

$$12\sum_{k=1}^{n-1}\zeta_1(k)\zeta_1(n-k)=5\zeta_3(n)-(6n-1)\zeta_1(n), \tag{11}$$

which in turn implies the following development,

$$12\left\{\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}\right\}^2=\sum_{n=1}^{\infty}(5\zeta_3(n)-(6n-1)\zeta_1(n))x^n. \tag{12}$$

Finally, in Theorem ($\delta$) place $G(x)=x$ to deduce

$$24\sum\zeta_1(n_1)\psi_1(n_2)=2\psi_3(n)-(6n-1)\psi_1(n)+3n\zeta_1'(n), \qquad n=2n_1+n_2,\ n_1,\ n_2>0. \tag{13}$$

This result may be interpreted so as to yield the following relation:

$$24\left\{\sum_{n=1}^{\infty}\frac{nx^{2n}}{1-x^{2n}}\right\}\left\{\sum_{n=1}^{\infty}\frac{nx^n}{1-x^{2n}}\right\}=2\sum_{n=1}^{\infty}\frac{n^3x^n}{1-x^{2n}}+\sum_{n=1}^{\infty}\frac{nx^n}{1-x^{2n}}-6\sum_{n=1}^{\infty}\frac{n^2(1+x^{2n})x^n}{(1-x^{2n})^2}+3\sum_{n=1}^{\infty}\frac{(2n-1)^2x^{2n-1}}{(1-x^{2n-1})^2}. \tag{14}$$

Results analogous to the preceding, but developed from an entirely different point of view, have been obtained by Glaisher.[10] Here, such results appear as very special consequences of the general theorems stated in §3. It is clear, of course, that similar results may be obtained through the full use of the expansions here given.

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[10] J. W. L. Glaisher, Messenger of Mathematics nos. 166, 169 (1885).

# THE BETTI GROUPS OF THE PRODUCT OF TWO NORMAL SPACES

C. E. CLARK

1. **Introduction.** Let $R_1$ and $R_2$ be homeomorphic to open sets of normal spaces. Furthermore, let each of $R_1$ and $R_2$ contain infinitely many points. For these spaces Alexandroff has defined inner Betti groups.[1] In this paper the inner Betti groups of the topological product $R_1 \otimes R_2$ are studied.

Let $\mathfrak{u}^r \in B^r(R_1)$, where $B^r$ denotes the $r$-dimensional inner cohomology group with the integers as coefficient domain, and let $\mathfrak{u}^\rho \in B^\rho(R_2)$. To these two elements there corresponds an element $\mathfrak{u}^r \times \mathfrak{u}^\rho \in B^{r+\rho}(R_1 \times R_2)$. All such elements with $r+\rho=n$ generate a subgroup $B_1^n(R_1 \times R_2) \subset B^n(R_1 \times R_2)$. We characterize this subgroup (Theorem 3). In addition, we characterize the factor-group $B_2^n(R_1 \times R_2) = B^n(R_1 \times R_2)/B_1^n(R_1 \times R_2)$ (Theorem 4). In doing so we show that if $\mathfrak{u}^s \in B^s(R_1)$ is of order $\mathfrak{e}^s \neq 0$, and if $\mathfrak{u}^\sigma \in B^\sigma(R_2)$ is of order $\mathfrak{e}^\sigma \neq 0$, then to these two elements there corresponds an element $(\mathfrak{u}^s, \mathfrak{u}^\sigma) \in B_2^{s+\sigma-1}(R_1 \times R_2)$.

To prove these results we employ Alexandroff's second definition of the inner Betti groups which uses barycentric subdivisions of coverings.[2] Furthermore, Freudenthal's simplicial division of the product of two simplexes is used.[3]

2. **The groups** $B_i^n(K^s \times K^\sigma \bmod C^{s\sigma})$, $i=1, 2$. In this section we state without proof some facts about products of complexes which are consequences of [3]. Let $K^s$ and $K^\sigma$ be finite complexes with subcomplexes $C^s$ and $C^\sigma$, respectively. Let $B^n$ denote the $n$-dimensional integral cohomology group. To $u^r \in B^r(K^s \bmod C^s)$ and $u^\rho \in B^\rho(K^\sigma \bmod C^\sigma)$ there corresponds the product of these cohomology classes $u^r \times u^\rho \in B^{r+\rho}(K^s \times K^\sigma \bmod C^{s\sigma})$. We define $B_1^n(K^s \times K^\sigma \bmod C^{s\sigma})$ to be the subgroup of $B^n(K^s \times K^\sigma \bmod C^{s\sigma})$ generated by—we could say consisting of—all $u^r \times u^\rho$ with $r+\rho=n$. Let $e^r$ and $e^\rho$ be the orders of $u^r$ and $u^\rho$, respectively, with the understanding that $e=0$ when $u$ is free. Let $(a, b)$ denote the greatest common divisor of $a$ and $b$ with the understanding that $(a, 0)=a$.

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[1] See [1, 7.2] (Numbers in brackets refer to references at end of paper).

[2] See [1, 9.22 and 9.4].

[3] See [2].

THEOREM 1. *A complete set of relations for the generators* $u^r\times u^\rho$ *of* $B_1^n(K^a\times K^\alpha \bmod C^{a\alpha})$ *is given by*

$$(e^r, e^\rho)(u^r \times u^\rho) = 0 \tag{1}$$

*and*

$$\sum p_i u_i^r + \sum q_j u_j^\rho = \sum p_i q_j (u_i^r \times u_j^\rho) \tag{2}$$

*where the* $p$*'s and* $q$*'s are integers.*

We define $B_2^n(K^a\times K^a \bmod C^{a\alpha}) = B^n(K^a\times K^\alpha \bmod C^{a\alpha})/B_1^n(K^a\times K^\alpha \bmod C^{a\alpha})$. This factor-group will be characterized in a way suitable for our later discussion.

Let $u^s\in B^s(K^a \bmod C^a)$ be of order $e^s\neq 0$ and $u^\sigma\in B^\sigma(K^\alpha \bmod C^\alpha)$ be of order $e^\sigma\neq 0$. Throughout the paper the superscripts $s$ and $\sigma$ will be used only for elements of order different from 0, while superscripts $r$ and $\rho$ indicate that no restriction is made on the orders of the elements. We shall associate with $u^s$ and $u^\sigma$ an element $(u^s, u^\sigma)\in B_2^{s+\sigma-1}(K^a\times K^\alpha \bmod C^{a\alpha})$. To do so let $z^s\in u^s$, $\dot{f}^{s-1}=e^s z^s$ where the dot denotes the coboundary operator, $z^\sigma\in u^\sigma$, and $\dot{f}^{\sigma-1}=e^\sigma z^\sigma$. For any cocycle $z$ let $H(z)$ denote the cohomology class with $z\in H(z)$. We define

$$H\left\{\frac{1}{(e^s, e^\sigma)}\,(f^{s-1}\times f^{\sigma-1})\cdot\right\} \in (u^s, u^\sigma). \tag{3}$$

We can show that $(u^s, u^\sigma)$ is independent of the choice of $z$'s and $f$'s.

THEOREM 2. *The cosets* $(u^s, u^\sigma)$, $s+\sigma-1=n$, *generate* $B_2^n(K^a\times K^\alpha \bmod C^{a\alpha})$, *and a complete set of relations for these generators consists of*

$$(e^s, e^\sigma)(u^s, u^\sigma) = 0 \tag{4}$$

*and*

$$\left(\sum p_i^s u_i^s,\ \sum p_j^\sigma u_j^\sigma\right) = \frac{m^s m^\sigma}{(m^s, m^\sigma)} \sum \frac{p_i^s p_j^\sigma (e_i^s, e_j^\sigma)}{e_i^s e_j^\sigma}\,(u_i^s, u_j^\sigma) \tag{5}$$

*where we consider only those sums* $\sum p_i^s u_i^s$ *and* $\sum p_j^\sigma u_j^\sigma$ *with the following properties: the* $p$*'s are integers, and the orders* $m^s$ *and* $m^\sigma$ *of* $\sum p_i^s u_i^s$ *and* $\sum p_j^\sigma u_j^\sigma$ *are the least common multiples of the orders of the terms* $p_i^s u_i^s$, $i=1, 2, \cdots$, *and* $p_j^\sigma u_j^\sigma$, $j=1, 2, \cdots$, *respectively.*

3. **Coverings and projections.** We first associate with cofinal systems of coverings[4] of $R_1$ and $R_2$ a cofinal system for $R_1\times R_2$ (our cover-

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[4] See [1, 3.5].

ings are finite and consist of open sets). With Alexandroff we say that a covering is multiplicative[5] if the intersection of any two sets of the covering is also a set of the covering. Let $\{\Omega^a\}$ and $\{\Omega^\alpha\}$ be cofinal systems of multiplicative coverings of $R_1$ and $R_2$, respectively. Let $\Omega^{a\alpha}=\Omega^a\times\Omega^\alpha$. Let $\Omega^{b\beta}$ follow $\Omega^{a\alpha}$ if both $\Omega^b$ follows $\Omega^a$ and $\Omega^\beta$ follows $\Omega^\alpha$. It is easily seen that $\{\Omega^{a\alpha}\}$ is a cofinal system of multiplicative coverings of $R_1\times R_2$.

If $\Omega^b$ follows $\Omega^a$, let $S_a^b$ be the canonical projection[6] of $\Omega^b$ into $\Omega^a$; that is, if $M^b\in\Omega^b$, then $S_a^bM^b$ is the smallest open set of $\Omega^a$ such that $M^b\supset S_a^bM^b$. Similarly we have the canonical projection $S_\alpha^\beta$ of $\Omega^\beta$ into $\Omega^\alpha$ if $\Omega^\beta$ follows $\Omega^\alpha$. With these two projections we shall associate the canonical projection $S_{a\alpha}^{b\beta}$ of $\Omega^{b\beta}$ into $\Omega^{a\alpha}$. Let $M^b\in\Omega^b$ and $M^\beta\in\Omega^\beta$. It is easily seen that

$$S_{a\alpha}^{b\beta}M^b\times M^\beta=S_a^bM^b\times S_\alpha^\beta M^\beta. \tag{6}$$

By the barycentric subdivision of a covering[7] is meant that subcomplex of the nerve of the covering made up of the simplexes whose vertices are associated with decreasing sequences of open sets of the covering. Let $K^a$, $K^\alpha$, and $K^{a\alpha}$ be the barycentric subdivisions of $\Omega^a$, $\Omega^\alpha$, and $\Omega^{a\alpha}$. We shall show that $K^{a\alpha}$ is a simplicial subdivision of $K^a\times K^\alpha$.

First, let the $r$-simplex $t^r$ of $K^a$ be associated with the following sets of $\Omega^a$:

$$M_0^a\supset\cdots\supset M_r^a. \tag{7}$$

Similarly, let $t^\rho$ of $K^\alpha$ be associated with

$$M_0^\alpha\supset\cdots\supset M_\rho^\alpha. \tag{8}$$

Corresponding to these two simplexes there is a set of $(r+\rho)$-simplexes of $K^{a\alpha}$ in (1-1)-correspondence with the sequences

$$M_0^a\times M_0^\alpha\supset\cdots\supset M_i^a\times M_j^\alpha\supset M_k^a\times M_l^\alpha\supset\cdots\supset M_r^a\times M_\rho^\alpha \tag{9}$$

where $k\geqq i$, $l\geqq j$, and $k+l=i+j+1$. But Freudenthal has shown[8] that this set of simplexes forms a simplicial division of $t^r\times t^\rho$. Furthermore, the cells $t^r\times t^\rho$ determined by (9) form a cell complex that is isomorphic with $K^a\times K^\alpha$.

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[5] See [1, 9.2].

[6] See [1, 9.3].

[7] See [1, 9.22].

[8] See [2].

Each of the complexes $K^a$, $K^\alpha$, and $K^{a\alpha}$ has a special subcomplex[9] $C^a$, $C^\alpha$, and $C^{a\alpha}$ made up of the simplexes all of whose vertices are associated with open sets whose closures are not bicompact. We shall show that $C^a \times K^\alpha + K^a \times C^\alpha \subset C^{a\alpha}$. Indeed, suppose the closure of each set of (7) is not bicompact. Then the closure of each set of any sequence of (9) is not bicompact (because the continuous image of a bicompact set is bicompact, and a factor of a product of two sets is a continuous image of the product). The same argument holds for a simplex of $C^\alpha$, and the above inclusion holds.

Next suppose $t^r$ is not in $C^a$. This means that $\bar{M}^a_r$, the closure of the final term of (7), is bicompact. Also suppose $t^\rho$ is not in $C^\alpha$. This means that $\bar{M}^\alpha_\rho$, from (8), is bicompact. Since the product of the closures of two sets is the closure of the product, and since the product of two bicompact sets is bicompact, the last term of (9) has a bicompact closure. This means that $t^r \times t^\rho$ is not in $C^{a\alpha}$. This with the above inclusion means that $C^a \times K^\alpha + K^a \times C^\alpha = C^{a\alpha}$. We have proved this lemma.

LEMMA. *The complex $K^{a\alpha}$ is a simplicial division of $K^a \times K^\alpha$, and $C^{a\alpha}$ is the division of $C^a \times K^\alpha + K^a \times C^\alpha$.*

The canonical projections $S^b_a$, $S^\beta_\alpha$ and $S^{b\beta}_{a\alpha}$ determine the usual homomorphic mappings $\rho^b_a$, $\rho^\beta_\alpha$ and $\rho^{b\beta}_{a\alpha}$ of the chains of $K^b$ mod $C^b$, and so on, into the chains of $K^a$ mod $C^a$, and so on.[10] We shall show that

$$\rho^{b\beta}_{a\alpha}(t^r \times t^\rho) = \rho^b_a t^r \times \rho^\beta_\alpha t^\rho. \tag{10}$$

In (7) replace the superscript $a$ by $b$ and let the resulting sequence denote the open sets associated with $t^r$. In the same way we replace $\alpha$ by $\beta$ in (8) and (9) to obtain sequences associated with $t^\rho$ and $t^r \times t^\rho$. From (6) and (9) we see that the left side of (10) is associated with the sequences $S^b_a M^b_0 \times S^\beta_\alpha M^\beta_0 \supset \cdots$. But these sequences are associated with $\rho^b_a t^r \times \rho^\beta_\alpha t^\rho$ because $\rho^b_a t^r$ is associated with $S^b_a M^b_0 \supset \cdots$ and $\rho^\beta_\alpha t^\rho$ is associated with $S^\beta_\alpha M^\beta_0 \supset \cdots$.

The canonical projections $S^b_a$, $S^\beta_\alpha$ and $S^{b\beta}_{a\alpha}$ determine the usual homomorphic mappings $\sigma^a_b$, $\sigma^\alpha_\beta$, and $\sigma^{a\alpha}_{b\beta}$ of the chains of $K^a$ mod $C^a$, and so on, into the chains of $K^b$ mod $C^b$, and so on.[10] From (10) we have

$$\sigma^{a\alpha}_{b\beta}(f^a \times f^\alpha) = \sigma^a_b f^a \times \sigma^\alpha_\beta f^\alpha. \tag{11}$$

Furthermore, these $\sigma$'s determine homomorphic mappings $\pi^a_b$ of

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[9] See [1, 9.1].

[10] See [1, 2.2].

$B^n(K^a \bmod C^a)$ into $B^n(K^b \bmod C^b)$, $\pi^\alpha_\beta$ of $B^n(K^\alpha \bmod C^\alpha)$ into $B^n(K^\beta \bmod C^\beta)$, and in view of the lemma $\pi^{a\alpha}_{b\beta}$ of $B^n(K^a \times K^\alpha \bmod C^{a\alpha})$ into $B^n(K^b \times K^\beta \bmod C^{b\beta})$.[11] From (11) we have

$$\pi^{a\alpha}_{b\beta}(\overset{a}{u} \times \overset{\alpha}{u}) = \pi^a_b \overset{a}{u} \times \pi^\alpha_\beta \overset{\alpha}{u}. \tag{12}$$

4. **The inner Betti groups of** $R_1 \times R_2$. Let $B^n(R_1)$, $B^n(R_2)$, and $B^n(R_1 \times R_2)$ be the inner Betti groups defined in §1. Let $\mathfrak{U}^r \in B^r(R_1)$ be of order $\mathfrak{e}^r$, and $\mathfrak{U}^\rho \in B^\rho(R_2)$ be of order $\mathfrak{e}^\rho$. We shall define $\mathfrak{U}^r \times \mathfrak{U}^\rho \in B^{r+\rho}(R_1 \times R_2)$. Let $u^{ra} \in \mathfrak{U}^r$ and $u^{\rho\alpha} \in \mathfrak{U}^\rho$ where $a$ and $\alpha$ are indices of coverings. From (12) we have $\pi^{a\alpha}_{b\beta}(u^{ra} \times u^{\rho\alpha}) = \pi^a_b u^{ra} \times \pi^\alpha_\beta u^{\rho\alpha}$. Hence we can define

$$u^{ra} \times u^{\rho\alpha} \in \mathfrak{U}^r \times \mathfrak{U}^\rho \tag{13}$$

and determine a bundle independent of the choice of coverings.

A consequence of (2) is

$$\sum p_i \mathfrak{U}^r_i \times \sum q_j \mathfrak{U}^\rho_j = \sum p_i q_j (\mathfrak{U}^r_i \times \mathfrak{U}^\rho_j). \tag{14}$$

It is possible to find coverings with indices $b$ and $\beta$ so that $u^{rb}$ and $u^{\rho\beta}$ are of order $\mathfrak{e}^r$ and $\mathfrak{e}^\rho$, respectively, (indeed, $\mathfrak{e}^r u^{ra}$ is in the zero bundle, and any cohomology class of the zero bundle can be projected into a zero cohomology class; furthermore, $u^{rb}$ cannot be of order less than $\mathfrak{e}^r$). This with (1) gives

$$(\mathfrak{e}^r, \mathfrak{e}^\rho)(\mathfrak{U}^r \times \mathfrak{U}^\rho) = 0. \tag{15}$$

Let $B^n_1(R_1 \times R_2)$ be the subgroup of $B^n(R_1 \times R_2)$ generated by the bundles (13) with $r + \rho = n$.

THEOREM 3. *Relations* (14) *and* (15) *are a complete set for the generators* (13) *of* $B^n_1(R_1 \times R_2)$.

PROOF. Let the finite sum $\sum p_{ij} \mathfrak{U}^r_i \times \mathfrak{U}^\rho_j = 0$, $r + \rho = n$, but $r$ and $\rho$ can vary from term to term. For some coverings with indices $a$ and $\alpha$ we have $\sum p_{ij}(u^{ra} \times u^{\rho\alpha}) = 0$. Theorem 1 proves that this latter relation is a consequence of (1) and (2). Hence the original relation is a consequence of (14) and (15) q.e.d.

Equation (12) shows that $\pi^{a\alpha}_{b\beta} B^n_1(K^a \times K^\alpha \bmod C^{a\alpha}) \subset B^n_1(K^b \times K^\beta \bmod C^{b\beta})$. Hence there is a natural homomorphism $\Pi^{a\alpha}_{b\beta}$ of $B^n_2(K^a \times K^\alpha \bmod C^{a\alpha})$ into $B^n_2(K^b \times K^\beta \bmod C^{b\beta})$. This natural homomorphism is defined as follows. If $\phi^{a\alpha} u^{a\alpha}$ is the coset of $B^n_2(K^a \times K^\alpha \bmod C^{a\alpha})$ that contains the cohomology class $u^{a\alpha} \in B^n(K^a \times K^\alpha \bmod C^{a\alpha})$, then

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[11] See [1, 2.33].

$$(16)\qquad \Pi^{a\alpha}_{b\beta}\phi^{a\alpha}u^{a\alpha} = \phi^{b\beta}\pi^{a\alpha}_{b\beta}u^{a\alpha}.$$

We shall show that $\Pi^{b\beta}_{c\gamma}\Pi^{a\alpha}_{b\beta}=\Pi^{a\alpha}_{c\gamma}$. Indeed, using (16) and the fact that $\pi^{b\beta}_{c\gamma}\pi^{a\alpha}_{b\beta}=\pi^{a\alpha}_{c\gamma}$, we have $\Pi^{a\alpha}_{c\gamma}\phi^{a\alpha}u^{a\alpha}=\phi^{c\gamma}\pi^{a\alpha}_{c\gamma}u^{a\alpha}=\phi^{c\gamma}\pi^{b\beta}_{c\gamma}\pi^{a\alpha}_{b\beta}u^{a\alpha}=\Pi^{b\beta}_{c\gamma}\phi^{b\beta}\pi^{a\alpha}_{b\beta}u^{a\alpha}$ $=\Pi^{b\beta}_{c\gamma}\Pi^{a\alpha}_{b\beta}\phi^{a\alpha}u^{a\alpha}$ proving the equality in question. We have shown that $[B^n_2(K^a\times K^\alpha \bmod C^{a\alpha});\Pi^{a\alpha}_{b\beta}]$ is a spectrum. Let the direct limit of this spectrum be denoted by $B^n_2(R_1\times R_2)$.

Let $\mathfrak{u}^s\in B^s(R_1)$ and $\mathfrak{u}^\sigma\in B^\sigma(R_2)$ have orders $\vartheta^s\neq 0$ and $\vartheta^\sigma\neq 0$. We shall associate with these two bundles an element $(\mathfrak{u}^s, \mathfrak{u}^\sigma)$ $\in B^{s+\sigma-1}_2(R_1\times R_2)$. Consider a cohomology class of the coset $(u^{sa}, u^{\sigma\alpha})$, $u^{sa}\in\mathfrak{u}^s$, $u^{\sigma\alpha}\in\mathfrak{u}^\sigma$, $u^{sa}$ of order $\vartheta^s$, and $u^{\sigma\alpha}$ of order $\vartheta^\sigma$. In view of (3) this cohomology class contains a cocycle which may be written

$$(17)\qquad \frac{1}{(\vartheta^s,\vartheta^\sigma)}\left(f^{s-1a}\times f^{\sigma-1\alpha}\right)^{\cdot}$$

with $f^{s-1a}=\vartheta^s z^{sa}$, $z^{sa}\in u^{sa}$, and similar relations with superscript $\sigma$. We have

$$(18)\qquad \sigma^{a\alpha}_{b\beta}(17) = \frac{1}{(\vartheta^s,\vartheta^\sigma)}\left(\sigma^a_b f^{s-1a}\times\sigma^\alpha_\beta f^{\sigma-1\alpha}\right)^{\cdot}$$

because of (11) and the fact that the operation of forming coboundary commutes with $\sigma^{a\alpha}_{b\beta}$.[12] But $(\sigma^a_b f^{s-1a})^{\cdot}=\vartheta^s\sigma^a_b z^{sa}$ with the same relation for superscript $\sigma$. Hence (18) is in a cohomology class of the coset $(\pi^a_b u^{sa}, \pi^\alpha_\beta u^{\sigma\alpha})$. Letting $H(z)$ still denote the cohomology class that contains the cocycle $z$, we have using these results and (16) that $\Pi^{a\alpha}_{b\beta}(u^{sa}, u^{\sigma\alpha})=\Pi^{a\alpha}_{b\beta}\phi^{a\alpha}H(17)=\phi^{b\beta}\pi^{a\alpha}_{b\beta}H(17)=\phi^{b\beta}H(18)=(\pi^a_b u^{sa}, \pi^\alpha_\beta u^{\sigma\alpha})$. Hence we have a unique bundle $(\mathfrak{u}^s, \mathfrak{u}^\sigma)\in B^{s+\sigma-1}_2(R_1\times R_2)$ defined by

$$(19)\qquad (u^{sa}, u^{\sigma\alpha})\in(\mathfrak{u}^s, \mathfrak{u}^\sigma).$$

From (4) we obtain

$$(20)\qquad (\vartheta^s,\vartheta^\sigma)(\mathfrak{u}^s,\mathfrak{u}^\sigma)=0.$$

Furthermore, let the sums $\sum p^s_i\mathfrak{u}^s_i$ and $\sum p^\sigma_j\mathfrak{u}^\sigma_j$ satisfy the conditions on the sums in (5). From (5) we have a relation which we label (21). The relation (21) is obtained from (5) by replacing each $u$ by $\mathfrak{u}$.

THEOREM 4. *The elements* $(\mathfrak{u}^s, \mathfrak{u}^\sigma)$, $s+\sigma-1=n$, *generate* $B^n_2(R_1\times R_2)$, *and relations* (20) *and* (21) *are a complete set for these generators.*

Theorem 4 is a consequence of Theorem 2 and the definition of

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[12] See [1, Theorem 2.221].

$B_2^n(R_1\times R_2)$ as a direct limit. The proof is similar to the proof of Theorem 3.

THEOREM 5. *We have* $B_2^n(R_1\times R_2)=B^n(R_1\times R_2)/B_1^n(R_1\times R_2)$.

PROOF. It follows from (16) that there is a homomorphic mapping of $B^n(R_1\times R_2)$ upon $B_2^n(R_1\times R_2)$.[13] The kernel of this homomorphism consists of the bundles of $B^n(R_1\times R_2)$ which contain elements of the groups $B_1^n(K^\alpha\times K^\alpha \bmod C^{\alpha\alpha})$.[13] From (12) and (18) we see that this kernel is precisely $B_1^n(R_1\times R_2)$.

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[13] See [1, Theorems 3.41 and 3.411].

# ON THE AVERAGE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION

M. KAC

1. **Introduction.** Consider the algebraic equation

$$X_0 + X_1x + X_2x^2 + \cdots + X_{n-1}x^{n-1} = 0, \tag{1}$$

where the $X$'s are independent random variables assuming real values only, and denote by $N_n = N(X_0, \cdots, X_{n-1})$ the number of real roots of (1). We want to determine the mean value (mathematical expectation = m.e.) of $N_n$ when all $X$'s have the same normal distribution with density

$$e^{-x^2}/\pi^{1/2}. \tag{2}$$

This problem was treated by Littlewood and Offord[1] who also considered the cases when the $X$'s are uniformly distributed in $(-1, 1)$ or assume only the values $+1$ and $-1$ with equal probabilities. Littlewood and Offord obtain in each case the estimate

$$\text{m.e. } \{N_n\} \leqq 25(\lg n)^2 + 12 \lg n, \qquad n \geqq 2000.$$

In our case of normally distributed $X$'s we shall be able to prove the exact formula

$$\text{m.e. } \{N_n\} = \frac{4}{\pi}\int_0^1 \frac{[1 - n^2[x^2(1 - x^2)/(1 - x^{2n})]^2]^{1/2}}{1 - x^2}\,dx \tag{3}$$

and then obtain the asymptotic relation

$$\text{m.e. } \{N_n\} \sim (2/\pi) \lg n \tag{4}$$

and the estimate

$$\text{m.e. } \{N_n\} \leqq (2/\pi) \lg n + 14/\pi, \qquad n \geqq 2. \tag{5}$$

In case the $X$'s are not normally distributed (but all have the same distribution with standard deviation 1) one can still prove (4). The necessary limiting processes can then be carried out by using the central limit theorem of the calculus of probability and, as one may expect, the computations will be quite lengthy. On the other hand they will contribute relatively little to the general picture and, what

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[1] J. London Math. Soc. vol. 13 (1938) pp. 288–295. No proofs are given in this paper, and the present author was unable to find them anywhere in print.

is worse, they may darken it by technicalities. We shall therefore assume in what follows that the $X$'s are normally distributed with density given by formula (2).

2. **A formula for the number of real roots.** Let $f(x)$ be a continuous function in $(-\infty, \infty)$ having a continuous first derivative $f'(x)$ and only a finite number of turning points in each finite interval.

Let $\psi_\epsilon(x)$ be 1 if $-\epsilon<x<\epsilon$ and 0 otherwise. We then have the following lemma.

LEMMA 1. *If neither $a$ nor $b$ is a zero of $f(x)$, then for sufficiently small $\epsilon$'s*

$$(1/2\epsilon)\int_a^b \psi_\epsilon(f(x))\,|f'(x)|\,dx$$

*is equal to the number of zeros of $f(x)$ inside the interval $(a, b)$. (Multiple roots are counted only once.)*[3]

We first notice that the set $E_\epsilon$ of those $x$'s for which $\psi_\epsilon(f(x))=1$ is an open set and is therefore a sum of disjoint open intervals $I_1, I_2, I_3, \cdots$.

We choose $\epsilon$ small enough so that (a) no turning point of $f(x)$ in $(a, b)$ lies in the strip $-\epsilon<y<\epsilon$ unless it happens to be at the same time a zero of $f(x)$; (b) no $I_i$ includes either $a$ or $b$.

Let $I_1, I_2, \cdots, I_r$, say, be totally contained in $(a, b)$. Then, since it is easily seen that

$$\int_{I_i} |f'(x)|\,dx = 2\epsilon, \qquad 1 \leqq i \leqq r,$$

we have

$$(1/2\epsilon)\int_a^b \psi_\epsilon(f(x))\,|f'(x)|\,dx = (1/2\epsilon)\sum_{i=1}^{r}\int_{I_i}|f'(x)|\,dx = r.$$

This proves Lemma 1.

REMARK 1. An easy extension of the above reasoning gives for sufficiently small $\epsilon$'s

$$(1/2\epsilon)\int_a^b \psi_\epsilon(f(x))\,|f'(x)|\,dx = \text{number of zeros of } f(x) \text{ inside } (a, b) + \eta,$$

where $\eta=0, 1/2, 1$ according as none, one or both of the numbers $a$, $b$ are zeros of $f(x)$.

REMARK 2. If $f(x)$ is a polynomial then for sufficiently small $\epsilon$'s

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[3] If $f(x)\equiv 0$ we consider $f(x)$ as not having any roots.

$$(1/2\epsilon)\int_{-\infty}^{+\infty}\psi_\epsilon(f(x))\,|f'(x)|\,dx$$

is equal to the number of real roots of $f(x)$. This is a trivial consequence of Lemma 1. One might notice that the limits of integration are really finite since $\psi_\epsilon(f(x))=0$ for sufficiently large $|x|$.

REMARK 3. The choice of $\epsilon$ in Remark 2 obviously depends on the coefficients of the polynomial. However, we can eliminate $\epsilon$ by restating the result of Remark 2 in the form:

$$\lim_{\epsilon\to 0}(1/2\epsilon)\int_{-\infty}^{+\infty}\psi_\epsilon(f(x))\,|f'(x)|\,dx = \text{number of real roots of } f(x).$$

If we put now

$$f(x) = X_0 + X_1 x + \cdots + X_{n-1}x^{n-1},$$

we get

$$N_n = \lim_{\epsilon\to 0}(1/2\epsilon)\int_{-\infty}^{+\infty}\psi_\epsilon(f(x))\,|f'(x)|\,dx. \tag{6}$$

We shall need this lemma in what follows.

LEMMA 2. *If $f(x)$ is a polynomial of degree $n-1$, then for every $\epsilon>0$*

$$(1/2\epsilon)\int_{-\infty}^{+\infty}\psi(f(x))\,|f'(x)|\,dx \leqq 3n-5.$$

In this case the set $E$ (see proof of Lemma 1) is a sum of at most $2n-3$ open intervals. Indeed, each $I_i$ contains either a real root of $f(x)$ or a turning point and there are at most $n-1$ real roots and at most $n-2$ turning points.

The proof of Lemma 2 follows now from the obvious remark that

$$\int_{I_i}|f'(x)|\,dx \leqq 2\epsilon(m_i+1), \qquad 1\leqq i\leqq 2n-3,$$

where $m_i$ is the number of turning points in $I_i$.

3. **Interchange of two limiting processes.** We now reduce the computation of m.e. $\{N_n\}$ to the computation of m.e. $\{\psi_\epsilon(f(x))\,|f'(x)|\}$ by means of this lemma.

LEMMA 3.

$$\text{m.e. }\{N_n\} = \lim_{\epsilon\to 0}(1/2\epsilon)\int_{-\infty}^{+\infty}\text{m.e. }\{\psi_\epsilon(f(x))\,|f'(x)|\}\,dx.$$

Random variables can be considered as measurable functions defined on a set $\Omega$ with a completely additive Lebesgue measure. (The measure of $\Omega$ is 1.) Mathematical expectation (m.e.) is nothing but a Lebesgue integral with respect to that measure. Both $N_n$ and $\psi_\epsilon(f(x))|f'(x)|$ are then measurable functions on $\Omega$ and they can be represented symbolically as

$$N_n(\mu) \quad \text{and} \quad g_\epsilon(x, \mu),$$

$x$ and $\epsilon$ being real parameters. We also have

$$\text{m.e. } \{N_n\} = \int_\Omega N_n(\mu) d\mu,$$

$$\text{m.e. } \{g_\epsilon(x, \mu)\} = \int_\Omega g_\epsilon(x, \mu) d\mu,$$

$d\mu$ indicating that integration is being performed with respect to the Lebesgue measure in $\Omega$. We first notice that

$$\text{m.e. } \left\{ \int_{-\infty}^{+\infty} g_\epsilon(x, \mu) dx \right\} = \int_{-\infty}^{+\infty} \text{m.e. } \{g_\epsilon(x, \mu)\} dx. \tag{7}$$

This follows from Fubini's theorem if one notices that the integral with respect to $dx$ is really an integral between finite limits (depending, of course, on $\epsilon$ and $\mu$). Furthermore, by Lemma 2 for every $\epsilon > 0$

$$(1/2\epsilon) \int_{-\infty}^{+\infty} g_\epsilon(x, \mu) dx < 3n - 5,$$

and hence, by a well known theorem from Lebesgue's theory,

$$\lim_{\epsilon \to 0} \int_\Omega \left[ (1/2\epsilon) \int_{-\infty}^{+\infty} g_\epsilon(x, \mu) dx \right] d\mu$$
$$= \int_\Omega \left[ \lim_{\epsilon \to 0} (1/2\epsilon) \int_{-\infty}^{+\infty} g_\epsilon(x, \mu) dx \right] d\mu.$$

This when combined with (6) and (7) completes the proof of Lemma 3.

4. **A formula for m.e.** $\{N_n\}$. In order to compute m.e. $\{\psi_\epsilon(f(x))|f'(x)|\}$ we need the following lemma.

LEMMA 4. *If* $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}, \beta_0, \beta_1, \cdots, \beta_{n-1}$ *are real numbers,* $\sum \alpha_i^2 = \alpha$, $\sum \beta_i^2 = \beta$, $\sum \alpha_i \beta_i = \gamma$ *and if* $\Delta = \alpha\beta - \gamma^2 > 0$, *then the density of the joint distribution of* $\alpha_0 X_0 + \cdots + \alpha_{n-1} X_{n-1}$ *and* $\beta_0 X_0 + \cdots + \beta_{n-1} X_{n-1}$ *is equal to*

$$(1/\pi\Delta^{1/2}) \exp\left\{-(\beta u^2 - 2\gamma uv + \alpha v^2)/\Delta\right\}.$$

This fact is well known.[3]

It also follows from well known facts that

$$\text{m.e. } \left\{\psi_\epsilon(\alpha_0 X_0 + \cdots + \alpha_{n-1}X_{n-1}) \mid \beta_0 X_0 + \cdots + \beta_{n-1}X_{n-1} \mid\right\}$$
$$= (1/\pi\Delta^{1/2}) \int_{-\infty}^{+\infty} \psi_\epsilon(u) \mid v \mid \exp\left\{-(\beta u^2 - 2\gamma uv + \alpha v^2)/\Delta\right\} du dv.$$

Let

$$\int_{-\infty}^{+\infty} \mid v \mid \exp\left\{-(\beta u^2 - 2\gamma uv + \alpha v^2)/\Delta\right\} dv = F(u),$$

then $F(u)$ is a continuous function of $u$ and we get

$$\lim_{\epsilon\to 0} (1/2\epsilon) \text{ m.e. } \left\{\psi_\epsilon\left(\sum \alpha_i X_i\right) \left|\sum \beta_i X_i\right|\right\}$$
$$= \lim_{\epsilon\to 0} (1/2\epsilon\pi\Delta^{1/2}) \int_{-\epsilon}^{\epsilon} F(u) du = F(0)/\pi\Delta^{1/2}.$$

But

$$F(0) = \int_{-\infty}^{+\infty} \mid v \mid \exp\left\{-\alpha v^2/\Delta\right\} dv = \Delta/\alpha$$

and finally

$$\lim_{\epsilon\to 0} (1/2\epsilon) \text{ m.e. } \left\{\psi_\epsilon\left(\sum \alpha_i X_i\right) \left|\sum \beta_i X_i\right|\right\} = \Delta^{1/2}/\alpha.$$

If we now put $\alpha_0 = 1$, $\alpha_1 = x$, $\cdots$, $\alpha_{n-1} = x^{n-1}$, $\beta_0 = 0$, $\beta_1 = 1$, $\beta_2 = 2x$, $\cdots$, $\beta_{n-1} = (n-1)x^{n-2}$, we obtain by an elementary computation

$$\Delta = \frac{x^{4n} - n^2 x^{2(n+1)} + 2(n^2-1)x^{2n} - n^2 x^{2(n-1)} + 1}{(x^2-1)^4}.$$

It is easy to prove that $\Delta > 0$ for every real $x$ and by combining the considerations of this section with Lemma 3 we obtain

$$\text{m.e. } \{N_n\}$$
$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{(x^{4n} - n^2 x^{2(n+1)} + 2(n^2-1)x^{2n} - n^2 x^{2(n-1)} + 1)^{1/2}}{(x^2-1)^2(1 + x^2 + x^4 + \cdots + x^{2n-2})} dx. \tag{8}$$

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[3] See for instance S. Bernstein, *Sur l'extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes*, Math. Ann. vol. 97 (1927) pp. 2–59, in particular, chap. 3 (pp. 43–59). This chapter contains limit theorems by means of which one can handle the case of not normally distributed $X$'s. Lemma 4 is but a simple consequence of the main results of that chapter.

5. **Estimates and the asymptotic formula.** By an elementary transformation we get

$$\text{m.e.}\ \{N_n\} = \frac{4}{\pi}\int_0^1 \frac{(1 - h_n^2(x))^{1/2}}{1 - x^2}\,dx,$$

where

$$h_n(x) = \frac{nx^{n-1}(1 - x^2)}{1 - x^{2n}}\,.$$

It should be mentioned that the value of the integrand at $x=1$ is $(n^2-1/12)^{1/2}$. We have

$$1 - x^{2n} = (1 - x)(1 + x + x^2 + \cdots + x^{2n-1}) < 2n(1 - x)$$

and therefore

$$\begin{aligned} h_n(x) &> x^{n-1}(1 + x)/2 \\ 1 - h_n^2(x) &< [1 - x^{n-1}(1 + x)/2][1 + x^{n-1}(1 + x)/2] \\ &< 2 - x^{n-1}(1 + x). \end{aligned}$$

Using the mean value theorem we get

$$\begin{aligned} 2 - x^{n-1}(1 + x) &= (1 - x)\left[\theta^{n-1} + (n - 1)\theta^{n-2}(1 + \theta)\right] \\ &< (2n - 1)(1 - x), \qquad x < \theta < 1. \end{aligned}$$

Finally,

$$(1 - h_n^2(x))^{1/2}/(1 - x^2) < (2n - 1)^{1/2}/(1 - x)^{1/2}, \qquad 0 \leqq x < 1.$$

On the other hand

$$(1 - h_n^2(x))^{1/2}/(1 - x^2) \leqq 1/1 - x^2, \qquad 0 \leqq x < 1,$$

and we can write

$$\int_0^1 \frac{(1 - h_n^2(x))^{1/2}}{1 - x^2}\,dx < \int_0^{1-1/n} \frac{dx}{1 - x^2} + \int_{1-1/n}^1 \frac{(2n - 1)^{1/2}}{(1 - x)^{1/2}}\,dx$$

$$= (1/2)\lg n + (1/2)\lg(2 - 1/n) + 2(2 - 1/n)^{1/2} < (1/2)\lg n + 3\cdot 5.$$

Finally,

$$\text{m.e.}\ \{N_n\} < (2/\pi)\lg n + 14/\pi, \qquad n \geqq 2. \tag{9}$$

In order to obtain a lower estimate let $\epsilon$ and $\delta$ be arbitrary positive numbers less than 1. We have

$$\int_0^1 \frac{\left(1 - k_n^2(x)\right)^{1/2}}{1 - x^2}\,dx > \int_0^{1-n^{\delta-1}} \frac{\left(1 - k_n^2(x)\right)^{1/2}}{1 - x^2}\,dx.$$

But for $0 \leqq x \leqq 1 - n^{\delta-1}$

$$k_n(x) < n x^{n-1} \leqq n(1 - n^{\delta-1})^{n-1}$$

and this last expression can be made smaller than $\epsilon^{1/2}$ if $n$ is sufficiently large. Hence, for sufficiently large $n$,

$$\int_0^1 \frac{\left(1 - k_n^2(x)\right)^{1/2}}{1 - x^2}\,dx > \int_0^{1-n^{\delta-1}} \frac{(1 - \epsilon)^{1/2}}{1 - x^2}\,dx$$

$$> \frac{(1 - \epsilon)^{1/2}(1 - \delta)}{2} \lg n.$$

Using (9) and the fact that $\epsilon$ and $\delta$ can be made arbitrarily small we obtain the asymptotic formula

$$\text{m.e. } \{N_n\} \sim (2/\pi) \lg n. \tag{10}$$

Let us add that it follows from (9) that the probability that (1) has more than $\sigma \lg n$ real roots is less than

$$2/\pi\sigma + 14/\pi\sigma \lg n.$$

This result is not trivial only in case $\sigma > 2/\pi$.

6. **Final remarks.** It is quite clear that the average number of real roots of (1) falling in the interval $(a, b)$ is given by formula (8) if one replaces $-\infty$ and $+\infty$ by $a$ and $b$, respectively. The proof of this statement does not differ from the one given above and one must only notice that the probability that either $a$ or $b$ is a root of (1) is 0.

One can also see almost immediately that if $(a, b)$ does not contain either 1 or $-1$ the average number of real roots of (1) falling within $(a, b)$ is $O(1)$. This means, roughly speaking, that most of the real roots of most of the equations cluster around 1 and $-1$. The problem of the exact determination of the average distribution of real roots of (1) on the real axis will, of course, depend on a more delicate treatment of the integral (8). It would also be interesting to know the higher moments of $N_n$. The present line of attack would lead to very complicated integrals, but it may be hoped that some other approach will furnish more information about the distribution of the number of real roots of equation (1).

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# SOME IMBEDDING THEOREMS AND CHARACTERIZATION PROBLEMS OF DISTANCE GEOMETRY

L. M. BLUMENTHAL

**Introduction.** The systematic development of abstract distance geometry was initiated by Menger's *Untersuchungen* [6][1] of 1928. The field opened up by these pioneering papers has been extensively cultivated during the past fourteen years with the result that not only have the boundaries of the subject been extended far beyond what was envisaged a decade and a half ago, but the territory gained has been regained and consolidated by new methods that attain their objectives more easily. It is the purpose of this paper to present some of these consolidations (Part I) and extensions (Part II)—at hand or in progress—and to point out a few promising regions for future exploration.

## PART I

1. **Preliminary definitions.** A *distance space*, in its most general aspects, arises upon associating with each ordered pair $p$, $q$ of elements of a "point" set an element $pq$ of a "distance" set, the association being conditioned only by certain very simple rules. If, in particular, the distance set is the set of non-negative real numbers, $pq=qp$, while $pq=0$ if and only if $p=q$, one obtains the class $\{\Sigma\}$ of *semimetric spaces*. A semimetric space is *metric* provided that for each three of its points $p$, $q$, $r$ the triangle inequality $pq+qr\geq pr$ is satisfied.

We shall have frequent occasion to consider the determinant

$$D(p_1, p_2, \cdots, p_k) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & p_1p_2^2 & \cdots & p_1p_k^2 \\ 1 & p_2p_1^2 & 0 & \cdots & p_2p_k^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & p_kp_1^2 & p_kp_2^2 & \cdots & 0 \end{vmatrix},$$

formed for $k$ points $p_1, p_2, \cdots, p_k$ of a semimetric space. With its use the triangle inequality takes the symmetric form $D(p, q, r)\leq 0$.

Two distance spaces (or subsets of the same distance space) are

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[1] See references at end of paper.

called *congruent* or *isometric* provided there exists a one-to-one, distance-preserving correspondence between their points, and a *distance geometry* studies those properties of a distance space which are invariant under the group of congruences.

2. **Problems for euclidean subsets and spaces.** It is observed that a semimetric space of $m$ elements $p_1, p_2, \cdots, p_m$ is given by a matrix $(p_ip_j)$ of non-negative real numbers—the mutual distances of the $m$ points—which, in general, is subject only to the requirements of being symmetric and having zeros along the principal diagonal. It is clear that if such a distance matrix be formed for $m$ points of a particular space (for example, euclidean) there is a marked decrease in the freedom with which the elements of the matrix may be selected. The distinctive character of the space is reflected in the structure (the inner relations) of this matrix, and it is natural to seek to classify and study distance spaces in terms of this structure.[2] The first problems to which these considerations gave rise were the following ones.

THE EUCLIDEAN SUBSET PROBLEM. *What is the structure of the distance matrix of a finite semimetric space which is necessary and sufficient to insure the euclidean character of the metric? On what finite subsets (if any) of an arbitrary semimetric $\Sigma$ is it necessary and sufficient to impose this matrix structure in order that $\Sigma$ may be congruently contained in a euclidean space of a given dimension?*[3]

THE EUCLIDEAN SPACE PROBLEM. *Let $\Sigma$ be any semimetric space. What are necessary and sufficient (metric) conditions for congruence of $\Sigma$ with a euclidean space of given dimension?*

The first solutions of both of these problems were given by Menger [6].

It is important to observe that the subset problem is the more general one. Of the many solutions that have been given for the space problem few advance the solution of the problem of congruent imbedding. This seems due to the fact that in characterizing a space one imposes from the outset certain obvious necessary conditions that profoundly affect the type of theorem obtained. So far as imbeddability is concerned, such theorems give only stringent sufficient conditions.

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[2] There are many ways in which this structure may be described. As seen later, one may base the description upon (1) the signs of principal minors of $D(p_1, \cdots, p_m)$ (Menger), (2) the character of the quadratic form with determinant $D$ (Morse), (3) the character of the quadratic form $(1/2)\sum_1^m(p_1p_i^2+p_1p_j^2-p_ip_j^2)x_ix_j$ (Schoenberg), (4) the behavior of the family of functions $\exp(-\lambda t^2)$, $\lambda$ positive, defined over the space of $m$ elements (E. H. Moore, Schoenberg), and so on. It would be useful to add to these means of interpreting the structure.

[3] It should be mentioned that this formulation of the euclidean subset problem is quite different from the original one.

Thus Busemann has shown that a convex, externally convex, finitely compact metric space is congruent to a euclidean or hyperbolic space provided the locus of points with equal distances from two distinct points contains with each distinct pair of its points each straight line through them.[4] This interesting result is, however, of little use in determining if an arbitrary semimetric space is congruently contained in a euclidean or hyperbolic space.

On the other hand, a solution of the euclidean subset problem does advance the solution of the space problem, for if subsets of $E_n$ are characterized metrically a characterization of $E_n$ itself is obtained by adjoining properties that distinguish $E_n$ among its subsets. This is a *euclidean* rather than an abstract space problem. It should be mentioned that a characterization of the $E_n$ by way of the subset problem is quite likely to contain redundancies in the hypotheses, for it may happen that some of the requirements which are essential for the imbedding of arbitrary semimetric spaces in $E_n$ might be dispensed with in the light of those adjoined conditions individualizing the $E_n$ among its subsets. We shall see an illustration of this later.

Before turning to the solutions of the two problems posed above, we state here the general characterization problem.

GENERAL CHARACTERIZATION PROBLEM. *Let* $\{\Delta\}$ *be a given class of distance spaces and* $\{\Delta^*\}$ *a given subclass of* $\{\Delta\}$. *The subclass* $\{\Delta^*\}$ *is characterized metrically with respect to the class* $\{\Delta\}$ *when metric conditions are obtained which are necessary and sufficient to insure that any space of* $\{\Delta\}$ *satisfying them be congruent with a member of* $\{\Delta^*\}$.

The problem of *isometric or congruent imbedding* consists in determining whether a given space $\Delta$ be congruently contained in a given space $\Delta^*$ (that is, congruent with a subset of $\Delta^*$). The congruence of the spaces $L^{(2)}$ and $l^{(2)}$ (1907), the isometric imbedding in Urysohn's space $U$ (or the space $C$ of continuous functions on $(0, 1)$) of each separable metric space are examples.

**3. Solution of euclidean subset and space problems.** The problem of congruent imbedding of $\Sigma$ in $E_n$ is reduced to a "finite" one by showing that *any semimetric space is congruently contained in* $E_n$ *whenever each* $n+3$ *of its points are.* This is described by saying that the $E_n$ has *congruence order* $n+3$ with respect to the class of semimetric spaces. In the following we give the imbedding theorems with weaker hypotheses than those originally imposed.[5]

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[4] For definitions of convexity, external convexity, and so on, see the writer's *Distance geometries*, University of Missouri Studies, vol. 13 (1938), which presents a survey of the development of abstract metrics.

The congruence order $n+3$ property of the $E_n$ is a consequence of the following easily proved theorem:

THEOREM 3.1. *A semimetric* $\Sigma$ *is congruently contained in* $E_n$ *if and only if there exists an integer* $r$, $r \leqq n$, *such that* (1) $\Sigma$ *contains* $r+1$ *points congruently imbeddable in* $E_r$, *not in* $E_{r-1}$, *and* (2) *each* $(r+3)$-*tuple of* $\Sigma$ *containing this* $(r+1)$-*tuple is imbeddable in* $E_r$. *Then* $\Sigma$ *is congruently contained in* $E_r$, *not in* $E_{r-1}$.

This theorem differs principally from Menger's congruence order theorem by not assuming that all $(r+3)$-tuples are imbeddable in $E_r$, but only those that contain a selected set of $r+1$ points. It thus permits the presence of *free* $(r+3)$-tuples; that is, sets of $r+3$ points which are not assumed imbeddable in $E_r$. We shall meet with other instances of this kind of freedom later. Further, the proof of the property as formulated above dispenses with a separation into cases and an induction which feature Menger's proof.

The euclidean subset problem is then solved when conditions for (1) the imbedding of $r+1$ points in $E_r$, not in $E_{r-1}$, and (2) the imbedding in $E_r$ of $(r+3)$-tuples containing such a set of $r+1$ points are obtained. Concerning (1) we have this statement:

THEOREM 3.2. *A semimetric* $(r+1)$-*tuple* $p_1, p_2, \cdots, p_{r+1}$ *is imbeddable in* $E_r$, *not in* $E_{r-1}$, *if and only if*

$$\operatorname{sgn} D(p_1, p_2, \cdots, p_{k+1}) = (-1)^{k+1}, \qquad k = 1, 2, \cdots, r.$$

This theorem was originally proved by Menger under the heavier assumptions that for each integer $k$ $(2 \leqq k \leqq r+1)$ and for each set of $k$ of the $r+1$ points, $\operatorname{sgn} D(p_{i_1}, p_{i_2}, \cdots, p_{i_k}) = (-1)^k$. Menger later observed that as a consequence of a remark of M. Morse it suffices to require merely the non-vanishing of the determinants of every $k$-tuple $(2 \leqq k \leqq r+1)$ of the points and $\operatorname{sgn} D(p_1, p_2, \cdots, p_{k+1}) = (-1)^{k+1}$ $(k=1, 2, \cdots, r)$ [5]. It turns out, however, that even the non-vanishing of these determinants need not be assumed.[6]

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[5] In a recent seminar course, the writer has completely revised Menger's treatment of the euclidean problems and obtained simpler proofs and stronger theorems. This applies particularly to the difficult matter of quasi-congruence order.

[6] This is implied also by Schoenberg's quadratic form criterion, but we shall use these results to obtain Schoenberg's theorem.

COROLLARY. *If $p_1, p_2, \cdots, p_r$ are congruently contained in $E_{r-1}$, not in $E_{r-2}$, then the semimetric $(r+1)$-tuple $p_1, p_2, \cdots, p_r, p_{r+1}$ is imbeddable in $E_r$, not in $E_{r-1}$, if and only if* sgn $D(p_1, p_2, \cdots, p_r, p_{r+1}) = (-1)^{r+1}$.

It is of interest that the inequality demanded of the determinant $D$ of the $r+1$ points is strong enough to force the selection of the $r$ distances of $p_{r+1}$ from the remaining $r$ points so that the $r$-tuples $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_r, p_{r+1}$ $(i=1, 2, \cdots, r)$ are euclidean. Turning to the imbedding of $r+3$ points, we have this theorem.

THEOREM 3.3. *If $p_1, \cdots, p_{r+1}$ are imbeddable in $E_r$, not in $E_{r-1}$, then the semimetric $(r+3)$-tuple $p_1, \cdots, p_{r+1}, p_{r+2}, p_{r+3}$ is imbeddable in $E_r$ if and only if*

$$\begin{aligned} D(p_1, \cdots, p_{r+1}, p_{r+2}) &= D(p_1, \cdots, p_{r+1}, p_{r+3}) \\ &= D(p_1, \cdots, p_{r+1}, p_{r+2}, p_{r+3}) = 0. \end{aligned}$$

The proof of this theorem follows familiar lines when it is established that $p_1, p_2, \cdots, p_{r+1}, p_{r+2}$ and $p_1, p_2, \cdots, p_{r+1}, p_{r+3}$ are imbeddable in $E_r$, not in $E_{r-1}$. This is done in a lemma, and the euclidean subset problem is completely solved.[7]

By despoiling $D(p_0, p_1, \cdots, p_k)$ of its bordering, we have

$$D(p_0, p_1, \cdots, p_k) = (-1)^{k+1} \cdot 2^k \left| p_{ij} \right|, \qquad i, j = 1, 2, \cdots, k,$$

where $p_{ij} = (1/2)\ (p_0p_i^2 + p_0p_j^2 - p_ip_j^2)$. It follows at once from the preceding theorems that a semimetric $(n+1)$-tuple $p_0, p_1, \cdots, p_n$ is imbeddable in $E_n$ if and only if there exists an integer $r$, $r \leqq n$, such that for $r+1$ of the $n+1$ points, say $p_0, p_1, \cdots, p_r$, $\left| p_{ij} \right| > 0$ $(i, j = 1, 2, \cdots, k; k \leqq r)$ while in case $r < n$, all $(r+1)$st and $(r+2)$d order determinants $\left| p_{ij} \right|$ containing $\left| p_{ij} \right|$ $(i, j = 1, 2, \cdots, r)$ vanish. *These are precisely the conditions that the quadratic form $\sum_{i,j=1}^{n} p_{ij}x_ix_j$ be positive definite of rank $r$, and we have the theorem of Schoenberg* [10]. It is easy to see that this condition is equivalent to the form $\sum_{i,j=0}^{n} (p_ip_j)^2 x_ix_j$ being negatively definite of rank $r$ on the hyperplane $x_0 + x_1 + \cdots + x_n = 0$, and this form of the condition has proved to be very useful.

Turning again to the congruence order $n+3$ property of the $E_n$, it is natural to inquire into the conditions that permit a reduction in the number $n+3$. That this number is in general the smallest integer $k$ for which the imbedding of each $k$-tuple of a semimetric space implies the imbedding of the whole space in $E_n$ follows, as Menger has shown, from the existence for each integer $n$ of semimetric spaces of $n+3$ points which are not imbeddable in $E_n$ though each $(n+2)$-tuple is.

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[7] We see that if $p_1, \cdots, p_r$ are imbeddable in $E_{r-1}$, not in $E_{r-2}$, then the semimetric $(r+1)$-tuple $p_1, \cdots, p_r, p_{r+1}$ is imbeddable in $E_r$ if and only if the sign of their determinant $D$ is *not* $(-1)^r$. An example shows this is no longer valid if the restriction that the $r$ points be not imbeddable in $E_{r-2}$ is dropped; for if $pq = qr = ps = 1$, $pr = 2$, $qs = 3$, $rs = (19)^{1/2}$, the triple $p, q, r$ is imbeddable in $E_1$ and $D(p, q, r, s) = 0$, but the four points are not congruently contained in a euclidean space.

The construction of these so-called pseudo-$E_n$ $(n+3)$-tuples is readily described, but only in the case of pseudo-linear quadruples $(n=1)$ have distance relations characterizing such sets been obtained. These quadruples have the form $pq=rs=a>0$, $qr=ps=b>0$, $pr=qs=a+b$, and $D(p, q, r, s)=-32$ (product of the six distances). It would be of interest to find the value of $D$ for a pseudo-$E_n$ $(n+3)$-tuple. So far it has been shown only that its sign is $(-1)^n$.

It follows from the preceding remark that pseudo-$E_n$ sets of $n+3$ points are not imbeddable in any euclidean (or Hilbert) space. Pseudo-linear quadruples are clearly imbeddable in a (convex) circle, but pseudo-planar quintuples are not imbeddable, so far as is known, in any singularity-free surface.[3] Recently spaces were constructed by Pepper which contain congruently all pseudo-$E_n$ $(n+3)$-tuples [7]. They are obtained by appropriately metrizing the union of two $n$-spaces "joined" along a simplex. For $n=2$ a simpler kind of space suffices. This is obtained by metrizing convexly the set consisting of the union of three half-planes with a common axis.

What of pseudo-$E_n$ sets of more than $n+3$ points? The most complicated part of the *Zweite Untersuchung* is devoted to showing that no such sets exist; that is, *if a semimetric space has more than $n+3$ points it is imbeddable in $E_n$ whenever each $n+2$ of its points has this property.* This is described by saying that the $E_n$ has *quasi-congruence order* $n+2$ with respect to the class of semimetric spaces. We shall return to this matter later.

Having solved the euclidean subset problem, the characterization of the whole $E_n$ is obtained by adjoining properties distinguishing it from its subsets. This yielded the following theorem.

THEOREM 3.4. *A semimetric space $\Sigma$ is congruent to $E_n$ if and only if $\Sigma$ is complete, convex, externally convex, has each $n+2$ of its points imbeddable in $E_n$ and at least one $(n+1)$-tuple not imbeddable in $E_{n-1}$.*

With this the principal objectives of the *Zweite Untersuchung* are attained. The reader is referred to *Distance geometries* for a discussion of the results of W. A. Wilson who, attacking the space problem directly, obtained a characterization theorem in terms of the imbedding of *quadruples* instead of $(n+2)$-tuples, as well as for the work of Aronszajn bearing upon the space problem.[9] By weakening Wilson's four-point property to require merely that each quadruple containing a linear triple be imbeddable (*weak four-point property*) the writer allowed the existence of *free* quadruples. Very recently B. E. Gillam characterized the $E_3$ by showing that a complete, convex, externally convex metric space with the weak four-point property, which contains at least one quadruple with non-vanishing determinant $D$, while $D$ vanishes for each five points of the space, *is logically equivalent to* $E_3$ [4]. This was accomplished by proving all of Hilbert's postulates for $E_3$ on the basis of the above assumptions, with lines, planes, betweenness and congruence appropriately defined.

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[3] The convex circle is the circle (circumference) with shorter arc metric.

## PART II

**Foreword.** As indicated in the introduction, this part of the paper is primarily concerned with extensions rather than consolidations. I propose to take up some recently established imbedding and characterization theorems, along with pertinent notions and concepts, and to raise certain questions suggested by them. No attempt is made to catalogue all of the new results; on the contrary, I limit myself to the discussion of the very few with which I have been connected.

**1. Concerning quasi-congruence order and free m-tuples.** We have seen that one of the important metric properties of the $E_n$ is its possession of quasi-congruence order $n+2$ with respect to semimetric spaces; that is, any semimetric space of more than $n+3$ distinct points is congruently contained in the $E_n$ whenever *each* $n+2$ of its points are. Several kinds of inquiries are suggested by this interesting concept.

(I). It has been observed in Part I that the requirement featuring the notion of congruence order $n+3$ (namely, that *each* $(n+3)$-tuple be imbeddable in $E_n$) has been weakened by allowing the presence of free $(n+3)$-tuples (that is, $(n+3)$-tuples which are not assumed congruently imbeddable in $E_n$) and the question arises whether a similar kind of weakening might be allowable here. Is the imbedding still valid if a certain number of the $(n+2)$-tuples are free? If so, how many of the $(n+2)$-tuples might be taken as free?

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[9] It should be noted, however, that Wilson's work is not entirely independent of the subset problem for it utilizes Menger's theorem on the imbedding of $n+1$ points in $E_n$.

In answering these questions it is first of all clear that it does not suffice to require merely that each $(n+2)$-tuple containing a selected set of $n+1$ independent points be imbeddable in $E_n$—a type of weakening immediately suggested by what proved to be effective in the case of congruence order. If $n=1$, for example, this demands only that every triple containing a given pair of distinct points be linear (that is, imbeddable in $E_1$). That the linearity of a space does not follow from this demand is seen by considering the convex circle, which is not imbeddable in the $E_1$ though each of its triples containing a given pair of diametral points is linear. There is, however, an analogue of the previous type of weakening. It is given in the following theorem.

THEOREM 1.1. *Let $\Sigma$ be a semimetric space of at least $n+4$ distinct points. $\Sigma$ is congruently contained in the $E_n$, not the $E_{n-1}$, if and only if* (a) *$\Sigma$ contains a set $S$ of $n+2$ distinct points of which at least one $(n+1)$-tuple is not imbeddable in $E_{n-1}$, and* (b) *each $n+2$ points of $\Sigma$ with at least $n$ points in common with $S$ is imbeddable in $E_n$.*

PROOF. Let $p_1, p_2, \cdots, p_{n+1}, p_{n+2}$ denote the points of $S$, with the first $n+1$ not imbeddable in $E_{n-1}$. Using (b) it follows that these $n+1$ points are imbeddable in $E_n$. If, now, $p_1, p_2, \cdots, p_{n+1}, r, s$ is any $(n+3)$-tuple of distinct points containing the independent $n+1$ points, consider the $n+4$ points formed by annexing the point $p_{n+2}$. (The case where either $r$ or $s$ is $p_{n+2}$ causes no difficulty.) By (b) each $n+2$ of these $n+4$ points is congruently contained in $E_n$, and hence, by the quasi-congruence order $n+2$ property of $E_n$, the $n+4$ points are imbeddable in $E_n$. Thus each $(n+3)$-tuple containing the independent $(n+1)$-tuple $p_1, p_2, \cdots, p_{n+1}$ is imbeddable in $E_n$ and it follows that the whole space is congruently contained in $E_n$.

An example shows that this theorem is no longer valid if the assumption that at least one $(n+1)$-tuple of $S$ be not imbeddable in $E_{n-1}$ is not made, suppressing at the same time, of course, the requirement that $\Sigma$ be not imbeddable in $E_{n-1}$. Let $p_1, p_2, p_3$, form a plane equilateral triple and label the in-center $p_4$ and $p_5$. The semimetric space $\Sigma$ whose points are $p_3, p_4, p_5$ together with the points of the straight line joining $p_1$ and $p_2$, with all distances euclidean *except the distance $p_4p_5$*, which is defined to be the radius of the circumcircle of $p_1, p_2, p_3$, contains a subset $S$ of four distinct points ($p_1, p_2, x, y$, where $x, y$ are any two points of the line other than $p_1, p_2$) such that every four points of $\Sigma$ with at least two points in common with $S$ are imbeddable in the plane $E_2$. But the space $\Sigma$ is obviously not imbeddable in $E_2$.

A more striking way of lifting the demand that every $(n+2)$-tuple be imbeddable in $E_n$ is given by the following theorems.

THEOREM 1.2. *Let $\Sigma$ be semimetric with at least $n+k+4$ distinct points. Then $\Sigma$ is imbeddable in $E_n$ if and only if $\Sigma$ contains at most $k$ free $(n+2)$-tuples.*

PROOF. We shall show that each $(n+2)$-tuple of $\Sigma$ is congruently contained in $E_n$.

Let, then, $p_1, p_2, \cdots, p_{n+2}$ be a free $(n+2)$-tuple. Since there are at most $k$ free $(n+2)$-tuples and at least $n+k+4$ points in $\Sigma$ it is readily seen that this $(n+2)$-tuple is contained in an $(n+5)$-tuple which has this $(n+2)$-tuple as its only free set of $n+2$ points. If $p_1, \cdots, p_{n+5}$ is such an $(n+5)$-tuple, each of the three sets of $n+4$ of these points obtained by omitting $p_n, p_{n+1}, p_{n+2}$ in turn have all $(n+2)$-tuples imbeddable in $E_n$, and hence the three $(n+4)$-tuples are themselves imbeddable in $E_n$. It follows that the rank of the determinant $D$ of the $n+5$ points is at most $n+2$, and so $D(p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2})=0$.

Now each $n+1$ points of $\Sigma$ are imbeddable in $E_n$, for each $(n+1)$-tuple is contained in at least $k+3$ $(n+2)$-tuples and not all of these are free since, by hypothesis, $\Sigma$ contains at most $k$ free $(n+2)$-tuples. Thus, each $n+1$ of the points $p_1, p_2, \cdots, p_{n+2}$ is imbeddable in $E_n$. Since the determinant $D$ of these $n+2$ points is zero, it follows that the points are imbeddable in $E_n$, and the theorem is proved.

The same kind of argument yields the next theorem.

THEOREM 1.3. *Let $\Sigma$ be semimetric with power exceeding $\aleph_0$. Then $\Sigma$ is congruently contained in $E_n$ if and only if $\Sigma$ has at most $\aleph_0$ free $(n+2)$-tuples.*

(II). The literature contains only two proofs of the quasi-congruence order $n+2$ property of the $E_n$, both of which are quite lengthy. A direct elementary proof is desired. Since the property is easily formulated in terms of determinant or quadratic form theory, attempts have been made to establish the result purely algebraically. For $n=1$, not a trivial case, a very simple proof has been found by the writer, but in general such efforts have so far not been effective. Since the complex space $K_2$ does not have quasi-congruence order 4 with respect to the class of complex distance spaces, a proof of the property for the $E_n$ must depend upon the reality of the field from which the distance elements are selected and hence is likely to be difficult. On the other hand, the writer has used algebraic methods to materially shorten the two earlier proofs. The question is, it seems, still open.

(III). It would be highly desirable to axiomatize the notion of quasi-congruence order or at least to express it in terms of more elementary properties of the space. A start in this direction has been made by C. V. Robinson and the writer by showing that for *complete, convex, externally convex metric spaces quasi-congruence order three (with respect to the class of semimetric spaces) is equivalent to the absence of equilateral triples of distinct points*. This was done by establishing the following very simple metric characterization of the line [3].

THEOREM 1.4. *A complete, convex, externally convex metric space, with at least two points, is a straight line if and only if it does not contain an equilateral triple of distinct points.*

This theorem has as immediate corollaries characterizations of the line due to Lindenbaum and to Menger.

More recently the writer showed that *a line segment is characterized among all compact and convex metric spaces by the absence of equilateral triples.*

Quasi-congruence order $n+2$ readily implies the absence of equilateral $(n+2)$-tuples, but not conversely. The half-line, for example, has neither equilateral triples nor quasi-congruence order three, and the $K_2$ has neither equilateral quadruples (with nonzero side) nor quasi-congruence order 4. *What is a large class of spaces for which the two notions are logically equivalent?*

In this connection, I would like to raise the question of interpreting *topologically* the absence of equilateral $k$-tuples. Biedermann has shown that a connected metric space in which each triple is linear (that is, a space *strongly* without equilateral triples) is homeomorphic with a line, a ray, or a segment. It was in fact this result that motivated Menger's characterization of euclidean subsets. (Menger showed that the homeomorphism of Biedermann is actually a congruence.)

If, now, one supposes merely that a connected metric space is without equilateral triples, what topological properties of the space ensue? *Is a metric Peano continuum without equilateral triples an arc?* If this is indeed the case (and I conjecture that it is) a topological property is given an elegant metric characterization.[10] Such problems are part of a general *program* which seeks to determine *what topologi-*

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[10] The writer has now established this conjecture. See *New characterizations of segments and arcs*, Proc. Nat. Acad. Sci. U. S. A. vol. 29 (1943) pp. 107–109.

*cal properties a space must possess in order that it may be homeomorphic to a space with given metric properties.*

2. **Congruence indices.** Increasing study of characterization problems has yielded results that the concepts of congruence and quasi-congruence orders are inadequate to describe. To meet this situation the notion of relative congruence indices was formulated [1]:

*Let* $\Gamma$ *and* $\Gamma^*$ *be two given spaces and* $\{\Delta\}$ *a given class of spaces. The space* $\Gamma$ *has* $\Gamma^*$*-relative congruence indices* $(n, k)$ *with respect to the class* $\{\Delta\}$ *provided any space* $\Delta$ *of the class with more than* $n+k$ *distinct points is imbeddable in* $\Gamma$ *whenever each* $n$ *of its points (not necessarily distinct) is imbeddable in* $\Gamma^*$.

The space $\Gamma^*$ is called $\Gamma$-*catalytic* of indices $(n, k)$ with respect to $\{\Delta\}$ since it (in general) facilitates the desired reaction of imbedding $\Delta$ in $\Gamma$ without participating in the result, for $\Delta$ is not necessarily imbeddable in $\Gamma^*$. If $\Gamma \equiv \Gamma^*$ the indices $(n, k)$ are called *congruence indices* of $\Gamma$ with respect to the class $\{\Delta\}$.

It is easy to see that congruence and quasi-congruence orders correspond to congruence indices $(k, 0)$ and $(k, 1)$, respectively, and that if $\Gamma$ has $\Gamma^*$-relative congruence indices $(n, k)$, then $\Gamma$ has $\Gamma^*$-relative indices $(n', k')$ if $n \leqq n'$ and $n+k \leqq n'+k'$, the class of comparison spaces being fixed. We order these indices lexicographically, and call those indices $(n, k)$ "best" which are not preceded by any indices $(n', k')$. It is important to observe that this ordering is *not* that of logical implication, for with respect to the latter criterion the indices do not form an ordered set. Thus a two-dimensional spherical cap of radius less than $\pi r/2$ has, with respect to the class of all semimetric spaces, congruence indices (4, 2) and also indices (5, 0), but neither set logically implies the other.

Keeping the comparison class of spaces fixed, let the best congruence indices of a space $\Gamma$ be $(n, k)$, with $(n', k')$ the best $\Gamma^*$-relative congruence indices. The space $\Gamma^*$ is *strongly* or *weakly* catalytic according as $n' < n$ or $n' = n$, $k' < k$, respectively. In case the indices $(n', k')$ are no better than $(n, k)$, $\Gamma^*$ is *formally* catalytic, while if $n' < n$ and $k' < k$, $\Gamma^*$ is *perfectly* catalytic. An open hemisphere of $S_{2,r}$, for example, is a strongly catalytic space for $S_{2,r}$, since by its use the best congruence indices (5, 0) of $S_{2,r}$ with respect to semimetric spaces are bettered by the hemisphere-relative congruence indices (4, 2). In fact, it has been shown that the $S_{n,r}$[11] has $\Gamma^*$-relative congruence indices $(n+2, n)$ with respect to all semimetric spaces, where $\Gamma^*$ denotes an open ($n$-dimensional) hemisphere of the $S_{n,r}$.

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[11] $S_{n,r}$ denotes the $n$-dimensional surface of a sphere of radius $r$ in $E_{n+1}$, with geodesic (shorter great circle arc) metric.

It is worth noting that pseudo-euclidean spaces fall under the notion of catalytic spaces. Clearly every subset of a catalytic space is catalytic (with the same indices) but the interest lies in enlarging rather than depleting these spaces, and the problem of finding saturated catalytic spaces of given indices arises.

The writer recently proposed the problems of finding the best congruence indices of the (closed) $n$-dimensional hemisphere and of small $n$-dimensional spherical caps with respect to (1) the class of semimetric spaces and (2) the set of all subsets of the containing $S_{n,r}$. The second inquiry led to posing the following question: *for what integer $k$ does the intersection of each $k$ members of a family of convex subsets of the $n$-dimensional spherical surface $S_{n,r}$ imply the existence of a common point for the family?*

In his Missouri dissertation, C. V. Robinson investigated these problems [9]. He showed that the best congruence indices of the $n$-dimensional hemisphere with respect to semimetric spaces are $(2n+1, 1)$. Even with respect to subsets of the containing $S_{n,r}$ these indices cannot be bettered. It follows that any subset $S$ of $S_{n,r}$ is coverable by a hemi-$S_{n,r}$ provided $S$ contains more than $2n+2$ distinct points and each $2n+1$ of its points is coverable by the hemi-$S_{n,r}$.

Of particular interest is the plane analogue of a cap theorem to the effect that a circular disc will cover a plane set $P$ if and only if each three points of $P$ are coverable by the disc. This behavior of the circular disc is all the more striking when contrasted with that of other convex subsets of the plane such as the square or elliptical disc. *For neither of these figures does any such integer $k$ exist*; for example, for any integer $k$ however large there are plane sets not coverable by a square (elliptical) disc even though each $k$ points of the set are so coverable. The same is true for any portion of the plane bounded by a broken-line curve. The possession of indices (3, 0) with respect to subsets of the plane has been made the basis of a characterization of the circular disc [8].

THEOREM 2.1. *The circular disc is the only connected, simply connected domain (closure of a bounded open set) of the plane with congruence indices* (3, 0) *with respect to plane sets.*

It would be highly desirable to make an exhaustive classification of plane sets according to their congruence indices with respect to subsets of the plane. The problems involved seem quite difficult and only the beginnings of a systematic investigation have been made. With respect to the class of linear subsets we have a partial enumeration of linear sets with fixed congruence orders (finite, hyperfinite, and transfinite).[12] Investigations aimed at completing these results and developing a similar theory for plane subsets are now in progress.

In this connection interest is attached to a new type of problem recently considered which I illustrate here for the simplest case. It is easily seen that corresponding to each integer $k$ there exists a bounded and closed subset of $E_1$ with best (finite) congruence order exceeding $k$. Thus if $P$ is an arbitrary bounded and closed linear set, one may not assert that any linear set $Q$ is imbeddable in $P$ whenever each $k$ of its points are, even for $k$ arbitrarily large. Note that, as is usual in imbedding theorems, the condition "every $k$ points of $Q$ are imbeddable in $P$" is a *unilateral* one. Suppose, on the other hand, it is also assumed that *every $k$ points of $P$ are imbeddable in $Q$*. Then one easily proves that $P$ and $Q$ are congruent when $k=4$; that is, *if $P$ and $Q$ are linear sets with $P$ or $Q$ bounded and closed, then $P \approx Q$ if and only if each four points of $P$ are imbeddable in $Q$ and each four points of $Q$ are imbeddable in $P$*. This bilateral type of condition gives rise to several new problems and causes the notion of congruence order (based upon unilateral conditions) to be viewed in a somewhat different light.

In answering the question raised concerning intersections of convex subsets of $S_{n,r}$, the following theorems were obtained.[13]

THEOREM 2.2. *If each $n+k+2$ members of a family of convex subsets of the sphere $S_{r,r}$ intersect, and if one member contains no $S_{k,r}$ $(0 \leq k \leq n)$, then there is a point common to all.*

THEOREM 2.3. *A family of more than $2n+2$ convex subsets of $S_{n,r}$ has a common point if each $2n+1$ of them intersect.*

Additional theorems are obtained when the diameters of the convex subsets are subject to certain restrictions.

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[12] A space $\Sigma$ has hyperfinite congruence order with respect to a class of spaces $\{\Delta\}$ if and only if each member of the class is imbeddable in $\Sigma$ whenever each of its finite subsets are. Thus hyperfinite congruence order is intermediate between finite and transfinite congruence orders. Each self-compact linear set has hyperfinite congruence order with respect to semimetric spaces; examples have been given of linear sets with hyperfinite and no finite congruence order, and transfinite but not hyperfinite congruence order. The complement of a finite or denumerable linear set has best congruence order $c$ (the power of the continuum). These notions were introduced in the Missouri dissertation of C. V. Robinson.

[13] A subset of $S_{n,r}$ is convex provided it is a product of hemispheres. This definition rules out the whole $S_{n,r}$ and includes a pair of diametral points. Thus from Theorem 2.2 it follows that a family of convex subsets of $S_{n,r}$ has a common point whenever each $2n+2$ of them intersect.

Intersection theorems for convex subsets of $E_n$ were obtained by Helly. Due to the variety of results such problems on the sphere turn out to be more interesting than their analogues in euclidean spaces. In contrast, for example, to the Helly theorem in $E_2$ which states that any family of convex subsets has a non-empty product whenever each three members of the family intersect, it is found that a family of convex subsets of $S_{2,r}$ have a common point if each *six* have or (if the family has more than six members) if each *five* have, or (if one member of the family contains no diametral point-pairs) if each *four* have. If each of the convex subsets has spherical diameter less than $2\pi r/3$ it suffices for each three of the sets to have a common point. It is clear how these results may be translated into "covering" theorems for spherical caps.

3. **$\delta$-supplementation and elliptic space.** If $\Sigma$ is semimetric with finite diameter $d$, denote the set of ***unordered*** pairs of its elements by $S^2$, and let $S^2 = S_1 + S_2$, $S_1 \cdot S_2 = 0$, be any decomposition of $S^2$ into two mutually exclusive subsets. Corresponding to such a decomposition, a real, non-negative function $F(x)$ is defined:

$$F(x) = x, \qquad x = pq;\ (p, q) \in S_1,$$

$$F(x) = \delta - x, \qquad x = pq;\ (p, q) \in S_2,$$

where $\delta$ is a fixed real number not less than $d$. Transforming metrically the space $\Sigma$ by this function $F$ yields a space which is in general not semimetric.[14] Nor does it necessarily become semimetric upon identification of those points $p$, $q$ (and only those) with $pq^* = F(pq) = 0$, for such an identification is justified (keeping to a single-valued metric) if and only if for each element $r$ of the space $pr^* = qr^*$.

If, however, $\delta > d$, such an identification does convert the transformed space into a semimetric one whose point set may, on account of the identification, differ from that of $\Sigma$. We denote such a space by $\Sigma^*$ and refer to it as a $\delta$-supplement of $\Sigma$ (more precisely, *the* $\delta_F$-supplement of $\Sigma$). We write $\Sigma^* = \sup_\delta \Sigma$ when there is no need to specify the function $F$.

An important example of a $\delta$-supplement is obtained by taking the

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[14] A space $F(\Sigma)$ which arises from $\Sigma$ by replacing the metric $pq$ of $\Sigma$ by $F(pq)$, where $F$ is a real, single-valued, non-negative function defined for every $x = pq$, $p, q \in \Sigma$, is called the metric transform of $\Sigma$ by $F$. A study of metric transforms was begun by the writer in 1934 (Ergebnisse eines der Mathematischen Kolloquiums, Wien, no. 7, pp. 8–10). Metric transforms of euclidean spaces into subsets of Hilbert space have been extensively investigated by Schoenberg and von Neumann (1938, 1942).

$n$-dimensional convex spherical surface $S_{n,r}$ as $\Sigma$, and transforming it metrically by the function $F$.

$$(\dagger)\qquad \begin{aligned} F(x) &= x, & x &= pq;\ pq \leqq \pi r/2,\\ F(x) &= \pi r - x, & x &= pq;\ pq > \pi r/2. \end{aligned}$$

(Here $\delta = d$.) The resulting semimetric space is the *$n$-dimensional elliptic space $\mathcal{E}_{n,r}$ with space constant $r$.*

In work still in progress, the notion of $\delta$-supplementation is made the basis for characterizing subsets of $\mathcal{E}_{n,r}$ among all semimetric spaces [2]. Elliptic space differs profoundly (both metrically and topologically) from all spaces heretofore treated and these differences make the older methods either inapplicable or extremely tedious. Fundamental in our work is the following theorem.

THEOREM 3.1. *A semimetric $\Sigma$ is congruently contained in $\mathcal{E}_{n,r}$ if and only if* (i) *no distance in $\Sigma$ exceeds $\pi r/2$ and* (ii) *there exists at least one function $F$ such that the $\delta_F$-supplement of $\Sigma$ is imbeddable congruently in $S_{n,1}$, where $\delta = \pi r$.*

PROOF. Let $\Sigma$ be congruently contained in $\mathcal{E}_{n,r}$. Then evidently $pq \leqq \pi r/2$ for each pair $p$, $q$ of elements of $\Sigma$. Denoting the metric transform of $S_{n,r}$ by the function $F$ defined in (†) by $\sup_{\pi r} S_{n,r}$, we have $\mathcal{E}_{n,r} = \sup_{\pi r} S_{n,r}$. Evidently

$$\text{subset } \mathcal{E}_{n,r} = \text{subset}(\sup_{\pi r} S_{n,r}) = \sup_{\pi r}(\text{subset } S_{n,r}),$$

and hence there exists a sup(subset $\mathcal{E}_{n,r}$) which is a subset of $S_{n,r}$. Since, now, $\Sigma$ is congruently contained in $\mathcal{E}_{n,r}$, there is a subset of $\mathcal{E}_{n,r}$ congruent with $\Sigma$, and, by the above, a supplement of this subset is a subset of $S_{n,r}$. Thus, a supplement of $\Sigma$ is imbeddable in $S_{n,r}$ and the necessity is proved.

On the other hand suppose that $\Sigma$ is semimetric with distances not exceeding $\pi r/2$ and that some $\pi r$-supplement of $\Sigma$ is congruently contained in $S_{n,r}$. Denote any such supplement by sup $\Sigma$. Then there exists a subset $S'_{n,r}$ of $S_{n,r}$ such that sup $\Sigma$ is congruent with $S'_{n,r}$. Now $\sup_{\pi r} S'_{n,r} = \text{subset } \mathcal{E}_{n,r}$ and hence $\sup_{\pi r}(\sup \Sigma)$ is congruently contained in $\mathcal{E}_{n,r}$.

The proof is complete when it is shown that for every sup $\Sigma$, $\sup_{\pi r}(\sup \Sigma)$ is identical with $\Sigma$. To this end, denote distances in sup $\Sigma$ by $d(p, q)$ and in $\sup_{\pi r}(\sup \Sigma)$ by $d_{\pi r}(p, q)$. The same argument shows that the two pointsets and the two distance-sets are identical, for let $p = q$ in $\Sigma$. Then $pq = 0$. If $d(p, q) = 0$, then $d_{\pi r}(p, q) = 0$, while if $d(p, q) = \pi r$, then $d_{\pi r}(p, q) = 0$. Hence $p = q$ in $\Sigma$ implies $p = q$ in

$\sup_{\pi r}(\sup \Sigma)$. Finally, if $p$ and $q$ are distinct points of $\Sigma$, then $0 < pq \leqq \pi r/2$. If $d(p, q) = pq$ then, since $pq \leqq \pi r/2$, $d_{\pi r}(p, q) = pq > 0$; while if $d(p, q) = \pi r - pq \geqq \pi r/2$, then $d_{\pi r}(p, q) = \pi r - (\pi r - pq) = pq > 0$, and the theorem is proved.

By virtue of this theorem, imbedding in $\mathcal{E}_{n,r}$ is referred to imbedding $\pi r$-supplements of semimetric spaces in $S_{n,r}$. In any such supplement the distance $pq$ of two points of $\Sigma$ is either unchanged or replaced by $\pi r - pq$, and *hence* $\cos(d(p, q)/r)$ *differs from* $\cos(pq/r)$ *at most in sign*. Applying known imbedding theorems for the $S_{n,r}$ we have the following theorem.

THEOREM 3.2. *A semimetric m-tuple* $p_1, p_2, \cdots, p_m$ *is imbeddable in* $\mathcal{E}_{n,r}$ *if and only if* (i) $0 < p_ip_j \leqq \pi r/2$ $(i, j = 1, 2, \cdots, m; i \neq j)$ *and* (ii) *there exists an m-rowed symmetric square matrix* $\epsilon = (\epsilon_{ij})$, $\epsilon_{ij} = \pm 1$, $\epsilon_{ii} = 1$ $(i, j = 1, 2, \cdots, m)$ *such that the determinant* $|\epsilon_{ij} \cos(p_ip_j/r)|$ $(i, j = 1, 2, \cdots, m)$ *has rank not exceeding* $n+1$, *with all nonvanishing principal minors positive.*

A determinant (symmetric) satisfying the condition (ii) of the theorem is said to have *positive* $\epsilon$*-rank not exceeding* $n+1$.

Now it is easy to show that each separable semimetric space is congruently contained in $\mathcal{E}_{n,r}$ whenever each of its finite subsets has this property, and hence we have obtained necessary and sufficient conditions for the imbedding of any separable semimetric space in a given $n$-dimensional elliptic space. It is desirable, however, to establish reduction theorems which express the imbedding in terms of the congruent mapping of subsets of a fixed number of points. Such theorems hinge upon the following query.

QUERY. *What is the smallest integer k such that the determinant* $|\cos(p_ip_j/r)|$ $(i, j = 1, 2, \cdots, m)$ *has positive* $\epsilon$*-rank not exceeding* $n+1$ *whenever each of its principal minors of order k has positive* $\epsilon$*-rank not exceeding* $n+1$?

It is conjectured that $k = n+3$, but so far *this has been proved only for* $n = 1$.[13] Thus any separable semimetric space is imbeddable in $\mathcal{E}_{1,r}$ if and only if each four of its points have this property. This can be established in other ways, and without the hypothesis of separability. But for $n > 1$, the algebraic method discussed here appears to be the only feasible approach. The above conjecture means (if verified) that $\mathcal{E}_{n,r}$ has (like euclidean, hyperbolic and spherical spaces of $n$-dimensions) congruence order $n+3$. It would be interesting if the different topological properties of $\mathcal{E}_{n,r}$ would be made manifest by a difference in the congruence order.

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[13] Examples show that $k$ is not less than $n+3$.

4. **Concluding remarks.** In conclusion it should be observed that this paper has been concerned with completely *global* imbeddings and characterizations though interesting results in which local properties are assumed have also been obtained. A completely local type of theorem is one characterizing the line segment among all arcs of a metric ptolemaic space by the vanishing of the metrically defined Menger curvature at each point, while Gauss surfaces have been characterized among all compact and convex metric spaces by the existence at each point of a metrically defined surface curvature. A different ("mixed") type of imbedding theorem is expressed in terms of the local property of imbedding neighborhoods in $E_n$ together with the global condition that each $n+3$ points be congruently contained in some euclidean space.

It has not been possible to present here any of the applications of metric imbedding theorems to other domains. Metric methods have already been successfully applied to many parts of analysis and algebra as well as to geometry, and there is reason to believe that continued cultivation of this field will abundantly increase both the number and the importance of these applications.

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8. C. V. Robinson, *A characterization of the disc*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 818, 819.

9. ———, *Spherical theorems of Helly type and congruence indices of spherical caps*, Amer. J. Math. vol. 64 (1942) pp. 260–272.

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The University of Missouri

# THE FEBRUARY MEETING IN NEW YORK

The three hundred ninety-fourth meeting of the American Mathematical Society was held at Columbia University on Saturday, February 27, 1943. The attendance was approximately two hundred including the following one hundred sixty-seven members of the Society:

C. F. Adler, R. L. Anderson, T. W. Anderson, R. G. Archibald, L. A. Aroian, F. E. Baker, S. F. Barber, M. F. Becker, E. G. Begle, Stefan Bergman, Felix Bernstein, G. D. Birkhoff, Gertrude Blanch, W. M. Bond, C. B. Boyer, H. W. Brinkmann, A. B. Brown, Hobart Bushey, J. H Bushey, S. S. Cairns, R. H. Cameron, M. D. Clement, L. M. Comer, T. F. Cope, Richard Courant, J. E Crawford, H. B. Curry, M. D. Darkow, J. A. Daum, Norman Davids, A. S. Day, Jesse Douglas, Arnold Dresden, Nelson Dunford, J. E. Eaton, M. L. Elveback, Paul Erdos, J. M. Feld, W. K. Feller, A. D. Fialkow, Edward Fleisher, W. B. Fite, Tomlinson Fort, R. M. Foster, A. H. Fox, G. A. Foyle, Hans Fried, Orrin Frink, D. L. Fuller, R. E. Fullerton, H. P. Geiringer, B. H. Gere, David Gilbarg, B. P. Gill, A. M. Gleason, M. C. Gray, H. J. Greenberg, C. C. Grove, Laura Guggenbuhl, Margaret Gurney, F. C. Hall, N. A. Hall, H. K. Hammer, Gerald Harrison, K. E. Hazard, Olaf Helmer, Einar Hille, T. R. Hollcroft, A. P. Holló, M. W. Hopkins, Harold Hotelling, S. E. Hotelling, E. M. Hull, Witold Hurewicz, W. H. Ingram, Nathan Jacobson, R. A. Johnson, H. A. Jordan, William Karush, Edward Kasner, S. C. Kleene, J. R. Kline, E. A. Knobelauch, B. O. Koopman, Mark Kormes, Arthur Korn, M. E. Ladue, Solomon Lefschetz, Marguerite Lehr, B. A. Lengyel, D. C. Lewis, Marie Litzinger, E. R. Lorch, Lee Lorch, A. N. Lowan, N. H. McCoy, R. S. McKee, C C. MacDuffee, H. B. Mann, C. W. Mathews, A. E. Meder, Karl Menger, E. C. Molina, L. W. Miller, H. L. Mintzer, R. E. von Mises, A. K. Mitchell, Deane Montgomery, Parry Moon, Richard Morris, E. J. Moulton, F. J. Murray, C. A. Nelson, Otto Neugebauer, E. N. Nilson, C. O. Oakley, A. F. O'Neill, Oystein Ore, J. C. Oxtoby, E. L. Post, A. L. Putnam, L. L. Rauch, Moses Richardson, R. G. D. Richardson, C. E. Rickart, J. F. Ritt, E. K. Ritter, P. C. Rosenbloom, Raphael Salem, Arthur Sard, Henry Scheffé, I. J. Schoenberg, Alberta Schuettler, I. E Segal, Seymour Sherman, Max Shiffman, James Singer, P. A. Smith, D. E. Spencer, V. E. Spencer, E. R. Stabler, W. J. Strange, W. C. Strodt, M. M. Sullivan, J. D. Tamarkin, Peter Treuenfels, A. W. Tucker, J. W. Tukey, D. F. Votaw, L. I. Wade, Abraham Wald, G. L. Walker, Wolfgang Wasow, Warren Weaver, André Weil, Louis Weisner, Hermann Weyl, George Whaples, A. P. Wheeler, D. V. Widder, V. A. Widder, John Williamson, H. P. Wirth, Jacob Wolfowitz, Alexander Wundheiler, E. H. Zarantonello, Antoni Zygmund.

The meeting opened at 10 A.M. in two sections, one for papers in Applied Mathematics, Professor J. W. Tukey presiding, and one for papers in Analysis, Vice President C. C. MacDuffee presiding. At the general session immediately following these sections, Professor Deane Montgomery gave an address entitled *Transformation groups and spheres*. This address had been originally scheduled for the 1942 Annual Meeting which was cancelled because of transportation difficulties. Vice President J. D. Tamarkin presided at this session.

The afternoon program began at 2 P.M. with a general session at which Professor Nathan Jacobson gave an address entitled *Some topics in the theory of semi-linear transformations*. At the close of this address, the presiding officer, Vice President C. C. MacDuffee, asked all to rise in memory of two well known mathematicians who had died recently, E. R. Hedrick and David Hilbert.

At 3:15 P.M. there were two sections, one for papers in Algebra and one for papers in Geometry and Statistics. Professors E. R. Lorch and E. J. Moulton were the presiding officers.

The Council met at 12:30 P.M. in the Men's Faculty Club of Columbia University.

The Secretary announced the election of the following fourteen persons to membership in the Society:

Mr. Louis Fein, Earlham College, Earlham, Ind.;
Professor Fred George Fender, Rutgers University;
Mr. N. Knoll Hersh, Tulane University;
Mr. Clarence Thomas Hill, Ponca City, Okla.;
Professor Carl Holtom, Purdue University;
Mr. Morris E. Levenson, Cooper Union, New York, N. Y.;
Mr. Robert Theodore Luginbuhl, University of Pennsylvania;
Professor Glen Treat Miller, Purdue University;
Professor Joseph Edward Morton, Knox College;
Mr. Kenneth Holden Murphy, West Virginia University;
Mr. Frederick Valentine Pohle, College of Engineering, New York University;
Mr. Edward John Specht, Washington University;
Mr. Allen Frederick Strehler, Ohio State University;
Mr. Joseph Simon Vigder, University of Saskatchewan.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

University of California: Messrs. Henry Ludwig Alder, Mark W. Eudey, and Herbert Federer, Misses Evelyn Fix and Mildred Lucille Gerini, Messrs. Eric Leo Lehmann and Robert Russell.
Cornell University: Messrs. Clifford D. Firestone and Alfred Milton Peiser.
Harvard University: Messrs. Richard F. Arens, Robert Creighton Buck, Lindley James Burton, Milton Frank Chauner, Carl Cohen, and James Bruce Crabtree, Miss Ellen Elizabeth Fedder, Messrs. Jaime Lifshitz, Roger Conant Lyndon, and Pesi Rustom Masani, Dr. Eduardo Hector Zarantonello.
Johns Hopkins University: Mrs. Ruth Bari.
University of Kentucky: Miss Elsie Temple Church.
Lehigh University: Messrs. James Levert Howell and Charles Miesse.
Northwestern University: Mr. Richard Lence, Miss Mary Ellen Patno.
Pennsylvania State College: Mr. Israel Harold Rose, Mrs. Alice Margaret Crawford Thorpe.
University of Toronto: Messrs. William James Robert Crosby and Bernard Noonan.
Wayne University: Miss Jane Cronin.
University of Wisconsin: Messrs. Richard V. Andree and Wulf Gotz.
Yale University: Mr. John George Pocock.

The following appointments by the President were reported: as auditors of the Society's books for 1943, Professors Samuel Borofsky and A. E. Meder, Jr.; as Chairman of the Committee on Places of Meetings for the period 1943–1945, Professor W. B. Carver; as a member of the Committee on Places of Meetings for 1943, Professor C. R. Adams; as a member of the Committee on the Award of the Bôcher Prize, Professor Emil Artin; as Committee on Arrangements for the 1943 Summer Meeting at Rutgers University, Professors Richard Morris (Chairman), T. R. Hollcroft, B. W. Jones, A. E. Meder, Jr., C. A. Nelson, and E. P. Starke; as members of the War Policy Committee (joint committee with the Mathematical Association of America), Professors M. H. Stone (Chairman), W. D. Cairns, G. C. Evans, L. M. Graves, Marston Morse, Dr. Warren Weaver, and Professor G. T. Whyburn; as members of a subcommittee of the War Policy Committee, to conduct a bureau of available teachers of mathematics for wartime needs, Professors W. D. Cairns, Arnold Dresden, and J. R. Kline; as Committee on Nomination of Officers and Members of the Council for 1944, Professors E. W. Chittenden (Chairman), E. T. Bell, Saunders MacLane, R. L. Moore, and Dean R. G. D. Richardson.

The Secretary reported that the following had been chosen as the voting representatives of editorial committees on the Council for the year 1943: Bulletin—Dean Tomlinson Fort; Transactions—Professor Einar Hille; Colloquium—Professor G. T. Whyburn; American Journal of Mathematics—Professor Hermann Weyl; Mathematical Reviews—Professor Otto Neugebauer. He also reported that Professor G. T. Whyburn had been selected Chairman of the Colloquium Editorial Committee for 1943.

The Council voted to cancel the meeting scheduled for June 16–17, 1943, at Oregon State College, in connection with the meetings of the Pacific Division of the American Association for the Advancement of Science.

The Council approved September 11–13 as the dates for the 1943 Summer Meeting at Rutgers University.

The Council adopted the following resolution on the death of Professor E. R. Hedrick:

> The Council of the American Mathematical Society records its deep sense of loss in the death on February 3, 1943, of Earle Raymond Hedrick. As an active member of the Society during four decades of its unprecedented growth and development, he made contributions which were great in number and varied in character. He gave abundantly of his time, thought, and energy to the Society and served it in the official capacities of Council member, Trustee, Vice President, President, and Editor-in-Chief of the Bulletin. Through membership on many important committees, both

within the Society and outside of it, he labored unceasingly toward the advancement of the interests and prestige of mathematics at all levels. His activities in the Mathematical Association of America, The National Council of Teachers of Mathematics, and in numerous engineering and other scientific societies were extensive and outstanding to a degree hard to comprehend in view of his heavy involvement in Society and other responsibilities. Professor Hedrick had a rare combination of broad interests, outstanding skill at logical and thorough analysis, good judgment, and ability to work effectively with other people. These characteristics invariably singled him out and placed him in a position of leadership.

The Secretary presented a report from President Stone, Chairman of the War Policy Committee, which had its first meeting in Philadelphia on February 21, 1943. Members of the Committee and officers of the Society and Association are maintaining effective contacts with various government agencies, with a view to initiating and accelerating actions which make the best use of mathematically trained and qualified personnel under the Army and the Navy Specialized Training Programs and in research connected with the war. The Committee has recommended that every effort should be made to continue scientific meetings, including those arranged on a national scale.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers numbered 1 to 5 were read in the section for Applied Mathematics, papers 6 to 11 in the section for Analysis, papers 12 to 16 in the section for Algebra, and papers 17 to 22 in the section for Geometry and Statistics; papers 23 to 35 were read by title. Paper 9 was read by Professor Cameron, paper 14 by Dr. Lengyel, paper 19 by Professor Kasner. Mr. Ettlinger was introduced by Professor R. L. Moore, and Dr. Schwerdtfeger by Dr. Peter Scherk.

1. H. J. Greenberg: *Application of a summability method in solving boundary value problems.* (Abstract 49-3-118.)

2. Hilda P. Geiringer: *New convergence cases for iteration methods applied to linear equations.* (Abstract 49-1-62.)

3. Arthur Korn: *On vibrational vortices.* (Abstract 49-1-67.)

4. Stefan Bergman: *A formula for the stream function in compressible fluid flow.* (Abstract 49-3-116.)

5. A. H. Fox: *Integral representation of the flow of a compressible fluid around a cylinder.* II. (Abstract 49-5-156.)

6. Wolfgang Wasow: *On a boundary layer problem for a certain linear partial differential equation.* (Abstract 49-1-50.)

7. C. E. Rickart: *Decomposition of additive set functions.* (Abstract 49-1-99.)

8. W. C. Strodt: *Analytic solutions of nonlinear difference equations.* Preliminary report. (Abstract 49-1-45.)

9. R. H. Cameron and W. T. Martin: *An expression for the solution of a class of nonlinear integral equations.* (Abstract 49-3-110.)

10. N. A. Hall: *Confluence of the basic hypergeometric series.* (Abstract 49-1-31.)

11. Raphael Salem: *Sets of uniqueness and sets of multiplicity.* (Abstract 49-3-114.)

12. Gerald Harrison: *The structure of algebraic moduls.* Preliminary report. (Abstract 49-1-4.)

13. Alexander Wundheiler: *An algebraic definition of affine space.* (Abstract 49-1-20.)

14. Saunders MacLane and B. A. Lengyel: *Integral invariants of a tensor under the symmetry operations of the* 32 *crystal classes.*Preliminary report. (Abstract 49-1-8.)

15. André Weil: *Differentiation in algebraic number-fields.* (Abstract 49-1-19.)

16. Olaf Helmer: *An extension of the elementary divisor theorem.* (Abstract 49-1-5.)

17. Mary E. Ladue: *Conformal geometry of horn angles of higher order.* (Abstract 49-1-74.)

18. Henry Scheffé: *On solutions of the Behrens-Fisher problem, based on the t-distribution.* (Abstract 49-3-121.)

19. Edward Kasner and John DeCicco: *Generalized dynamical trajectories in space.* (Abstract 49-3-120.)

20. W. K. Feller: *On the general form of the so-called law of the iterated logarithm.* (Abstract 49-1-87.)

21. Domina E. Spencer: *The tensor representation of the figures of Study's "Geometrie der Dynamen."* (Abstract 49-1-79.)

22. Jesse Douglas: *Point transformations and isothermal families of curves.* II. (Abstract 49-5-160.)

23. E. F. Beckenbach: *On conjugate harmonic functions.* (Abstract 49-3-109-*t*.)

24. Joseph Bowden: *The quaternary permutation function and a generalization of Newton's binomial theorem and Vandermonde's permutation theorem.* (Abstract 49-3-104-*t*.)

25. Nathaniel Coburn: *Boundary value problems in plane plasticity.* Preliminary report. (Abstract 49-3-117-*t*.)

26. L. L. Dines: *On linear combinations of quadratic forms.* (Abstract 49-3-105-*t*.)

27. M. G. Ettlinger: *On irreducible continuous curves.* (Abstract 49-3-122-*t*.)

28. W. K. Feller: *On a probability limit theorem of H. Cramér.* (Abstract 49-1-86-*t*.)

29. K. O. Friedrichs: *On nonlinear vibrations of third order.* (Abstract 49-1-61-*t*.)

30. O. G. Harrold: *A decomposition theorem for certain compacta.* (Abstract 49-3-123-*t*.)

31. Einar Hille and Gabor Szegö: *Remarks on ergodic theorems.* (Abstract 49-5-138-*t*.)

32. Fritz John: *Instability of certain nonlinear vibrations.* (Abstract 49-1-65-*t*.)

33. H. Schwerdtfeger: *Identities between skew-symmetric matrices.* (Abstract 49-3-106-*t*.)

34. Gabor Szegö: *On the oscillation of differential transforms.* IV. *Jacobi polynomials.* (Abstract 49-3-115-*t*.)

35. J. W. T. Youngs: *On the additivity of the Lebesgue area.* (Abstract 49-1-56-*t*.)

T. R. Hollcroft,
*Associate Secretary*

# EARLE RAYMOND HEDRICK—IN MEMORIAM

On the morning of February 3, 1943, the announcement was made from the Rhode Island Hospital at Providence that Dr. Hedrick had just died. He had been a patient with an infected lung since early in November.

When he retired from the office of Provost and the Vice President of the University of California at Los Angeles he accepted a position as Visiting Professor of Mathematics at Brown University, and had just begun his duties there when he was stricken. The shock to the University can well be understood; it was felt throughout the academic world.

It is fitting that we pause to meditate on the life and work of this remarkable man.

Earle Raymond Hedrick was born at Union City, Indiana, September 27, 1876 of Dutch and German ancertry (Hetrig was the earlier spelling). The family migrated to Pennsylvania about 1670.

The principal biographical facts are given adequately in the *Semicentennial history of the American Mathematical Society*, 1888–1938, *with biographies and bibliographies of the past presidents*, by Professor R. C. Archibald, for 21 years librarian of the Society. This volume contains a complete list of his mathematical writings to that time, and much of the history of his contribution to the development of mathematics in America. The following statements are taken from it.

When the late Frank Nelson Cole retired from his various offices in the Society in 1921 an opportunity was afforded to make numerous fundamental changes in the organization and administration of the Society.

Hedrick was appointed editor-in-chief of the BULLETIN. This office he held until his appointment to that of Provost and Vice President of the University of California at Los Angeles in 1937. It was a period of unparalleled mathematical development in America, and with it were evolved many details in arrangement, style of type, etc., that have contributed in no small degree to the usefulness of the periodical, such as the listings of New Publications, and differentiating in the contents of odd and even numbers. In this process the size of each volume was practically doubled.

Simultaneously with this development in the make-up of the BULLETIN the standard for the acceptance of material became necessarily high, and the effective presentation of the complex material notable. All this required an enormous amount of personal attention to detail. The total effects of these efforts on the place and the tone of mathematics in America can hardly be overestimated.

In appreciation of this service the Council of the American Mathematical Society directed that volume 44 (1938) of the BULLETIN be dedicated to him. The frontispiece of the volume bears his portrait.

From his earliest years Dr. Hedrick always showed a keen interest in teaching mathematics from the preparatory school to the graduate school. His translation of Goursat's *Cours d'analyse* and (in collaboration with the late C. A. Noble) of the first third of Klein's *Elementarmathematik vom höheren Standpunkte aus* rendered great service to students and college teachers. In 1916 the Mathematical Association of America was founded and Hedrick was elected as its first president.

The effort to improve the quality of teaching was always in the foreground during his professorship at the University of Missouri. A notable contribution is the Series of Mathematical Texts, about forty volumes, of which he was general editor. In addition to secondary school and elementary college texts, this series includes several important works of a more advanced nature.

His interest in mechanics early resulted (1909) in the publication (with the late O. D. Kellogg) of the *Applications of the calculus to mechanics* which proved to be a particularly useful book for those teaching calculus to engineers. Besides his own contributions, he has also been editor of 34 volumes by various authors in the notable Engineering Science Series. Moreover, for five years he was secretary of Section A of A.A.A.S.

His services as administrative officer of the University of California at Los Angeles will be fittingly treated on another occasion.

During the last year he was active in the affairs of the School of Mechanics at Brown University.

In all his numerous and exacting duties, each performed with great patience and consumate skill, the main characteristic of this man was his consideration of human attributes. He was always interested, he was always sympathetic and he was always helpful. Probably no one has put a deeper imprint on the development of numerous lines mentioned here than he.

VIRGIL SNYDER

# BOOK REVIEWS

*Methods of correlation analysis*. By Mordecai Ezekiel. 2d edition. New York, Wiley, 1941. 19+531 pp. $5.00.

The first edition of this well known work appeared in 1930 and had a marked influence upon the users of correlation theory in this country. Before its appearance attention had been centered upon the calculation of correlation coefficients to many decimal places with too much reliance upon probable errors and too much faith in Blakeman's tests. Ezekiel's book emphasized the regression side of correlation with particular stress on nonlinear regression functions. The use of free hand fitting of regression curves in simple and multiple correlation is clearly and quite completely discussed. As a result of the simplicity of explanations and the careful description of calculation procedures, it soon became one of the most important books in the field of correlation.

The difference between the two editions is a survey of the history of the advances in correlation theory during the intervening decade. As stated in the preface of the second edition these major changes have been "first, in the interpretation of the meaning of standard errors and, second in the application of logical limitations to the flexibility of graphic curves. Other significant developments have been in the perfection of new and speedier methods of estimating the reliability of an individual estimate or forecast." Remaining portions of the subject matter are left practically the same.

In general these changes are well made. Treatment of the reliability of an individual forecast is given in Chapter 19. Probability statements arising in the interpretations of standard errors have been correctly made, but the author did not introduce the terminology of "confidence interval" and "fiducial limits." This would be advisable. In discussing logical limitations of graphic curves it is carefully noted that extrapolation is based on these logical considerations rather than the statistical analysis. Samples give most reliable information for the ranges of the variables included in the sample.

Since the general features of the first edition are so well known, it seems most important to mention here some of the detail in the new edition where particular comment is pertinent.

The method of identifying classes by open class limits is superior to the one used. For example (p. 5) change the notation from 22.5–25.4 bushels to 23–25 bushels. The latter form gives the lowest and highest values included in the class and indicates the accuracy of the measurements.

On p. 10 the text states "Most of the reports come at about the middle values and then thin out to both ends (that is, the distribution approximates normality). In such cases the standard deviation gives a measure of the range within which a definite proportion of the cases will be included." This is followed by a statement of the common area values for the normal curve of error. The author should explain that these area values are only approximate for non-normal distributions which satisfy the given conditions, and that normality requires a particular functional relationship, not merely a graphic similarity.

On p. 156 the following statement appears. "Since the index of determination is simply $\bar{\rho}^2_{yx}$, it is 71.0 per cent." According to the definition given it should be .710.

In another discussion involving the coefficient of determination (p. 159) the following statement is made. "Since the coefficient is a ratio, it is a "pure number," that is, it is an arbitrary mathematical measure, whose values fall within a certain limited range, and it can be compared only with other constants like itself, derived from similar problems." The arithmetic mean is a ratio, and it depends on unit! Furthermore not all ratios are arbitrary! The meaning of the quotation is vague. The point in question is an important one, and a clear explanation should be given.

On pp. 160–162 the author discusses the interpretation of the three types of measure of correlation; regression coefficient, correlation coefficient, and standard error of estimate. He states that the most accurate estimate of values of the dependent variable made from the regression equation calculated from a sample will be made from that sample for which $S_y$, the standard error of estimate, is the smallest. The accuracy of an estimate of $y$ should be interpreted with reference to the importance of a unit change in $y$. In a sample with small variation in the dependent variable two situations may be considered: (a) a small variation may be important in case the population itself has small variation, or (b) the dependent variable has been so controlled in sampling that the variation is materially biased more than the usual sampling bias. It is well known that sampling under case (b) tends to lower the correlation coefficient and thus to make the whole correlation analysis unreliable. In case (a) it is clear that $S_y$ must be interpreted in relation to $\sigma_y$. The factor $(1-r^2)^{1/2}$ measures the improvement in estimation of $y$ due to the use of regression, and the smaller the value of $r$, disregarding sign, the smaller is the amount of information supplied by the regression.

In introducing linear multiple regression (p. 164) it should be pointed out that the form of the equation assumes the relationship between the variables to be additive. The discussion given might lead the reader to believe that the linear equation includes all cases.

Statisticians interested in factor analysis will question the recommendation that not more than ten, usually five, variables be used in a study.

To obtain such quantities as $\sum XY$ and $\sum X^2$ it should be mentioned that the extensions as made in the text are not needed. An explanation of the use of a calculating machine to shorten the work might well be given in the text at an early point.

One rather general criticism of the book should be made. There is throughout the tendency to over-correct calculated constants and to over-refine tests of significance. More care should be taken to explain the fact that experimental data frequently do not justify the use of many of these refinements. To an untrained reader they may imply an accuracy of analysis not actually present. In some cases the significance tests suggested are actually incorrect, as for example the use of the standard error of the coefficient of multiple correlation.

In spite of the above criticisms the reviewer considers this book still to be the best in its field.

E. L. WELKER

*Finite dimensional vector spaces.* By Paul R. Halmos. (Annals of Mathematics Studies, no. 7.) Princeton University Press, 1942. 5+196 pp.

In this book the author presents the topics covered usually in an introductory course in algebra (matrices, linear equations, linear transformations, and so on) from the point of view of a modern analyst interested in general vector spaces.

The ever-growing interest in Hilbert and more general linear spaces makes the appearance of the book very timely, especially since it furnishes an excellent introduction to the subject certainly within the grasp of a first-year graduate student or even a good senior or junior.

The topics are treated in such a manner as to make future generalizations look both natural and suggestive. This sometimes is done at the expense of the shortness of exposition. Some theorems, as the author himself confesses, could be proved in fewer lines. He prefers, however, longer proofs that admit a generalization to shorter ones that do not.

The reviewer finds himself in complete agreement with this method of presentation, since in the long run it is more economical as far as "economy of thought" is concerned.

The book is divided into three chapters: Chapter I, Spaces, Chapter II, Transformations, and Chapter III, Orthogonality. There are also three appendices: on the classical canonical form, on the direct products, and on Hilbert space. This last appendix is just a short preview of what the reader will get into when he takes up further study.

The chapters are divided into sections, and this division was done with great care and didactical skill. The headings of the sections indicate by themselves the impressive variety of topics treated. These range from the relatively familiar ones like linear dependence (§§4, 5), linear manifolds (§9), or proper values of Hermitean and unitary transformations (§62), to those that are of a definitely less familiar and more difficult character like reflexivity of finite dimensional spaces (§15), polar decomposition (§67), or the ergodic theorem for unitary transformations (§76).

However, the reviewer doubts whether this book could be used successfully in a course on matrices and linear equations. The presentation is definitely that of an analyst and the "algebraic" point of view is purposely avoided. The author states in the preface that his purpose is " . . . to emphasize the simple geometric notions common to many parts of mathematics, and to do it in a language which gives away the trade secrets and tells the students what is in the back of the minds of people proving theorems about integral equations and Banach spaces."

The reviewer thinks that the author succeeded admirably in this respect.

MARK KAC

*The non-singular cubic surfaces*. By B. Segre. Oxford, The Clarendon Press, 1942. 11+180 pp. 63 figures. $4.50.

The extensive literature on the cubic surface would hardly justify further additions unless such additions contributed a novel approach. The novelty of the approach in the case of the volume under review justifies its inclusion.

A non-singular cubic surface $F$ and a triad of planes not belonging to a pencil form a pencil the surfaces of which remain non-singular in all the intermediate positions from $F$ to the triad of planes. Under these circumstances the lines on the surface are always distinct and preserve their incidence relations, thus giving rise to the group.

The book is divided into four chapters devoted to (1) the discussion of the lines and the configurations formed by them, (2) the group, (3) the properties of the surface under the restriction that it shall be real, with all possibilities of real and imaginary lines, the latter occurring in pairs, (4) the exceptional cases of the pentahedroid.

In the limiting triad of planes, intersecting by pairs in three lines, each of which meets $F$ in three points, the 27 lines are represented by joining these points in pairs. Consider a symbol $abc$, one of which one is zero, and each of the others may be 1 or 2, or 3, independently of each other. The numerical value and the position of each letter in the symbol has meaning. Consider the points 1, 2, 3 on each line $r_1$, $r_2$, $r_3$ in which two planes of the triad intersect. The position of $a$ refers to points on $r_1$, and so on. Thus, the symbol 012 is the (image) line having no point on $r_1$, having point 1 on line $r_2$ and the point 2 on $r_3$. Two lines $abc$ and $a'b'c'$ intersect when and only when these symbols have one and only one element as $a=a'$, or $b=b'$, or $c=c'$ alike. The configuration of the lines of $F$ can now be expressed in terms of simple combinatorial relations in these symbols and expressed graphically in axonometric projection. Thus, the following theorems are samples. They are immediate consequences of the symbolism.

Every line meets ten others.

Given a line $r$, and any two others, skew to it and to each other; then three other lines can be found having similar relations among themselves, but each new line meeting all the lines of the first group.

Given any line $r$, there are five others skew to it and to each other.

The properties of the double sixes and of the tritangent planes follow at once. A plane through a line $r$ meets $F$ in a conic which meets $r$ in two points of an involution, the double points of which on the three lines of any tangent plane lie by threes on the sides of a complete quadrilateral.

The second chapter discusses the group of the 27 lines from the property that incident relations are preserved. It is shown to have six independent generators, directly suggested from the graphical representation. It contains a conjugate set of 36 involutorial operations, called $\sigma$-transformations, each of which interchanges the conjugate lines of two complementary sextuples and leaves unaltered the remaining 15. The 15 pairs of parabolic points of the 15 lines of $F$ residual to a double six are conjugate with respect to a Schur quadric of the double six. The six pairs of parabolic points of the lines of any two complementary triplets lie on a space quartic curve of the first kind.

The third chapter—on real non-singular cubic surfaces—is the main theme discussed in the book; it contains about two-thirds of the entire volume. Consider the pencil $F+\lambda\phi=0$, wherein $\phi$ is a nodal cubic. The six lines of $F$ through the node replace the 12 lines of a double six; the incidence relations of the remaining 15 are those of $F$. The classification of surfaces $F$ into four distinct types, according to the number of pairs of conjugate imaginary lines, and the addition of one more type with 12 pairs of conjugate imaginary lines, now proceeds as in the Schläfli enumeration, supplemented by topological consideration of the space [19] of all cubic surfaces and the primal representing the discriminant of the surface. It is shown that these five types comprise all possible forms of non-singular real cubic surfaces. Each of these types is then considered in more detail, the graphical representation being applied to each, with special devices for representing the imaginary lines of the first and second kind. Expressed in this way, the configurations become obvious; the various kinds of double sixes are immediately apparent. The groups of the various types are considered further, featuring the differences between right-hand and left-hand lines as determined by the sequence of parabolic planes and the planes of composite residual section, as the plane turns about the line. This study is then applied to the configurations of the real sextuples of lines of each type, and finally to the topology of real cubic surfaces in connection with the real lines. With this preparation the form of the parabolic curve on each type is determined.

The last chapter is analytic. It is devoted to the Sylvester pentahedron and to those cubic surfaces invariant under homographic transformations. The general cubic surfaces can be expressed by an equation involving five cubes of linear forms in one and only one way. The only non-singular cubic surfaces for which there is a point $V$ such that all their sections by planes through $V$ have the same modulus are the cyclic surfaces having $V$ as center. Conversely, all the plane sections of the non-conical cyclic cubic surfaces are equianharmonic. A real cubic surface with 27 distinct real lines has always a well determined Sylvester pentahedron, consisting of 5 distinct real planes, no four of which are dependent.

Three short appendices appear at the end of the book, devoted respectively to the degeneration of a cubic surface into a plane and a quadric, the degeneration of a non-singular cubic primal of [4] into three primes, and on singular surfaces of a pencil of cubic surfaces.

The press-work is of the expected excellence of the Clarendon Press. The 63 figures are particularly good. The volume is strikingly free of typographical errors.

VIRGIL SNYDER

## NOTES

A copy of G. N. Watson's *Theory of Bessel's Functions* is urgently needed in connection with important war research. Anyone who has a copy he would be willing to sell, or to lend for the duration, is requested to communicate with Dr. Warren Weaver, Chief of the Applied Mathematics Panel, National Defense Research Committee, Room 5500, 49 West 49th Street, New York, N. Y.

Nature reports a tercentenary celebration of Isaac Newton in Great Britain in January. Professor Max Born of the University of Edinburgh was chairman of the celebration and the program included an address by Professor E. T. Whittaker.

The Association for Symbolic Logic has announced the election of the following officers for the year 1943: President, Professor C. H. Langford; Vice President, Professor C. A. Baylis; Secretary-Treasurer, Miss Helen C. Brodie; Members of Executive Committee, Professor Arnold Dresden and Professor F. B. Fitch; Member of Council, Dr. A. M. Turing.

The 1942 Lamme Medal of the American Institute of Electrical Engineers has been awarded to Dr. Joseph Slepian of the Westinghouse Electric and Manufacturing Company.

Dr. H. G. Barone has been appointed to an assistant professorship at Pennsylvania State College.

Dr. H. F. Bright has been appointed to an assistant professorship at Denison University.

Professor Emeritus W. D. Cairns of Oberlin College has been appointed visiting professor of mathematics at the University of New Mexico.

Dr. Nancy Cole of Sweet Briar College is visiting assistant professor at Kenyon College.

Dr. D. W. Hall of Brown University has been appointed to an assistant professorship at the University of Maryland.

Dr. J. F. Heyda of Michigan State College has been appointed to an assistant professorship at Denison University.

Associate Professor R. C. Hildner of Mt. Union College has been appointed to an assistant professorship at the College of Wooster.

Assistant Professor R. N. Johanson of Bradley Polytechnic Institute has been appointed to an assistant professorship at Hamilton College.

Assistant Professor E. M. Justin of the Case School of Applied Science is on leave of absence and is at the University of California

Dr. Joseph Lehner of Cornell University has accepted a position with the Kellex Corporation of New York City.

Professor E. J. Moulton of Northwestern University is on leave of absence, and Professor H. T. Davis is acting head of the department of mathematics.

Assistant Professor W. H. Myers of San Jose State College has been appointed acting head of the department of mathematics.

Mr. N. D. Nelson of the University of Wisconsin has been appointed to an assistant professorship at Amherst College.

Dr. E. A. Nordhaus of the University of Wisconsin at Milwaukee has been appointed to an assistant professorship at Michigan State College.

The following appointments to instructorships are announced: Allegheny College: Mr. R. E. Smith; Brown University: Mr. D. A. Jonah, Mr. P. C. Rosenbloom; University of Chicago: Mr. H. L. Meyer. Cornell University: Mr. F. W. Dittman; Denison University: Mr. Paul Cramer, Mr. A. C. Moeller, Mr. Robert Sutton, Miss Elaine Van Aken, Mr. Z. A. Wilson; Hamilton College: Dr. V. O. McBrien, Mr. Fritz Steinhardt; Illinois Institute of Technology: Dr. Edward Helly; University of North Carolina: Mrs. F. D. Jacobson, Mr. J. G. McAllister, Mr. E. P. Pearce, Mr. C. A. Phillips, Dr. C. M. Smith, Mr. C. B. Toxey; Pennsylvania State College: Dr. Helen B. Owens, Mr. G. B. Snyder, Mrs. Alice C. Thorpe; University of Rochester: Mr. Frederick Bagemihl; Rutgers University: Mr. J. N. B. Livingood.

The death of Professor David Hilbert of Göttingen University has been reported. He was eighty-one years old.

Dr. W. A. Granville died February 4, 1943, at the age of 79 years. He had been a member of the Society from 1897–1934.

Miss Lilian Hackney of Marshall College died February 4, 1943. She had been a member of the Society since 1923.

Professor H. F. Minssen, chairman of the department of mathematics and vice president of San Jose State College, died February 7, 1943.

Professor H. H. Mitchell of the University of Pennsylvania died March 13, 1943. He had been a member of the Society since 1909.

Professor Emeritus L. M. Passano of Massachusetts Institute of Technology died January 30, 1943.

Assistant Professor N. S. Risley of Fenn College died December 30, 1942.

The death of Mr. Oscar Schmiedel has been reported. He had been a member of the Society since 1892.

Professor M. H. Tyler of Rhode Island State College died December 16, 1942.

Professor J. J. Westemeier of Dowling College, Des Moines, Iowa, died February 1, 1943.

The following seventy-five doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1942 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which degree was conferred, minor subject (other than mathematics), and the title of the dissertation are given in each case if available.

Claire F. Adler, New York University, June, minor in physics, *An isoperimetric problem with an inequality.*

B. H. Arnold, Princeton, June, *Rings of operators on vector spaces.*

S. P. Avann, California Institute of Technology, June, minor in physics, I. *Theory of the* $\tau$- *and* $\sigma$-*functions in semi-modular lattices.* II. *Lattice automorphisms.*

Sister Saint Augustine Ball, Catholic, November, minor in education, *Power functions and distribution functions related to "Student's test."*

J. C. Bell, Illinois, October, minors in astronomy and physics, *An extension of the parallelism of Cliffird.*

T. A. Botts, Virginia, June, *Sufficient conditions for generalized-curve problems in the calculus of variations.*

H. K. Brown, Michigan, May, *The resolution of boundary value problems by means of the finite Fourier transformation,*

H. E. Burns, Northwestern, June, *Linear operators expressed as Green's functions with special application to the elastic plate problem.*

W. Z. Chien, Toronto, December, minor in civil engineering, *The intrinsic theory of elastic shells and plates.*

D. E. Christie, Princeton, June, *Net homotopy for compacta.*

Paul Civin, Duke, May, minor in physics, *Two-to-one mappings of manifolds.*

W. G. Clark, Kentucky, June, *On generalized quaternion algebras and universal Hermitian forms.*

I. S. Cohen, Johns Hopkins, May, *The structure and ideal theory of local rings.*

L. L. Cronvich, Wisconsin, June, *Deformation of naturally twisted compound beams.*

W. F. Eberlein, Harvard, June, *Closure, convexity, and linearity in Banach spaces.*

W. C. G. Fraser, Toronto, June, *The factorial transform.*

Louis Garfin, Iowa, May, *A comparative study of the underlying principles of certain pension schemes for a staff of employees with special reference to teachers and public employees.*

T. S. George, Duke, May, minor in physics, *Concerning the equilibrium point of the Green's function for an n-dimensional spherical annulus.*

H. F. Gingerich, Illinois, June, minor in physics, *Generalized fields and Desargues configurations.*

Caspar Goffman, Ohio State, June, *On the converses of certain theorems on the symmetric structure of sets and functions.*

Saul Gorn, Columbia, May, *Homomorphisms and modular functionals.*

R. W. Hamming, Illinois, June, minor in physics, *Some problems in the boundary value theory of linear differential equations.*

C. A. Hayes, California (Berkeley), May, *Boundary value problems for elliptic systems of linear partial differential equations.*

L. B. Hedge, Brown, May, I. *Moment problem for a bounded region.* II. *Transformations of multiple Fourier series.*

R. G. Helsel, Ohio State, August, *A geometrical application of integral means.*

Edwin Hewitt, Harvard, June, *A problem of set-theoretic topology.*

Carl Holtom, Chicago, September, *Permanent configurations in the n-body problem.*

Robert Hooke, Princeton, January, *Linear p-adic groups and their Lie algebras.*

R. P. Isaacs, Columbia, June, *A finite difference function theory.*

G. K. Kalisch, Chicago, June, *On special Jordan algebras.*

Samuel Kaplan, Michigan, May, *Homologies in metric separable spaces.*

William Karush, Chicago, June, *Isoperimetric problems and index theorems in the calculus of variations.*

P. J. Kelly, Wisconsin, June, *On isometric transformations.*

D. E. Kibbey, Illinois, February, minor in physics, *Boundary values of analytic functions.*

Mary E. Ladue, Columbia, June, *The conformal geometry of horn angles of higher order.*

P. A. Lagerstrom, Princeton, June, *Measure and integral in partially ordered spaces.*

J. A. Larrivee, Catholic, May, minor in physics, *Effect of the singularities of a rational curve on the symmetric correspondence determined by residuals on tangents.*

Howard Levi, Columbia, May, *On the structure of differential polynomials and on their theory of ideals.*

J. V. Lewis, California (Berkeley), October, *The existence of solutions to Lagrange problems for multiple integrals.*

L. H. Loomis, Harvard, February, *Some studies on simply-connected Riemann surfaces:* I. *The problem of imbedding.* II. *Mapping on the boundary for two classes of surfaces.*

R. R. R. Luckey, Cornell, June, *Certain applications of Fourier integrals.*

V. O. McBrien, Catholic, May, minor in physics, *On some systems of orthopolar and Kantor lines related to a quadrangle.*

G. W. Mackey, Harvard, June, *The subspaces of the conjugate of an abstract linear space.*

Murray Mannos, Harvard, June, *Vector lattices of finite dimension.*

N. S. Mendelsohn, Toronto, June, *A group-theoretic characterization of the general projective collineation group.*

C. F. Meyer, George Washington, June, *The theory of the motion and capture of free electrons in crystal dielectric media.*

Josephine M. Mitchell, Bryn Mawr, June, *On double Sturm-Liouville series.*

Vladimir Morkovin, Wisconsin, June, *On the deflection of anisotropic thin plates.*

F. G. Myers, Virginia, June, *On sufficiency conditions for the problem of Lagrange.*

Anne F. O'Neill, Radcliffe, June, *Contributions to the theory of derivates.*

C. R. Phelps, Harvard. June, *A homogeneous algebra with limited associativity.*

E. J. Pinney, California Institute of Technology, June, minor in physics, *Calculus of variations in abstract spaces and related topics.*

Harry Pollard, Harvard, February, *Studies on the Stieltjes transform.*

A. L. Putnam, Harvard, June, *Integral domains complete in a valuation.*

S. E. Rauch, Stanford, June, minor in physics, *Mapping properties of the second arithmetic mean of the geometric series.*

D. L. Robb, Catholic, May, minor in physics, *Special cases of the symmetric correspondence $C_6$ and their relations to the general $C_6$.*

Sister Mary de Pazzi Rochford, Notre Dame, May, *Integrability conditions without differentiability assumptions.*

J. D. Rommel, Kentucky, August, *On conservative transformations of functions of two variables.*

R. D. Schafer, Chicago, June, *Alternative algebras over an arbitrary field.*

Robert Schatten, Columbia, December, *On the direct product of Banach spaces.*

C. W. Seekins, California Institute of Technology, June, minor in physics, *Lattice multiplications and residuations.*

H. L. Shih, Harvard, February, *Mappings of a 2-manifold into a space.*

R. D. Specht, Wisconsin, June, *Secondary effects in the simple bending of elliptical beams.*

Domina E. Spencer, Massachusetts Institute of Technology, June, minor in electrical engineering, *A tensor interpretation of Study's "Geometrie der Dynamen."*

Zenon Szatrowski, Northwestern, September, minor in economics, *Solutions of some integro-differential equations.*

H. S. Thurston, Wisconsin, June, *The P-adic numbers of Hensel.*

R. H. Tripp, Iowa State, March, minors in physics and theoretical and applied mechanics, *Statical equilibrium of skew and sector shaped plates.*

Alice E. Turner, Chicago, September, *Singularities of space curves.*

Bernard Vinograde, Michigan, May, *Split rings and their representation theory.*

G. L. Walker, Cornell, February, *Direct product and Lorentz matrices.*

W. R. Wasow, New York University, June, minor in physics, *On boundary layer problems in the theory of ordinary differential equations.*

P. A. White, Virginia, June, *r-regular convergence spaces.*

J. E. Wilkins, Chicago, December, *Multiple integral problems in parametric form in the calculus of variations.*

Jacob Wolfowitz, New York University, June, minor in physics, *Additive partition functions and a class of statistical hypotheses.*

G. S. Young, Texas, June, *Concerning the outer boundaries of certain connected domains.*

The following doctorates were conferred in 1941, but were not included in the list in the preceding volume of this BULLETIN (vol. 48, pp. 346–352).

J. A. Daum, Nebraska, August, minor in physics, *Basic hypergeometric series.*

C. H. Frick, North Carolina, June, *The development of formulas for columns with varying moment of inertia and end conditions.*

D. C. May, Princeton, October, *An integral formula for analytic functions of k variables with some applications.*

A. H. Stone, Princeton, October, *Connectedness and coherence.*

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of the issue, and the serial number of the abstract.

### ALGEBRA AND THEORY OF NUMBERS

125. A. T. Brauer: *On the non-existence of odd perfect numbers of form* $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$.

If an odd perfect number exists, it must have the form $n = p^\alpha q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}$ where $p, q_1, q_2, \cdots, q_t$ are primes and $p \equiv \alpha \equiv 1 \pmod 4$. This was proved by Euler. Sylvester obtained estimates for $t$, in particular $t \geqq 4$, and $t \geqq 7$ if $n \not\equiv 0 \pmod 3$. Recently, it was shown by R. Steuerwald (Sitzungsberichte der Bayerischen Akademie der Wissenschaften 1937) that the case $\beta_1 = \beta_2 = \cdots = \beta_t = 1$ is impossible, and by H. J. Kanold (J. Reine Angew. Math. vol. 183 (1941)) that the same is true for $\beta_1 = \beta_2 = \cdots = \beta_t = 2$. Moreover Kanold proved that $n$ is not perfect if the greatest common divisor $d$ of $2\beta_1+1, 2\beta_2+1, \cdots, 2\beta_t+1$ is divisible by 9, 15, 21, or 33, and some similar results. All these results deal with the case $d > 1$. In this paper it is proved that no odd number of form $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$ exists; here $d = 1$. For the proof some theorems of T. Nagell on Diophantine equations are used (Norsk Matematisk Forenings Skrifter 1921). (Received March 24, 1943.)

126. R. P. Dilworth: *Lattices with unique complements.*

For some time an outstanding problem in lattice theory has been the following: Is every lattice having unique complements a Boolean algebra? It is shown here that the statement is *not* true. Indeed the following counter theorem is proved: Every lattice is a sublattice of a lattice with unique complements. (Received March 20, 1943.)

127. J. E. Eaton: *A Galois theory for differential fields.*

Let $\Lambda$ be an algebraically transcendental extension of the partial differential field $\Gamma$, and consider all isomorphisms of $\Lambda$ leaving $\Gamma$ invariant. These isomorphisms may be grouped into a finite number of disjoint sets $\{m_i\} = \mathfrak{M}$ which, with a suitably defined multiplication, is a multigroup called the Galois multigroup of $\Lambda$ over $\Gamma$. There is a 1-1 correspondence between the subfields of $\Lambda$ containing $\Gamma$ and the submultigroups of $\mathfrak{M}$ such that if $\Delta \rightleftarrows \mathfrak{H}$, then $\mathfrak{H}$ is the Galois multigroup of $\Lambda$ over $\Delta$ and $\mathfrak{M}//\mathfrak{H}$ (the multiplicative system of the double coset decomposition of $\mathfrak{M}$ with respect to $\mathfrak{H}$) is the Galois multigroup of $\Delta$ over $\Gamma$. These results are an extension of the work of H. W. Raudenbush (*Hypertranscendental extensions of partial differential fields*, Bull. Amer. Math. Soc. vol. 40, (1934) pp. 714–720) and E. R. Kolchin (*Extensions of differential fields*, I, Ann. of Math. vol. 43 (1942) pp. 724–729). (Received March 25, 1943.)

### 128. D. H. Lehmer: *On Ramanujan's numerical function $\tau(n)$.*

Ramanujan's numerical function $\tau(n)$ may be defined by $\sum\tau(n)x^{n-1} = \{\pi(1-x)^n\}^{24}$. Among several unsolved questions about $\tau(n)$ is the so-called Ramanujan hypothesis to the effect that if $p$ is a prime $|\tau(p)\cdot p^{-11/2}| < 2$, which Ramanujan verified for primes $p < 30$. The present writer, in attempting to disprove this important hypothesis, has examined all primes $p < 300$, as well as $p = 571$, and finds that in all these 47 cases the hypothesis holds true. It nearly fails for $p = 103$ when $\tau(103)\cdot 103^{-11/2} = -1.918\cdots$. In connection with this hypothesis Rankin proved in 1939 that as $x\to\infty$, $x^{-12}\sum_{n\leq x}\tau(n)^2$ tends to a limit represented by a certain double integral extended over the fundamental region of the full modular group. In this paper a practical method is devised for evaluating this integral whose value is found to be $.0320047918814\cdots$. Various congruence and divisibility properties of $\tau(n)$ are also discussed. For example $\tau(n)$ is composite for $1 < n < 7921$. (Received March 26, 1943.)

### 129. A. E. Ross: *Positive quaternary quadratic forms representing all integers with at most k exceptions.*

In this paper it is shown that there are a finite number of classes of positive quaternary quadratic forms which represent all integers with at most $k$ exceptions. The determinants of such forms have an upper bound $B_k$ depending on $k$. This is a generalization of the results of Ramanujan (Proc. Cambridge Philos. Soc. vol. 19 (1917) pp. 11–21), Ross (Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) p. 607) and Halmos (Bull. Amer. Math. Soc. vol. 44 (1938) pp. 141–144). Ross gives $B_1 = 112$ for the classic case and Halmos' results imply that $B_1 \geqq 240$ in the classic case. (Received March 26, 1943.)

### 130. L. I. Wade (National Research Fellow): *Transcendence properties of the Carlitz $\psi$-function.*

The paper is concerned with quantities transcendental over the field $GF(p^n, x)$. For the Carlitz $\psi$-function (L. Carlitz, Duke Math. J. vol. 1 (1935) pp. 137–168) and its inverse $\lambda(t)$ the following theorem is proved. If $\beta$ is algebraic (over $GF(p^n, x)$) and irrational and if $\alpha\neq 0$ is algebraic, then $\psi(\beta\lambda(\alpha))$ is transcendental over $GF(p^n, x)$. In a sense this is an analogue of Hilbert's seventh problem for the transcendence of $\alpha^\beta = e^{\beta\log\alpha}$ over the rational number field. (Received March 26, 1943.)

## ANALYSIS

### 131. L. W. Cohen: *On linear equations in Hilbert space.*

If the rows of the infinite matrix $A = \|a_{ij}\|$ are points in Hilbert space and $a^{i_1\cdots i_m}_{j_1\cdots j_m}$ are the $m$-rowed determinants with elements in $A$, it is shown that $\det A_{i_1\cdots i_m}B'_{i_1\cdots i_m} = \sum_{j_1\cdots j_m} a^{i_1\cdots i_m}_{j_1\cdots j_m} b^{i_1\cdots i_m}_{j_1\cdots j_m}$ where $A_{i_1\cdots i_m}$, $B_{i_1\cdots i_m}$ are $m$-rowed minors of $A$, $B$ respectively. The series is summed over all combinations of integers $j_1, \cdots, j_m$ and converges absolutely. This identity is used to establish sufficient conditions in order that the linear system represented by $Ax = y$ have a solution in Hilbert space for all $y$ in that space. (Received March 24, 1943.)

### 132. R. J. Duffin: *Some representations for Fourier transforms.*

Let $\phi(x)$ be an arbitrary function and let $f(x)$ and $g(x)$ be defined by the series: $f(cx) = \sum_1^\infty(-1)^n\phi((2n-1)/x)/x$, $g(cx) = \sum_1^\infty(-1)^n\phi(x/(2n-1))/(2n-1)$. Then if

$c=x^{1/2}/2^{1/2}$, $g(x)$ is the Fourier sine transform of $f(x)$. The purpose of this paper is to give several conditions for the validity of these formulas. (Received March 26, 1943.)

### 133. F. A. Ficken: *Note on the existence of scalar products in normed linear spaces.*

In order that a normed linear space $\Sigma$ with complex scalars may permit the definition of a scalar product, it is necessary and sufficient that, whenever $a$ and $b$ are real scalars and $A$ and $B$ are vectors, $|A|=|B|$ shall imply ($\alpha$): $|aA+bB|=|bA+aB|$. The necessity of this condition is easily verified. To establish its sufficiency ($\alpha$) is used to prove, for any vectors $A$ and $B$, that ($\beta$): $|A+B|^2+|A-B|^2=2|A|^2+2|B|^2$. The sufficiency (and necessity) of condition ($\beta$) for the existence of a scalar product was established by J. von Neumann and P. Jordan (Ann. of Math. vol. 36 (1935) pp. 719–723). (Received March 29, 1943.)

### 134. F. A. Ficken: *On two criteria for convexity in a symmetric Banach space.*

The results refer to a Banach space $\Sigma$ with real scalars and a scalar product. It has been shown by Moskovitz and Dines (Duke Math. J. vol. 5 (1939) pp. 520–534; Bull. Amer. Math. Soc. vol. 46 (1940) pp. 482–489) that: If a set $E$ is closed and has interior points, then $E$ is convex if and only if $E$ has a supporting plane at each point of a set which is everywhere dense in its frontier $E'$. In their work a supporting plane is required to contain a point of $E$, and hence can exist only at points of $E'$ which belong to $E$. Dropping this requirement, their results are used to obtain slightly stronger conclusions without assuming $E$ to be closed. Another criterion, established by Jessen for Euclidean $n$-space (cf. Math. Rev. vol. 2 (1941), p. 261) requires the presence, for each point of the complement of $E$, of a unique nearest point on the closure of $E$. By generalizing the familiar geometric transformation of inversion, this criterion is established for the convexity of a compact set, or of any set if $\Sigma$ has the property that each bounded subset is compact. (Received March 29, 1943.)

### 135. P. R. Halmos: *Approximation theories for measure preserving transformations.*

Three topologies (called neighborhood, uniform, and metric) are introduced into the set $G$ of all (1, 1) measure preserving transformations of a finite measure space onto itself. The first two of these are the analogues of the strong and the uniform topologies, respectively, of the set of bounded operators on Hilbert space and the third is defined by the distance function $d(S, T)=m\{x|Sx\neq Tx\}$. It is proved that the uniform and the metric topologies are identical; in both the neighborhood and the metric topologies $G$ becomes a complete topological group. The central results in both cases assert that arbitrary measure preserving transformations may be approximated by transformations with comparatively simple properties (for example, by transformations of finite period). It follows from these approximation theorems that the adage according to which "in general a measure preserving transformation is ergodic" is true in the neighborhood topology (that is, the set of ergodic transformations is a residual $G_\delta$) and false in the metric topology (that is, the set of ergodic transformations is nowhere dense). (Received March 2, 1943.)

### 136. R. W. Hamming: *The asymptotic location of the roots of exponential sums having polynomial exponents.*

This paper considers the asymptotic location of the zeros of $f(z)=\sum_{i=1}^{n}P_i(z)$ $\exp Q_i(z)$, where $P_i(z)$ and $Q_i(z)$ are polynomials in $z$. An inequality concerning certain regions covering the zeros is also given. (Received March 1, 1943.)

### 137. R. G. Helsel and Tibor Radó: *The transformation of double integrals.*

Radó and Reichelderfer have established, for certain wide classes of continuous transformations in the plane, extremely general topological transformation formulas for double integrals. The purpose of the present paper is to show that their results imply all of the results on the transformation of double integrals in the literature of which the authors are aware. Particular attention is given to the work that W. H. Young has done on the transformation of double integrals. The paper also contains new results on approximation by integral means. (Received February 19, 1943.)

### 138. Einar Hille and Gabor Szegö: *Remarks on ergodic theorems.*

Ergodic theorems are considered from the point of view of classical theory of summability. In particular, Abelian and Tauberian theorems give relations between the existence of $(A)$-lim $T^n$ and $(C, k)$-lim $T^n$ in a given topology. A detailed study is made of the transformation $T=T_\alpha=I-U_\alpha$ where $U_\alpha f$ is the fractional integral of $f$ of order $\alpha$. Here the resolvent has an isolated essential singular point at $\lambda=1$, but $(A)$-lim $T_\alpha^n=0$ for $0<\alpha<2$ in the strong (but not in the uniform) topology of $L_1(0, 1)$ as well as almost everywhere. For $0<\alpha<1$ even lim $T_\alpha^n=0$ and for $\alpha=1$, $(C, k)$-lim $T_1^n=0$ for $k>1/2$. (Received February 27, 1943.)

### 139. H. K. Hughes: *On a theorem of Newsom.*

A theorem proved by Newsom (Amer. J. Math. vol. 60 (1938) pp. 561–572) furnishes an expression from which it is often possible to obtain the asymptotic expansions for large values of $|z|$ of the integral function $f(z)=\sum_{n=0}^{\infty}g(n)z^n$ (radius of convergence $=\infty$), the series being regarded as given. It is assumed that the function $g(w)$, where $w=x+iy$, is single-valued and analytic in the finite $w$ plane, and is less in absolute value than a constant multiple of $e^{k\pi|y|}$, $k$ being a positive integer. The present paper considers the situation when $g(w)$ has a singularity in the finite plane, while the inequality remains satisfied for sufficiently large values of $|w|$. The result is that a certain loop integral must be subtracted from the expression obtained by Newsom. A method of obtaining the asymptotic expansion of the loop integral is indicated. (Received March 25, 1943.)

### 140. H. D. Huskey: *Contributions to the problem of Geöcze.*

Consider a continuous surface $S$: $z=f(x, y)$ where $f(x, y)$ is continuous on the unit square $I$: $0\leqq x\leqq 1$, $0\leqq y\leqq 1$. The area of such a surface is given by $A(S)=$ greatest lower bound lim inf $A(P_n)$ where $P_n$: $z=p_n(x, y)$, $p_n(x, y)\rightarrow f(x, y)$ uniformly on $I$, and the greatest lower bound is taken with respect to all such sequences of polyhedra (not necessarily inscribed). Let $A^*(S)$ denote the value obtained in the preceding definition if the polyhedra are required to be inscribed. The problem of Geöcze asks: When is $A(S)=A^*(S)$? The author has used integral means and results on convergence in area (Radó and Reichelderfer, *Convergence in length and convergence in area*, Duke Math. J. vol 9 (1942) pp. 527–565) to obtain two different proofs that $A(S)=A^*(S)$ if $f(x, y)$ is absolutely continuous in the sense of Tonelli. (Received February 19, 1943.)

### 141. D. H. Hyers: *A note on the Michal differential in Banach spaces.*

In this note an example is given which shows that the Michal $M$-differential (Proc. Nat. Acad. Sci. U.S.A. vol. 24 (1938) pp. 340–342 and Bull. Amer. Math. Soc. vol. 45 (1939) pp. 529–563) is more general than the Fréchet differential in Banach spaces. (Received March 27, 1943.)

### 142. R. E. Lane: *The values of continued fractions and their computation.*

This paper gives a method of computing the differences between the values of successive approximants to a continued fraction and shows how it is often possible to find the maximum possible absolute value of the difference between the value of the $n$th approximant and the value of the continued fraction. The principal result, the two-circle theorem, gives a necessary and sufficient condition on $s$ for a certain linear fractional transformation, $v=t_s(u)$, to carry $|u-c|\leq|c|$ into some portion of $|v-d|\leq|d|$. The Worpitzky circle, the Paydon-Scott-Wall parabolas, and various other element-regions for continued fractions arise as special cases. Among the other applications of the two-circle theorem is a simple proof that if the elements of a continued fraction lie in a certain parabola, the approximants lie in the inner loop of a certain limacon. A generalization of this result is given. (Received March 27, 1943.)

### 143. L. H. Loomis: *Spherical volume as a coordinate-free basis for Lebesgue measure theory.*

The formula for the volume of a sphere is deduced in metric spaces having certain homogeneity properties in common with Euclidean spaces. The fundamental properties of spherical volume are established, and the whole of classical measure theory which does not concern a specific dimension number is then immediately available. The development of Lebesgue measure theory is thus freed from the background of a coordinate representation as the $n$th power (in the sense of Cartesian product) of the real number system. (Received March 15, 1943.)

### 144. Glynn Owens: *A boundary value problem for an ordinary nonlinear differential equation of the second order.*

The uniqueness and existence in the infinite region $I[0\leq x<\infty]$ of the solution of the boundary value problem (A): $d^2y/dx^2-\lambda f(x, y)y=q(x, y)$, $y(0)=0$, $\lim_{x\to\infty} y(x)=0$, is to be demonstrated in this paper. A fundamental assumption will be that $f(x,y)$ is positively and uniformly bounded away from zero, and it is necessary to assume that the parameter $\lambda$ is positive and sufficiently great. In the first section of the paper the uniqueness and existence of the solution of a certain linear equation is considered. In addition certain bounds for the solution are derived which are valid respectively in the region $I$ and in a suitable neighborhood of infinity. The principal result, namely the treatment of problem (A), is dealt with in section two. There, by linearizing the problem (A), subjecting the coefficients of (A) to certain conditions, such as Lipschitz requirements and the existence of certain integrals taken over the region $I$, it is proved, with the aid of the results of section one, that the method of successive approximations used gives a sequence of functions which converge uniformly in $I$ to the desired solution of (A). (Received March 24, 1943.)

### 145. George Pólya: *Inequalities for the area of the ellipsoid.*

Let $E = E(a, b, c)$ denote the area of the ellipsoid with semiaxes $a$, $b$, $c$. Let $\lambda$, $\mu$, $\nu$ satisfy the conditions $\lambda \geqq \mu \geqq \nu \geqq 0$, $\lambda + \mu + \nu = 2$. Define $P_{\lambda\mu\nu} = P_{\lambda\mu\nu}(a, b, c) = 4\pi \sum a^{\lambda} b^{\mu} c^{\nu}/6$; the sum has 6 terms, obtained from $a^{\lambda} b^{\mu} c^{\nu}$ by performing all possible permutations of $a$, $b$, $c$. Consider linear combinations $P = kP_{\lambda\mu\nu} + k'P_{\lambda'\mu'\nu'} + \cdots$, assume that $k > 0$, $k' > 0$, $\cdots$ and $k + k' + \cdots = 1$; the number of the terms is finite, $\geqq 1$. Now $P$ is just as $E$, a continuous function of $a$, $b$, $c$ for $a \geqq 0$, $b \geqq 0$, $c \geqq 0$, positive for $a > 0$, $b > 0$, $c > 0$, homogeneous of degree 2, symmetric in $a$, $b$, $c$, and it has the same value as $E$ for $a = b = c$ (for spheres). The problem is to find all linear combinations $P$ "comparable" with $E$ (see G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, p. 5). A complete solution is given an essential part of which can be stated thus: The inequalities $E > P_{1,1,0}$, $E < (P_{2,0,0} + P_{1,1,0})/2$ hold for $a > 0$, $b > 0$, $c > 0$ and are the best of their kind. The linear combination $(64P_{1,1,0} - 2P_{2,0,0} - 27P_{2/3,2/3,2/3})/35$ is not a $P$, because it has negative coefficients; but it gives a remarkably good numerical approximation to $E$ if the ellipsoid is not very different from a sphere. (Received March 26, 1943.)

### 146. Robert Schatten: *On certain properties of crossnorms.*

For two Banach spaces $E_1$, $E_2$, a Banach space $E_1 \otimes E_2$ is constructed with a crossnorm $N$. With $E_1 \otimes E_2$, there is considered the associate Banach space $\overline{E}_1 \otimes \overline{E}_2$ with a norm $\overline{N}$, associate with $N$. Similarly $\overline{\overline{N}}$ is defined in $\overline{\overline{E}}_1 \otimes \overline{\overline{E}}_2$. $N$ is minimal if $\overline{N^*} = \overline{N}$ implies $N^* \geqq N$. $N$ is reflexive if $\overline{\overline{N}} = N$. $N$ has an "associate property" if for a certain $N^*$, $\overline{N^*} = N$. It is shown that minimal and reflexive norms and those having an associate property are identical. The existence of a least crossnorm is proved, and that the associate with every crossnorm is also a crossnorm. It is also shown that a uniformly convex crossnorm sets up the relation $\overline{E_1 \otimes E_2} = \overline{E}_1 \otimes \overline{E}_2$, if and only if it is minimal. Let $k$ denote a natural number. The values of a crossnorm for all expressions of rank at most $k$ do not necessarily determine the crossnorm. Finally "semi-self-associate" crossnorms are constructed. In particular, when $E_1$, $E_2$ denote two Hilbert spaces, for every natural $k$, minimal crossnorms $S_k$ are constructed, such that $S_k \neq S$, and $S_k = \overline{S}_k = S$ for all expressions of rank at most $k$, where $S$ denotes the self-associate crossnorm for Hilbert spaces constructed by F. J. Murray and J. von Neumann (Ann. of Math. (2) vol. 37 (1936) pp. 118–125). (Received February 16, 1943.)

### 147. W. S. Snyder: *A note on the interiority of real functions.* Preliminary report.

Let $F$ be a continuous real-valued function defined on a space $X$. Let $A$ be the set of points of $X$ where $F$ has a local maximum and let $B$ denote the set of points of $X$ where $F$ has a local minimum. If $X$ is locally connected then the function (transformation) $F$ is interior at $p \in X$ if and only if $p \in X - (A + B)$. If $X$ has a countable basis then $F(A + B)$ is countable. These results together constitute an extension of a theorem of Whyburn (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 942–945). (Received March 25, 1943.)

### 148. Otto Szász: *On some trigonometric summability methods and Gibbs phenomenon.*

In this paper certain summability methods of trigonometric type are discussed in relation to Cesàro summability. Applied to Fourier series they lead to new formulae for the determination of the jump at a point, and to a generalized Gibbs phenomenon. In particular, the Fourier series of an integrable function $f(t)$ presents a Gibbs phenomenon at $\theta=0$, whenever for some $j\neq 0$, $(1/\theta)\int_0^\theta |f(t)-j/2|\,dt\to 0$ as $\theta\to 0$. (Received March 1, 1943.)

### 149. A. E. Taylor: *Banach spaces of functions representable by Cauchy's integral formula.*

Let $I[\phi; z]$ denote Cauchy's integral $(2\pi i)^{-1}\int\phi(t)(t-z)^{-1}dt$, the integration extended over $|t|=1$ in the counterclockwise sense. Let $E^p$ be the class of functions $f(z)$ representable when $|z|<1$ by $f(z)=I[\phi; z]$, where $\phi(t)$ is an arbitrary function of class $L^p$ on $|t|=1$. Let $G^p$ denote the class of functions representable in the same way when $|z|>1$. Let $H^p$ denote the class of functions $f(z)$ analytic when $|z|<1$, and such that the mean value of $|f(re^{i\theta})|^p$ on $|z|=r$ is bounded when $r<1$. It is assumed that $p>1$. The author proves that $E^p$ and $H^p$ are identical, and that $f(z)$ is in $G^p$ if and only if it is analytic when $|z|>1$, zero at infinity, and such that the mean value of $|f(re^{i\theta})|^p$ on $|z|=r$ is bounded when $r>1$. Any function $F(t)$ of class $L^p$ on $|t|=1$ is representable uniquely in the form $f(t)+g(t)$, almost everywhere on $|t|=1$, where $f\in E^p$ and $g\in G^p$. $I[F; z]$ is equal to $f(z)$ if $|z|<1$, and to $-g(z)$ if $|z|>1$. $E^p$ and $G^p$ are, in a natural sense, complex Banach spaces equivalent to complementary closed linear manifolds $M$ and $N$, respectively, in $L^p$. The form of linear functionals on $E^p$ is determined, and it is proved that $(E^p)^+$ is isomorphic with $E^q$, where $p^{-1}+q^{-1}=1$. A sequence $\{f_n(z)\}$ in $E^p$ converges weakly to $f(z)$ if and only if $f_n(z)\to f(z)$, $|z|<1$, and the mean value of $|f_n(re^{i\theta})|^p$ on $|z|=r$ is bounded uniformly as to $n$ and $r$, $r<1$. (Received February 23, 1943.)

### 150. W. J. Thron: *Twin convergence regions for continued fractions.*

Two regions $B_0$ and $B_1$ in the complex number plane are called twin convergence regions for the continued fraction $1+K(b_n^2/1)$ if the conditions $b_{2n}\in B_0$ and $b_{2n-1}\in B_1$ (for all $n\geqq 1$) insure its convergence. It is shown that the regions $z\in B_0$, if $|z|\leqq M$; $z\in B_1$, if $|z-i|\geqq M+\epsilon$ and $|z+i|\geqq M+\epsilon$, are twin convergence regions for every arbitrary small $\epsilon>0$ and every positive $M<1$. Another family of twin convergence regions is formed by sets $B_0$ and $B_1$, if both sets are closed and bounded and if $z=x+iy\in B_0$, if $ax-b<y<ax+b$; and $z\in B_1$, if $ax-1+b<y<ax+1-b$. The number $a$ here can be an arbitrary real number and $b$ any number $0<b<1$. Concerning the bestness of these results the following theorem has been proved. Two regions $B_0$ and $B_1$ can be twin convergence regions only if for every $z\in B_0$ the four points $z+i$, $z-i$, $-z+i$, $-z-i$ are not interior points of $B_1$. These results contain and improve certain earlier results by Leighton and Wall (Amer. J. Math. vol. 38 (1936) ), Paydon and Wall (Duke Math. J. vol. 9 (1942) ) and Leighton and Thron (Duke Math. J. vol. 9 (1942) ). (Received March 1, 1943.)

### 151. S. M. Ulam: *Theory of the operation of product of sets.* II. Preliminary report.

A projective algebra is a Boolean algebra of sets contained in a direct product $E^n$ of a set $E$ with itself and closed with respect to the operations of projection of a set into the component sets $E$ (considered as a subset of $E^n$) and the direct product of $n$ sets of the algebra contained in $E$. A systematic investigation of structures of projective algebras and the properties of projective isomorphisms is undertaken. The cases of infinite projective algebras generated from a finite number of elements by the Boolean operations and the operation of direct product and projection, and the question of existence of finite bases in the above sense for countable algebras are studied. Particular Boolean algebras studied in connection with these ideas include: the algebra of all sets of integers modulo finite sets, the algebra of sets of integers modulo the sets of density 0, the algebra of finite sums of dyadic intervals on (0, 1), the algebra of Borel sets modulo sets of measure 0. (Received March 25, 1943.)

### 152. F. A. Valentine: *A Lipschitz condition preserving extension for a vector function.*

The function $f(x)$ mapping a set $S$ in the $n$-dimensional Euclidean space $E_n$ into a set $S'$ in $E_n$, and satisfying the condition $|f(x_1)-f(x_2)| \leq K|x_1-x_2|$ on $S$, can be extended to any set $T\supset S$ so as to preserve the Lipschitz condition. This is a generalization of a previous result given by the author for the plane (see Bull. Amer. Math. Soc. vol. 49 (1943) p. 100). This extension is a consequence of the following fact: Consider in $E_n$ a set of hyperspheres which have a point in common. Move these hyperpsheres to new positions in $E_n$, subject to the condition that the distance between any pair of centers is not increased. Then all the hyperspheres in the new positions will still have a point in common. This is first proved for $n+1$ hyperspheres. A theorem of Helly is used to proceed from a set containing $n+1$ hyperspheres to an arbitrary set. Corresponding theorems are obtained for the surface of an $n$-dimensional hyperhemisphere and for a Hilbert space. (Received March 17, 1943.)

### 153. H. S. Wall and Marion D. Wetzel: *Contributions to the analytic theory of J-fractions.* Preliminary report.

The authors develop the theory of the class of $J$-fractions $1/(b_1+z)-a_1^2/(b_2+z)-a_2^2(b_3+z)-\cdots$, $a_p\neq 0$, all of whose approximants $f_p(z)$ satisfy the condition $\mathfrak{I}[f_p(z)]<0$ for $\mathfrak{I}(z)>0$. These $J$-fractions are completely characterized by the condition: $\sum_{r=1}^{p}\mathfrak{I}(b_r)x_r^2-\sum_{r=1}^{p-1}\mathfrak{I}(a_r)x_rx_{r+1}\geq 0$, $p=1, 2, 3, \cdots$, for all real $x_r$. Hence they include the classical $J$-fractions ($\mathfrak{I}(b_r)=\mathfrak{I}(a_r)=0$) and also those discussed in a recent paper of Hellinger and Wall (Ann. of Math. vol. 44 (1943) pp. 103–127) where $\mathfrak{I}(b_r)\geq 0$, $\mathfrak{I}(a_r)=0$. It is shown that all the poles of $f_p(z)$ lie in the lower half-plane $\mathfrak{I}(z)\leq 0$ and there exists a nondecreasing function $\phi_p(u)$, $0\leq\phi_p(u)\leq 1$, such that $f_p(z)=\int_{-\infty}^{\infty}d\phi_p(u)/(z+u)$; therefore the sequence $\{f_p(z)\}$ is uniformly bounded for $\mathfrak{I}(z)\geq\delta>0$ and contains an infinite subsequence converging uniformly on the interior of the upper half-plane to an integral of the same form. The older theories, including asymptotic properties and the moment problem, are extended to this general class of $J$-fractions. (Received March 26, 1943.)

### 154. Alexander Weinstein: *Differential equations with general boundary conditions.*

The following problem, discussed in this paper, is a typical example of a new kind of differential eigenvalue problems. Let $g_1, g_2, \cdots, g_m$ be a basis of a linear manifold $L_m$ of functions defined on the boundary $C$ of a domain $S$. It is required to find the eigenvalues of the partial differential equation $\Delta u+\lambda u=0$ with the boundary conditions: $u$ is orthogonal to all elements of $L_m$ and $du/dn$ is an element of $L_m$. Other examples of the same kind have been introduced by the author in previous papers. The corresponding problems for ordinary differential equations are connected with the classical theory. (Received March 23, 1943.)

## APPLIED MATHEMATICS

### 155. H. R. Branson: *On the difference equation of a general quantum mechanical problem.*

The problems of quantum mechanics may be derived from the classical functions either in the $q$-language or the $p$-language. In the $q$-language $F(q, p)=F(q, hi^{-1}\partial/\partial q)$; in the $p$-language $F(q, p)=F(-hi^{-1}\partial/\partial p, p)$. With certain types of potential function, the $p$-language leads to difference equations. Because of the great popularity of quantum mechanics the expression of a general problem in difference equations may aid in "the creation of a usable mathematics of the discrete". The usual equation in one dimension is $H(q, p)\psi=E\psi$ where $H(q, p)=(1/2m)p^2+V(q)$ The equation becomes in the $p$-language $(1/2m)p^2\psi+V(-hi^{-1}\partial/\partial p)\psi=E\psi$. Limiting potentials $V(q)$ to those which may be expanded in Fourier series, it follows that: $V(-hi^{-1}\partial/\partial p)\psi=\sum_{k=0}^{\infty}[a_k\psi(p+kh)+b_k\psi(p-kh)]$. With some obvious simplification, the general difference equation for this type of one dimensional quantum mechanical problem may be written in the form $\sum_{k=-\infty}^{\infty}C_k\psi(p+kh)=0$, wherein all $C_k$ are constants except $C_0=(p^2/2m+a_0+b_0-E)$. Some examples and extensions are discussed. (Received March 25, 1943.)

### 156. A. H. Fox: *Integral representation of the flow of a compressible fluid around a cylinder.* II.

The method outlined in Part I (Bull. Amer. Math. Soc. abstract 49-1-60) is applied for values of $q=0$ and $q=1/2$ to the flow of a perfect gas around a circular obstacle. The nature of the variations in the boundary of the flow are discussed, and the effect of compressibility on the pressure is considered. (Received February 22, 1943.)

### 157. J. F. Harding and Isaac Opatowski: *An approximate formula for the Legendre elliptic integral of the second kind.*

By means of ultraspherical polynomials $E=\int_0^{\pi/2}(1-k^2\sin^2\phi)^{1/2}d\phi$ is expanded in $E=(1+k')\sum a_nk_1^n$ where $k'=(1-k^2)^{1/2}$, $k_1=(1-k')/(1+k')$. By a suitable change of the series $\sum a_nk_1^n$ the formula $E=(\pi/4)(1+k')[0.75+0.1875k_1^2+(4-k_1^2)^{-1}]+\epsilon$ is obtained, where $0\leqq\epsilon\leqq 1-(61\pi/192)=0.00189\cdots$, which is a simpler and a closer approximation than the formulas of G. A. Grunberg (Applied Mathematics and Mechanics vol. 1 (1933) pp. 61–69) or those given by J. Thomae (*Formeln und Satze aus dem Gebiete der elliptischen Funktionen*, Leipzig, 1905, p. 29) or by W. Laska (Sammlung von Formeln, pp. 501–502). (Received March 26, 1943.)

### 158. Wilfred Kaplan and Max Dresden: *Topology of the molecular N-body problem.*

The $N$ molecules of a gas are considered as mass-particles exerting forces on each other derived from a potential of the form $ar^{-n}-br^{-m}$, $n>m>0$, $a>0$, $b>0$. The force is thus highly repulsive for small $r$ and weakly attractive for large $r$. The energy integral: Total Kinetic Energy plus Total Potential Energy $=$ Const. $=C$ is then interpreted as restricting the trajectories of the system, in the corresponding $6N$-dimensional phase space, to a $(6N-1)$-dimensional hypersurface $M(C)$. It is shown that for $C\geqq 0$, $M(C)$ has the topological structure of a $(6N-1)$-sphere, minus certain $(6N-4)$-spheres corresponding to collisions. For $-\delta<C<0$, where $\delta$ is a certain explicitly known function of $a$, $b$, $n$, $m$ (not of $N$), $M(C)$ is homeomorphic to the topological product of a $(3N-1)$-sphere and a $3N$-sphere, minus certain $(6N-4)$-dimensional loci corresponding to collisions. For $C\leqq -\delta$ the structure of $M(C)$ has yet to be determined. It is planned to correlate the critical values $C=0$ and $C=-\delta$ with the physically known transition temperatures such as the critical temperature. (Received March 23, 1943.)

### 159. Isaac Opatowski: *An explicit formula for the refractive index in electron optics.*

The refractive index $\mu$ is expressed in electron optics (W. Glaser, Zeitschrift fur Physik vol. 81 (1933) pp. 647–686) in terms of the electrostatic potential $V$, the magnetic vector potential $A$ and the unit vector $s$, which is defined as tangent to the electron trajectory. Since $s$ is not known a priori and is a function of $V$ and $A$, the elimination of $s$ from the expression of $\mu$ is of advantage. This is done in the paper for a very ample class of fields in which a momentum integral of the equations of motion exists (Bull. Amer. Math. Soc. vol. 46 (1940) p. 887 and Journal of Mathematics and Physics vol. 20 (1941) pp. 418–424). (Received March 26, 1943.)

## GEOMETRY

### 160. Jesse Douglas: *Point transformations and isothermal families of curves.* II.

This paper is a continuation of one with the same title (see Bull. Amer. Math. Soc. abstract 49-1-71). Its new feature is the principal use of synthetic rather than analytic methods. The problem is referred to the investigation of certain properties of a hexagonal web. (Received February 27, 1943.)

### 161. Jacques Dutka: *Transversality in higher space.*

In this paper, a geometric criterion for transversality developed by Kasner in his paper *Transversality in space of three dimensions* (Trans. Amer. Math. Soc. vol. 30 (1928) pp. 447–452) is generalized for $n$-dimensional Euclidean space. It is shown here that a necessary and sufficient condition for a given correspondence between a lineal element and a hypersurface element to be a transversality is that a certain induced correlation be a polarity. A principle of transference connecting simple and $(n-1)$-fold integrals in the calculus of variations when they produce equivalent transversalities is established. The result obtained is applied to the theory of infinitesimal contact transformations from which are derived analytic tests equivalent to the above-mentioned geometric criterion. Actual examples of transversalities in addition to the well known condition of orthogonality are also given. (Received March 25, 1943.)

### 162. Jacques Hadamard: *On fractional iteration and connected questions.*

The author presents some results communicated to him by two younger geometers on fractional iteration and permutable transformations in one variable. This subject is connected with group theory or, more precisely, with Cartan's conception of geodesics in a group-space. (Received March 27, 1943.)

### 163. T. R. Hollcroft: *Plane curve systems with distinct nodes and cusps and of negative virtual dimension.*

In this paper a system $C$ of plane curves is defined by plane sections of a tangent cone of species $r-2$ from an $S_{r-3}$ to a nonsingular, irreducible primal $V_{r-1}$ of order $\nu$ in $S_r$. The only singularities of $C$ are distinct nodes and cusps. The system $C$ has a negative virtual dimension for certain limiting values of $r$ and $\nu$. The above has been presented to the Society (Bull. Amer. Math. Soc. abstract 44-11-412). At that time, the system $C$ had not been proved irreducible. The irreducibility of $C$ has been established. These systems are at present the only plane curve systems known to exist of negative virtual dimension with no other singularities than distinct nodes and cusps. (Received March 26, 1943.)

### 164. Edward Kasner and John DeCicco: *Generalized transformation theory of isothermal families.*

Kasner has proved that the complete group of lineal-element transformations which send every isothermal family of curves into an isothermal family is the product of the conformal group by the non-contact group $U=u$, $V=v$, $\Theta=a\theta+h(u)+k(v)$. (Note that $(u, v)$ are the minimal coordinates of the point and $\theta$ is the inclination.) Further generalizations have been given by the authors to first order field-element transformations and curvature-element transformations (Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 406–412, vol. 28 (1942) pp. 52–55 and pp. 328–338). In this paper, the authors determine all $n$th order field-element to lineal-element transformations which preserve the isothermal character. Denote $r_\alpha=\partial^\alpha\theta/\partial u^\alpha$, $t_\beta=\partial^\beta\theta/\partial v^\beta$, $s_{\alpha\beta}=\partial^{\alpha+\beta}\theta/\partial u^\alpha\partial v^\beta$ for $\alpha, \beta=1, 2, \cdots, n$. Any transformation of our set must be of the form $U=\phi(u, r_\alpha, s_{\alpha\beta})$, $V=\psi(v, s_{\alpha\beta}, t_\beta)$, $\Theta=a\theta(s_{\alpha\beta})+h(u, r_\alpha, s_{\alpha\beta})+k(v, s_{\alpha\beta}, t_\beta)$, or this type followed by a reflection in the $X$-axis. Finally it is shown that the total set of all $n$th order differential-element to lineal-element transformations preserving the class of all isothermal families is exactly the Kasner group. (Received March 23, 1943.)

### 165. Edward Kasner and John DeCicco: *The congruence of element-series associated with a polygenic function.*

The derivative $\gamma=dw/dz$ of the polygenic function $w$ with respect to $z$ induces a correspondence $T$ between the lineal-elements $(x, y, \theta)$ of the $z$-plane and the points $\gamma=\alpha+i\beta$ of the $\gamma$-plane. This associated transformation $T$ carries the $\infty^1$ elements at a point in the $z$-plane into a circle (the Kasner circle) in the $\gamma$-plane. However, a point in the $\gamma$-plane does not correspond to a single element but to a series of elements. Therefore, in general, there exists a congruence $(\infty^2)$ of series in the $z$-plane which by $T$ is converted into the set of $\infty^2$ points in the $\gamma$-plane. Kasner has already determined those polygenic functions for which the associated $\infty^2$ series are all unions. In that event, the unions are all circles passing through a fixed point (Bull. Amer. Math. Soc. vol. 44 (1938) pp. 726–732). In this paper, a complete analytic characterization of the congruence of element-series associated with any polygenic function is given. One geometric property is that, if the tangent turbines of the $\infty^1$ series which pass through a given point $z$ are constructed at $z$, the centers of these turbines will describe a conic section. (March 23, 1943.)

### 166. J. E. Wilkins: *A special class of surfaces in projective differential geometry.*

It is known that the asymptotic osculating quadrics at a point of a curve on a surface coincide if and only if the curve is tangent to a curve of Darboux and the surface satisfies the relation (1) $\beta\psi^3=\gamma\phi^3$. This paper investigates the properties of surfaces which satisfy this relation indentically. In particular coincidence surfaces possess all of these properties. There exist, however, surfaces satisfying the relation (1) which are not coincidence surfaces. A necessary and sufficient condition that a surface which satisfies (1) be also a cubic surface is given and used to prove that there is, in the sense of projective equivalence, only one cubic coincidence surface. (Received March 11, 1943.)

## LOGIC AND FOUNDATIONS

### 167. B. A. Bernstein: *Postulate sets for Boolean rings.*

The author gives nine sets of postulates for Boolean rings in terms of ring operations. Each set is independent, and remains independent when a unit-element postulate is added. (Received March 25, 1943.)

### 168. Max Zorn: *Informal note on the second underivability theorem.*

The content of this note may be condensed into one question: In which sense does a formula like $(Ex)$ (Form $(x)$ & $\overline{\text{Bew}}$ $(x)$), which from the formalist point of view has no independent meaning, "represent" the consistency of a formalism in the sense of Hilbert? The answer to this question is expected from those who insist that the underivability of such formulae constitutes evidence in support of the opinion that finitary consistency proofs of the type which so far have been employed by the Hilbert school probably cannot be found for the arithmetic formalism or *Principia Mathematica.* (Received March 27, 1943.)

## STATISTICS AND PROBABILITY

### 169. Leon Alaoglu: *Harmonic analysis of stochastic processes.* Preliminary report.

If $\Omega$ is a differential stochastic process of elements $s(t)$ which are complex-valued functions of a real variable $t$, such that the distribution function of the variable $e^{i\theta}(s(t+h)-s(t))$ is independent of $\theta$ ($\theta$ real) and if the expectations $F(t)=\int_\Omega|s(t)-s(0)|^2dP$, $m(t)=\int_\Omega(s(t)-s(0))dP$ exist, the first being bounded and the second vanishing, then the Fourier transform of the function $s(t)$ exists for almost all $s$ and defines a stationary stochastic process. (Received March 26, 1943.)

## TOPOLOGY

### 170. B. H. Arnold: *On decompositions of $T_1$ spaces.*

Several authors (Banach, *Théorie des Opérations Linéaires*, p. 170; Eilenberg, Ann. of Math. vol. 43 (1942) pp. 568–579; Eidelheit, Studia Mathematica vol. 9 (1940) pp. 97–105) have proved theorems of the form: The "structure" of a certain class of transformations defined on a suitable space $A$ to a fixed suitable space $B$ determines the space $A$. In the present paper the author proves an analogous result which is valid for a very wide class of spaces $A$, but at the expense of allowing $B$ to become variable. If two $T_1$ spaces, $A$, $A'$ are such that the ordered system $M$ of the upper semicontinuous decompositions of $A$ is isomorphic to that of $A'$, then $A$ and $A'$ are homeomorphic. Separation, connectedness, and compactness properties of the space

$A$ are expressed in terms of order properties of $M$ and a procedure is given which allows $A$ to be constructed from $M$, again using only order properties of $M$. (Received March 6, 1943.)

### 171. D. G. Bourgin: *On a quasi linear operation.*

Let $E$ be a linear topological space which is convex (though this last restriction can be weakened). The operators referred to below are on a subset of $E$ to $E$, satisfy $U(\alpha_1 x+\alpha_2 y)=\alpha_1 U(x)+\alpha_2 U(y)$ for $\alpha_i \geqq 0$, $\alpha_1+\alpha_2=1$, and are continuous. A typical result is that if $\phi$ induces an automorphism of the convex bicompact set $K\subset E$ then the relation $\phi=(\psi+\gamma)/2$ where $\psi(K)=K$ implies $\gamma=\psi=\phi$. An incidental result is that under certain fairly general conditions $U$ may be extended to $U'$ on the closed linear extension of the original domain and then satisfies $U'(ax+by)=aU'(x)+bU'(y)+(a+b-1)U'(\theta)$ and is merely a translation plus a linear transformation. (Received March 26, 1943.)

### 172. R. H. Fox: *Natural systems of homomorphisms.* Preliminary report.

Let $P_{3k+1}$ denote the $k$th homotopy group of a complex $Y$, $P_{3k+2}$ the $k$th homotopy group of the $n$-dimensional skeleton $X$, and $P_{3k}$ the $k$th homotopy group of $Y$ mod $X$. Let $Q_{3k+1}$, $Q_{3k+2}$ and $Q_{3k}$ denote the corresponding continuous homology groups with integer coefficients. The groups $P_m$, $Q_m$, together with the natural homomorphisms $r=r_m^{(P)}$ of $P_m$ into $P_{m-1}$, $r=r_m^{(Q)}$ of $Q_m$ into $Q_{m-1}$, and $h=h_m$ of $P_m$ into $Q_m$ constitute the natural system of $Y, X$. The nucleus of $r_m$ is the image of $r_{m+1}$, and $hr=rh$. These two simple facts lead to the following theorem: If $Y$ is simply connected and acyclic in the dimensions less than or equal to $n-2$, then every $n$-cycle is spherical. Other consequences are a generalization of a theorem of S. Eilenberg (Bull. Amer. Math. Soc. vol. 47 (1941) p. 432) and the recent theorem of H. Hopf (Comment. Math. Helv. vol. 14 (1942) p. 257). (Received March 25, 1943.)

### 173. R. C. James: *Orthogonality in normed linear topological spaces.*

The purpose of this paper is to investigate certain possible definitions of orthogonality in normed linear topological spaces, properties possessed by elements orthogonal in each sense, and conditions that each type of orthogonality have certain common and desirable properties. These properties are: (1) If $x$ is orthogonal to $y(x\perp y)$, then $y\perp x$; (2) if $x\perp y$, then $ax\perp by$ for all $a$ and $b$; (3) if $x\perp y$ and $x\perp z$, then $x\perp(ay+bz)$ for all $a$ and $b$; (4) for any elements $x$ and $y$ there exists a number $a$ such that $x\perp(ax+y)$. The three definitions of orthogonality which are used are: (1) $x\perp y$ if $\|x-y\|=\|x+y\|$; (2) $x\perp y$ if $\|x\|^2+\|y\|^2=\|x-y\|^2$; (3) $x\perp y$ if $\|x+ay\|\geqq\|x\|$ for all $a$. All three types of orthogonality have the fourth property, while it is shown that if orthogonality of either of the first two definitions has all four properties the norm can be defined by a bilinear and symmetric inner product. All three definitions are equivalent and have all four properties when the norm is given by such an inner product. (Received March 12, 1943.)

### 174. Tibor Radó: *On continuous path-surfaces of zero area.*

Let $S: x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $(u, v)\in Q: 0\leqq u\leqq 1$, $0\leqq v\leqq 1$, be a continuous path-surface. Let $[S]$ denote the point-set in $(x, y, z)$ space that corresponds to $Q$ by means of the equations of $S$. The Lebesgue area $A(S)$ of $S$ is not determined by

$[S]$, but rather by the manner in which $[S]$ is described while $(u, v)$ describes $Q$. In fact, simple examples, due to Geöcze, show that $A(S)$ may equal zero even though $[S]$ is a solid cube. Geöcze derived various necessary and sufficient conditions for $A(S)=0$. The purpose of this paper is to establish further conditions of this type. The methods used are essentially topological. (Received February 19, 1943.)

### 175. G. E. Schweigert: *Fixed elements and periodic types for homeomorphisms on semi-locally-connected continua.*

Let $S$ be a semi-locally-connected continuum and $T(S)=S$ a homeomorphism. If $N \neq S$ is an invariant node then there exists in $S$ another invariant cyclic element $E \neq N$. This leads to equivalence of five properties of Ayres and Whyburn (cf. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications vol. 28, 1942, chap. 12, Theorems 4.21–4.5); pointwise almost periodicity is disregarded, second property "localized" to invariant $A$-set. Componentwise periodicity means, for every true invariant $A$-set $A$, each component of $S-A$ has finite period. In order that $T(S)=S$ be elementwise periodic on cyclic elements of complement of end points of $S$ it is necessary and sufficient that one of the following conditions holds: (a) $T$ is componentwise periodic, and for every positive integer $n$, $T^n$ has one of five properties above, (b) $T$ is componentwise periodic and each $T^n$-invariant $A$-set $A$, with no points of $A$ on true cyclic element of $S-A$, admits a contracting $T^n$-approximation at $A$, that is, for every $A$-set $C$ with $A \subset \operatorname{Int} C$ there is a third $T^n$-invariant $A$-set $B$ such that $A \subset \operatorname{Int} B \subset C$; (c) Theorem (4.7), chap. 12, holds. Part (b) is applied to give a certain continuous orbit decomposition. (Received February 27, 1943.)

### 176. G. E. Schweigert: *Minimal A-sets, infinite orbits, and fixed elements.*

If $E$ is a cyclic element of a semi-locally-connected continuum $S$ with infinite period relative to the homeomorphism $T(S)=S$ and $B$ is the least invariant $A$-set which contains the orbit of $E$, then $B=C(A, D)+0$, where $0$ is the closure of the orbit of a cyclic chain $C(E, Y)$ with $C(E, Y)\cdot C(A, D)=Y$ and $A$ and $D$ are the invariant cyclic elements of $B$ determining the cyclic chain $C(A, D)$. If $A \neq D$, then $A$ and $D$ are fixed end points of $C(A, D)$; if $A=D$, then $C(A, D)=Y$ is an unique invariant cyclic element called center of rotation. If $E=Y$, then $B=C(A, D)$. As a corollary it is sufficient to get the Ayres' property (Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications vol. 28, 1942, chap. 12, Theorem 4.5) to assume $C(E, T(E))$ contains an invariant element in (and only in) case $E$ has an infinite orbit. If $T(S)=S$ is elementwise periodic on all cyclic elements except the end points of $S$, the least invariant $A$-set $M$ containing the infinite orbit of an end point $p$ of $S$ has an unique invariant element, and the dendritic form of the cyclic element hyperspace of $M$ is similar to the dyadic tree. (Received February 27, 1943.)

### 177. J. W. T. Youngs: *On parametric representations of surfaces.* III.

A representation is a continuous transformation $R(A)=B$ from the 2-sphere $A$ to a set $B$ in 3-space. $R(A)=L(M(A))$, where $M(A)=\Sigma$ is monotone and $L(\Sigma)=B$ is light. $\Sigma$ is imbedded in 3-space so that $R$ is of degree 1 on each true cyclic element of $\Sigma$. Two representations $R_1$ and $R_2$ are $F^+$ equivalent if the Fréchet distance between them is zero and the homeomorphisms of the definition are of degree $+1$. They are $K^+$ equivalent if there is a homeomorphism $H(\Sigma_1)=\Sigma_2$ such that $L_1(\Sigma_1)=L_2(H(\Sigma_1))$

and $H$ is of degree $+1$ on each true cyclic element of $\mathbf{Z}_2$. $F^-$ and $K^-$ equivalence are defined similarly. $R_1$ is Fréchet equivalent to $R_2$ if $R_1$ is either $F^+$ or $F^-$ equivalent to $R_2$, and similarly for $K$ equivalence. Various relations between these definitions are investigated. Applications to the theory of area are made on the basis of the fact that $F$ equivalence is the same as $K$ equivalence. A general method for obtaining smooth representations is developed. The content of the paper is examined from the point of view of the $\mu$-length of Morse. (Received March 26, 1943.)

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# ON SEQUENCES OF POLYNOMIALS AND THE DISTRIBUTION OF THEIR ZEROS

OTTO SZÁSZ

The first results on this subject are due to Laguerre (1882); they were generalized to a remarkable degree by Pólya and in a joint paper by Lindwart and Pólya. I quote the following theorems [2].[1]

THEOREM 1. *If a sequence of polynomials*

$$(1)\qquad P_n(z) = 1 + \sum_1^n c_{n\nu} z^\nu = \prod_\nu (1 - z z_{n\nu}^{-1})$$

*converges uniformly in a circle* $|z| < R$, *and if for some integer* $k$

$$(2)\qquad \sum_1^n |z_{n\nu}|^{-k} < M, \qquad M \text{ independent of } n,$$

*then the sequence* (1) *converges uniformly in every finite domain to an entire function* $F(z)$ *which is the product of a function of genus at most* $k-1$ *and of* $e^{\gamma z^k}$, $\gamma$ *a constant.*

THEOREM 2. *If the sequence* (1) *converges uniformly in a circle* $|z| < R$, *and if the roots* $z_{n\nu}$ *lie in the half-plane* $\Re z \geqq 0$ *for each* $n$, *then the sequence* (1) *converges uniformly in every finite domain to an entire function* $F(z)$ *which is at most of genus* 2, *and the roots* $z_\nu$ *of* $F(z)$ *satisfy* $\sum |z_\nu|^{-2} < \infty$.

While in Theorem 1 the assumption of uniform convergence could be replaced by convergence at infinitely many points with a finite limit point and by boundedness of the sequences: $|c_{n1}|, \cdots, |c_{n\,k-1}|$, $n = 1, 2, \cdots$, the deduction of Theorem 2 required uniform convergence in $|z| < R$. We give here a new proof for Theorem 2 with a weaker hypothesis assuming instead of uniform convergence only convergence at infinitely many points in some finite domain and boundedness of the sequences $|c_{n1}|$, $|c_{n2}|$. We further generalize the assumption on the location of the zeros (following a similar remark of Weisner [5]), assuming only that the zeros of $P_n(z)$ lie in a half-plane containing the origin on its boundary, but otherwise varying with $n$. Finally we extend the results to certain sequences of entire functions.

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

LEMMA 1. *Given n complex numbers* $\xi_1, \xi_2, \cdots, \xi_n$, *which lie in a half-plane*: $\Re e^{i\theta}\xi_\nu \geqq 0$ *for some* $\theta$. *Suppose further*

$$\left|\sum_1^n \xi_\nu\right| \leqq a, \qquad \left|\sum_1^n \xi_\nu^2\right| \leqq b; \tag{3}$$

*then*

$$\sum_1^n |\xi_\nu|^2 \leqq 2a^2 + b.$$

For a similar statement see [4, vol. 1, p. 90, problem 37]. For the proof let first $\theta = 0$, $\xi_\nu = u_\nu + iv_\nu$, so that $u_\nu \geqq 0$; it then follows that

$$\sum_1^n u_\nu^2 \leqq \left(\sum u_\nu\right)^2 \leqq a^2.$$

Also from (3)

$$\left|\sum (v_\nu^2 - u_\nu^2)\right| \leqq b,$$

hence

$$\sum v_\nu^2 \leqq b + a^2, \qquad \sum |\xi_\nu|^2 = \sum u_\nu^2 + \sum v_\nu^2 \leqq 2a^2 + b.$$

In the general case let $\xi_\nu = e^{-i\theta}\eta_\nu$, then $\Re\eta_\nu \geqq 0$, $\left|\sum\eta_\nu\right| = \left|\sum\xi_\nu\right| \leqq a$, $\left|\sum\eta_\nu^2\right| = \left|\sum\xi_\nu^2\right| \leqq b$, and from the previous result

$$\sum |\eta_\nu|^2 = \sum |\xi_\nu|^2 \leqq 2a^2 + b.$$

LEMMA 2. *For any complex* $z$

$$|(1 - z)e^z| \leqq e^{r^2} \quad \textit{for} \quad |z| \leqq r.$$

Elementary calculus yields easily this inequality.

LEMMA 3. *If a polynomial*

$$P(z) = \sum_0^n c_\nu z^\nu, \qquad c_0 \neq 0,\ c_n \neq 0,$$

*has all its roots in a half-plane* $\Re e^{i\theta}z \geqq 0$, *then*

$$|P(z)| \leqq |c_0| \exp\left(r\left|\frac{c_1}{c_0}\right| + 3r^2\left(\left|\frac{c_1}{c_0}\right|^2 + \left|\frac{c_2}{c_0}\right|\right)\right) \qquad \textit{for } |z| \leqq r.$$

Let

$$P(z) = c_0\prod_1^n\left(1 - \frac{z}{z_\nu}\right), \qquad P(z)P(-z) = c_0^2\prod_1^n\left(1 - \frac{z^2}{z_\nu^2}\right),$$

so that

$$(4) \qquad -c_0\sum \frac{1}{z_\nu} = c_1, \qquad -c_0^2\sum \frac{1}{z_\nu^2} = 2c_0c_2 - c_1^2.$$

The numbers $1/z_\nu = \xi_\nu$ lie in the half-plane $\Re e^{-i\theta}z \geqq 0$, hence Lemma 1 and (4) yield

$$(5) \qquad \sum \frac{1}{|z_\nu|^2} \leqq 3\left|\frac{c_1}{c_0}\right|^2 + 2\left|\frac{c_2}{c_0}\right|.$$

Now

$$P(z) = c_0 \exp\left(-z\sum \frac{1}{z_\nu}\right)\prod\left(1 - \frac{z}{z_\nu}\right)e^{z/z_\nu},$$

and using Lemma 2 we have

$$|P(z)| \leqq |c_0| \exp\left(r\left|\frac{c_1}{c_0}\right| + r^2\left(-3\left|\frac{c_1}{c_0}\right|^2 + 2\left|\frac{c_2}{c_0}\right|\right)\right) \quad \text{for } |z| \leqq r.$$

This proves Lemma 3. An immediate consequence is the following theorem:

THEOREM I. *Given a sequence of polynomials*

$$(6) \qquad P_n(z) = \sum_{\nu=0}^{m} c_{n\nu}z^\nu, \qquad m = m_n \uparrow \infty,\ c_{n0} \neq 0,\ c_{nm} \neq 0;$$

*suppose that the roots of* $P_n(z)$ *lie in a half-plane* $\Re e^{i\theta_n}z \geqq 0$, *and suppose that for some constants* $\alpha_0$, $\alpha_1$,

$$(7) \qquad 0 < \alpha_0 \leqq |c_{n0}| \leqq \alpha_1, \quad |c_{n1}| \leqq \alpha_1, \quad |c_{n2}| \leqq \alpha_1 < \infty, \quad \textit{for all } n.$$

*Then the sequence* (6) *is uniformly bounded in any circle* $|z| \leqq r$; *in fact*

$$(8) \qquad |P_n(z)| \leqq \alpha_1 \exp\left(r\frac{\alpha_1}{\alpha_0} + 3r^2\left(\frac{\alpha_1^2}{\alpha_0^2} + \frac{\alpha_1}{\alpha_0}\right)\right).$$

The roots $z_{n\nu}$, $\nu = 1, 2, \cdots, m$ of $P_n(z)$ satisfy the inequality

$$(9) \qquad \sum_{\nu=1}^{m} \frac{1}{|z_{n\nu}|^2} \leqq 3\left(\frac{\alpha_1^2}{\alpha_0^2} + \frac{\alpha_1}{\alpha_0}\right).$$

This inequality follows from (5) and (7).

Another way of getting Theorem I is by applying a theorem of Hermite and Biehler [4, vol. 1, p. 88, problem 25].

Using well known results (see for example [3, pp. 21–30] or Vitali's convergence theorem), it follows from Theorem I that every subsequence of (6) contains a subsequence convergent everywhere, and uniformly in every circle $|z| < r$. Thus the sequence (6) has either one or several limit functions $F(z)$, and $F(z)$ is an entire function of the form (cp. (8) and (9))

$$(10) \qquad F(z) = e^{\gamma_0 + \gamma_1 z + \gamma_2 z^2} \prod \left(1 - \frac{z}{z_\nu}\right) e^{z/z_\nu}, \qquad \sum \frac{1}{|z_\nu|^2} < \infty.$$

In particular $F(z)$ is unique if the sequence (6) converges at infinitely many points with a finite limit point. A necessary and sufficient condition for this case is (see for a similar situation [1, §§1 and 2]) that each of the limits

$$(11) \qquad \lim_{n \to \infty} c_{n\nu} = c_\nu \qquad \nu = 0, 1, 2, \cdots,$$

exists. In this case (7) can be replaced by $c_0 \neq 0$. For (11) we can write:

$$\lim_{n \to \infty} P_n^{(\nu)}(0) = \nu! c_\nu, \qquad \nu = 0, 1, 2, \cdots,$$

exists. It then follows from our result that the limits

$$\lim_{n \to \infty} P_n^{(\nu)}(z) = F^{(\nu)}(z), \qquad \nu = 0, 1, 2, \cdots,$$

exist uniformly in any finite domain.

An immediate corollary of our result is this theorem.

THEOREM II. *If a formal power series* $\sum c_\nu z^\nu$ *has infinitely many partial sums* $s_n(z) = \sum_0^n c_\nu z^\nu$, *such that the roots of* $s_n(z)$ *lie in a half-plane* $\Re e^{i\vartheta_n} z \geqq 0$, *then the power series represents an entire function of the form* (10).[2]

A linear transformation enables us to shift the role of the point $z = 0$ in our results to any point $z_0$ in the plane. Thus let $w = z - z_0$, $P_n(z) = P_n(w + z_0) = Q_n(w)$. If the roots of $P_n(z)$ lie in a half-plane $\Re e^{i\vartheta_n}(z - z_0) \geqq 0$, then the roots of $Q_n(w)$ lie in a half-plane $\Re e^{i\vartheta_n} w \geqq 0$. If further

$$(12) \qquad \lim_{n \to \infty} P_n^{(\nu)}(z_0) = \nu! c_\nu(z_0) \text{ exists for all } \nu, \text{ and } c_0(z_0) \neq 0,$$

then

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[2] In this connection I refer to a paper by E. Benz, *Über lineare, verschiebungstreue Funktionaloperationen und die Nullstellen ganzer Funktionen*, Comment. Math. Helv. vol. 7 (1935) pp. 243–289, which was pointed out to me by the referee.

$$\lim_{n\to\infty} Q_n^{(\nu)}(0) = \nu! c_\nu(z_0).$$

We thus get this theorem:

THEOREM III. *If the sequence of polynomials* (6) *is such that the roots of* $P_n(z)$ *lie in a half-plane* $\Re e^{i\vartheta_n}(z-z_0)\geqq 0$, *if moreover* (a) *either* (12) *holds, or* (b) $0<\beta_0\leqq |P_n(z_0)| \leqq\beta_1$, $|P_n'(z_0)| \leqq\beta_1$, $|P_n''(z_0)| \leqq\beta_1$ *for some constants* $\beta_0$, $\beta_1$, *and for all* $n$ *and* $\lim_{n\to\infty}P_n(z)$ *exists at infinitely many points in a finite domain, then* $P_n(z)$ *converges uniformly in every finite domain to an entire function of the form* (10).

Instead of polynomials we may consider more generally primitive entire functions of genus 0, thus

$$(13)\quad F_n(z) = \sum_{\nu=0}^{\infty} c_{n\nu}z^\nu = c_{n0}\prod_\nu\left(1-\frac{z}{z_{n\nu}}\right),\qquad \sum_\nu \frac{1}{|z_{n\nu}|}<\infty,$$
$$n = 1, 2, 3, \cdots .$$

Theorem I remains true if we replace $P_n$ by $F_n$, assuming accordingly that the roots of $F_n(z)$ lie in a half-plane $\Re e^{i\vartheta_n}z\geqq 0$ and that (7) holds. To prove this we note that

$$c_{n1} = -c_{n0}\sum_\nu \frac{1}{z_{n\nu}},\qquad -c_{n0}^2\sum \frac{1}{z_{n\nu}^2} = 2c_{n0}c_{n2}-c_{n1}^2,$$

hence

$$\left|\sum_\nu z_{n\nu}^{-1}\right|\leqq\frac{\alpha_1}{\alpha_0},\qquad \left|\sum_\nu z_{n\nu}^{-2}\right|\leqq 2\frac{\alpha_1}{\alpha_0}+\frac{\alpha_1^2}{\alpha_0^2}.$$

Lemma 1 evidently applies to absolutely convergent series, and yields

$$\sum_\nu |z_{n\nu}|^{-2}\leqq 3\frac{\alpha_1^2}{\alpha_0^2}+2\frac{\alpha_1}{\alpha_0}.$$

The corresponding extension of Lemma 3 yields

$$|F_n(z)|\leqq c_0\exp\left\{r\frac{\alpha_1}{\alpha_0}+3r^2\left(\frac{\alpha_1^2}{\alpha_0^2}+\frac{\alpha_1}{\alpha_0}\right)\right\}\qquad\text{for } |z|\leqq r;$$

thus the sequence (13) is uniformly bounded in a given circle $|z|\leqq r$. The rest of the argument is the same as in the case of polynomials.

A similar reasoning holds for a more general class of entire functions. We assume that $F_n(z)$ is of the form

$$(14) \qquad F_n(z) = \sum_\nu c_{n\nu} z^\nu = c_{n0} e^{\gamma_n z} \prod_\nu (1 - z z_{n\nu}^{-1}) e^{z z_{n\nu}^{-1}},$$

where $\Re e^{i\vartheta_n} z_{n\nu} \geqq 0$, $\sum_\nu |z_{n\nu}|^{-2} < \infty$. We assume furthermore that the series $\sum_\nu (\Re e^{-i\vartheta_n} z_{n\nu}^{-1}) = \beta_n$ converges and that $\beta_n \leqq \beta$ for all $n$. Assuming (7) we shall prove that the sequence (14) is uniformly bounded in any circle $|z| < r$.

We have

$$\frac{F_n'(z)}{F_n(z)} = \gamma_n + \sum_\nu \left( \frac{1}{z - z_{n\nu}} + \frac{1}{z_{n\nu}} \right),$$

$$F_n''(z) F_n(z) - (F_n'(z))^2 = - F_n^2(z) \sum_\nu \frac{1}{(z - z_{n\nu})^2},$$

and, putting $z = 0$,

$$c_{n1} = c_{n0} \gamma_n, \qquad 2c_{n0}c_{n2} - c_{n1}^2 = - c_{n0}^2 \sum_\nu z_{n\nu}^{-2}.$$

Now from (7)

$$|\gamma_n| \leqq \frac{\alpha_1}{\alpha_0}, \qquad \left| \sum_\nu z_{n\nu}^{-2} \right| \leqq \left( \frac{\alpha_1}{\alpha_0} \right)^2 + 2 \frac{\alpha_1}{\alpha_0};$$

furthermore putting $e^{-i\vartheta_n} z_{n\nu}^{-1} = u_{n\nu} + i v_{n\nu}$, we have $u_{n\nu} \geqq 0$, and (as in Lemma 1)

$$\sum_\nu |z_{n\nu}|^{-2} = \sum_\nu | e^{-i\vartheta_n} z_{n\nu}^{-1} |^2 = 2 \sum_\nu u_{n\nu}^2 - \sum_\nu \Re(e^{-i\vartheta_n} z_{n\nu}^{-1})^2$$

$$\leqq 2\beta^2 + \left( \frac{\alpha_1}{\alpha_0} \right)^2 + 2 \frac{\alpha_1}{\alpha_0} \cdot$$

Thus, using Lemma 2, we have

$$|F_n(z)| < \alpha_1 \exp \left\{ \frac{\alpha_1}{\alpha_0} r + \left( 2\beta^2 + 2 \frac{\alpha_1}{\alpha_0} + \frac{\alpha_1^2}{\alpha_0^2} \right) r^2 \right\} \qquad \text{for } |z| < r,$$

which proves our assertion.

If we assume that $\sum_\nu z_{n\nu}^{-1} = \xi_n$ converges, and put $\gamma_n = -\xi_n$, then the assumption on the $\beta_n$ is superfluous.

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# A NOTE ON SEPARATION AXIOMS AND THEIR APPLICATION IN THE THEORY OF A LOCALLY CONNECTED TOPOLOGICAL SPACE

J. W. T. YOUNGS

In a recent paper [1][1] G. E. Albert and the author attempt a comprehensive study of a locally connected (1.8) topological space from the point of view of Peano space theory [2]. Cyclic elements are defined (2.15) and are themselves found to be locally connected and topological (2.29). Moreover, it is shown that under proper and very natural topologization (3.3) the class of all cyclic elements (the hyperspace) becomes a locally connected topological space (3.3 and 3.8). In fact, this hyperspace has no nondegenerate[2] cyclic elements (3.17).

For the purposes of this note it is the concept of a hereditary class of spaces which is important (4.1). A subclass $\mathcal{H}$ of the class $\mathfrak{X}$ of all locally connected topological spaces is hereditary if, whenever $X$ is a space of the class $\mathcal{H}$: (1) each true cyclic element (2.15) of $X$ is a member of $\mathcal{H}$; and (2) the hyperspace $X_\lambda$ is in $\mathcal{H}$. (It should be remembered that the first condition is the one required of a class for it to be cyclicly reducible in the classical Peano space theory.)

The problem is to define small hereditary classes (4.1). In fact, though there is a smallest hereditary class, an intrinsic definition of it is lacking (4.2–4.5).

In this connection the main results are that: (1) the class of all locally connected $T_0$-spaces is a hereditary class (4.10); (2) the class of all locally connected $T_1$-spaces is *not* a hereditary class (4.1).

It is the purpose of this note: (1) to define, by means of a separation axiom, a new hereditary class; (2) to place this separation axiom

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[1] Numbers in brackets refer to the bibliography; numbers in parentheses to appropriate paragraphs in [1].

[2] A set is degenerate if it is vacuous or contains but a single point.

in the spectrum of available separation axioms; (3) to point out the fitting character of the axiom from the point of view of Peano space theory.

First the axiom:

$T$: *If* $x\cdot y = 0$, *then* $\bar{x}\cdot\bar{y}$ *is degenerate.*[2]

(The reader is reminded that in a topological space the closure of a point may contain many points.)

THEOREM. *The class of all locally connected $T$-spaces is hereditary.*

PROOF. Suppose that $X$ is such a space. It is clear that any true cyclic element $M$ of $X$ is a $T$-space. That $M$ is locally connected has been shown elsewhere (2.29).

Consider the hyperspace $X_h$ consisting of the totality of cyclic elements of $X$. Let $\xi$ and $\eta$ be any two distinct points of $X_h$.[3]

(1) Suppose that $T^{-1}(\xi)$ is nondegenerate. (For a definition of the transformation $T(X) = X_h$ see 3.2.) Then there is a true cyclic element $M$ such that $T^{-1}(\xi) = k(M)$, (2.14).

From the definition of the topologization (3.2) and the fact that $M$ is a closed inverse set, it follows that $T(M)$ is closed. Moreover, $\xi \subset T(M)$, therefore $\bar{\xi} \subset T(M)$, and $T^{-1}(\bar{\xi}) \subset M$.

If $M\cdot T^{-1}(\eta) = 0$, let $S$ be the component of $X - M$ which contains $T^{-1}(\eta)$. The existence of $S$ is doubtful only in case $T^{-1}(\eta)$ is nondegenerate. But then $T^{-1}(\eta)$ is contained in some true cyclic element and so lies in a component $S$ of $X - M$ (2.29 and 2.15). Now $T^{-1}(\eta) \subset \bar{S} = S + F(S)$, and this is an inverse set (3.10 and 3.11). Therefore, $\bar{\eta} \subset T(\bar{S})$, and $T^{-1}(\bar{\eta}) \subset \bar{S}$. Hence, $T^{-1}(\bar{\xi}\cdot\bar{\eta}) = T^{-1}(\bar{\xi})\cdot T^{-1}(\bar{\eta}) \subset M\cdot\bar{S} = F(S)$ which is degenerate (2.15). Therefore $\bar{\xi}\cdot\bar{\eta}$ is degenerate.

If $M\cdot T^{-1}(\eta) \neq 0$, then $T^{-1}(\eta) = y \in M - k(M)$ (2.12 and 2.15). Hence $y$ is a cut point of the space, and as such $\bar{y} = y$ (2.2). Therefore $\bar{\eta} = \eta$ and $\bar{\xi}\ \bar{\eta} = \bar{\xi}\cdot\eta$ is certainly degenerate.

(2) A similar argument applies in the event $T^{-1}(\eta)$ is nondegenerate.

(3) If $T^{-1}(\xi) = x$ and $T^{-1}(\eta) = y$, then $\bar{x}$ and $\bar{y}$ are inverse sets (4.9) and so $T(\bar{x})$ and $T(\bar{y})$ are closed in $X_h$. Hence these sets contain $\bar{\xi}$ and $\bar{\eta}$, respectively; moreover, $T^{-1}(\bar{\xi}) \subset \bar{x}$ and $T^{-1}(\bar{\eta}) \subset \bar{y}$. Therefore $T^{-1}(\bar{\xi}\cdot\bar{\eta}) = T^{-1}(\bar{\xi})\cdot T^{-1}(\bar{\eta}) \subset \bar{x}\cdot\bar{y}$ *which is degenerate because $X$ is a*

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[3] The reader who is unfamiliar with [1] can still follow the spirit of the argument, if he is acquainted with [2], by thinking of $X$ as a Peano space. In case $T^{-1}(\xi)$ is degenerate think of it as a cut point or an end point. If it is nondegenerate, think of it as a true cyclic element less all the cut points of $X$. As for the topologization in $X_h$, a set is open if and only if its inverse under $T$ is open in $X$.

*T-space*. This implies $\xi \cdot \bar{\eta}$ is degenerate, and the proof is complete on observing that $X_k$ is locally connected and topological (3.3 and 3.8).

The separation axioms mentioned in the course of this note may be tabulated as follows:

$T_0$: $x \cdot \bar{y} + \bar{x} \cdot y$ is degenerate.
$T$ : $\bar{x}\;\bar{y}$ is degenerate.

If $x \cdot y = 0$, then

$T_1$: $x \cdot \bar{y} + \bar{x} \cdot y = 0$.
$T_1$: $\bar{x} \cdot \bar{y} = 0$ (alternate form).

THEOREM. *A $T_1$-space is a $T$-space; a $T$-space is a $T_0$-space.*

The proof is left as an entertaining exercise for the reader. An immediate consequence is that the $T$-axiom might very well have attached to the letter $T$ as a subscript any positive number less than one.

In conclusion it might be interesting to note that the $T$-axiom fits the phenomenon of Peano space theory relative to intersections of cyclic elements. The theorem that *two cyclic elements which have more than one point in common are identical* has as its counterpart the statement that *two points whose closures have more than one point in common are identical.*

## BIBLIOGRAPHY

1. G. E. Albert and J. W. T. Youngs, *The structure of locally connected topological spaces*, Trans. Amer. Math. Soc. vol. 51 (1942) p. 637.

2. C. Kuratowski and G. T. Whyburn, *Sur les éléments cycliques et leurs applications*, Fund. Math. vol. 16 (1930) p. 305.

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# INTERSECTIONS OF CONTRACTIBLE POLYHEDRA

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Let the finite polyhedron $C$ be expressed as a sum of two polyhedra, $A$ and $B$. Using the Mayer-Vietoris formula, it is easy to prove the following two propositions:

(1) *If $A\cap B$, the intersection of $A$ and $B$, is acyclic,*[2] *then $C$ is acyclic if and only if both $A$ and $B$ are acyclic.*

(2) *If $A$, $B$, and $C$ are all acyclic, so is $A\cap B$.*

Aronszajn and Borsuk have shown[3] that proposition (1) is also true if acyclic is replaced by contractible.[4] They left open the question as to whether or not this is true for proposition (2). We show by means of an example that it is not true. The example is constructed as follows:

Let $P$ be a Poincaré sphere, that is, a 3-dimensional polyhedron with the homology groups of a 3-sphere and with a non-vanishing fundamental group. Let $K$ be the polyhedron obtained by removing an open 3-simplex from $P$. It is easy to see that $K$ is acyclic and that its fundamental group is the same as that of $P$, so that $K$ is not contractible.

Let $C$ be the join[5] of $K$ with two points, $q$ and $q'$. Denote the join of $K$ with $q$ by $A$, and the join of $K$ with $q'$ by $B$. Let $C=A\cup B$ the sum of $A$ and $B$. Then $A\cap B=K$. It is clear that $A$ can be continuously deformed into the point $q$, and $B$ into $q'$, so both these polyhedra are contractible. Hence it remains to show only that $C$ is contractible. By proposition (1), $C$ is acyclic. Hurewicz has shown[6] that if the fundamental group of an acyclic polyhedron vanishes,

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[1] National Research Fellow.

[2] A polyhedron is said to be acyclic if all its homology groups vanish for all coefficient groups.

[3] N. Aronszajn, and K. Borsuk, *Sur la somme et le produit combinatoire des retracts absolus*, Fund. Math. vol. 18 (1932) pp. 193–197.

[4] A set is said to be contractible if it can be deformed over itself into a point. A contractible polyhedron is acyclic.

[5] The join of two sets, $X$ and $Y$, is the collection of line segments joining each point of $X$ with each point of $Y$ and disjoint except for their common end points. If $X$ and $Y$ are polyhedra, so is their join.

[6] W. Hurewicz, *Beiträge zur Topologie der Deformationen*, Neder. Akad. Wetensch. vol. 38 (1935) pp. 521–528, Theorem IV.

then the polyhedron is contractible. Therefore we are reduced to showing that the fundamental group of $C$ is trivial. Let the origin for this group be chosen in $K$. Using the fact that $C$ is a polyhedron, it is easy to see that each closed path in $C$ is homotopic to a sum of closed paths, each one entirely in $A$ or in $B$. But both $A$ and $B$ are contractible, so each of these paths is homotopic to a point, which completes the proof.[7]

REMARKS. I.[8] The dimension of the polyhedron $C$ is 4, but a similar example can be constructed for which the dimension is 3. This follows from the fact that $K$ can be retracted by deformation into a part, $K'$, of its 2-dimensional skeleton. Using $K'$ in place of $K$ in the above construction, the resulting polyhedron will be of dimension 3.

II. By the usual method of condensation of singularities, compact metric spaces $A'$ and $B'$ can be constructed such that $A'$, $B'$, and $A' \cup B'$ are all locally contractible,[9] while $A' \cap B'$ is not. However, it is relatively easy to prove the generalization of proposition (2) to the case of locally acyclic compact metric spaces.

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[7] Cf. §52 of H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig, 1934.

[8] This remark is due to S. Eilenberg.

[9] A space is said to be locally contractible at a point $p$ if each neighborhood $U$ of $p$ contains a neighborhood $V$ of $p$ such that $V$ can be deformed inside $U$ into a point. The space is locally contractible if it is locally contractible at each point. The definition of a locally acyclic space is entirely analogous.

# ON LINEAR COMBINATIONS OF QUADRATIC FORMS

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The characteristics of linear combinations $\sum \lambda_i Q_i(x)$ of a given set of real quadratic forms

$$(1) \qquad Q_i(x) \equiv \sum_{k,l=1}^{n} a_{kl}^{i} x_k x_l, \qquad i = 1, 2, \cdots, m,$$

have been considered in several recent papers.[1]

One of the theorems in my earlier paper may be stated as follows:

*A necessary and sufficient condition that there exist a linear combination $\sum \lambda_i Q_i(x)$ which is positive definite is that there exist no set of points $x^j = (x_1^j, x_2^j, \cdots, x_n^j) \neq (0, \cdots, 0)$ $(j = 1, 2, \cdots, r)$ such that*

$$(2) \qquad \sum_{j=1}^{r} \mu_j Q_i(x^j) = 0, \qquad i = 1, 2, \cdots, m,$$

*the coefficients $\mu_j$ being positive.*

Shortly after the publication of this paper, Fritz John kindly called my attention to the fact that a closely related result is contained in an earlier paper of his.[2]

Certainly John's paper contains essentially the "sufficiency" half of the theorem quoted above. Furthermore it introduces a very interesting suggestion in noting that the validity of relations (2) implies the existence of a quadratic form

$$B(x) \equiv \sum_{k,l=1}^{n} b_{kl} x_k x_l$$

which is definite or semi-definite, and such that

$$\sum_{k,l=1}^{n} a_{kl}^{i} \cdot b_{kl} = 0, \qquad i = 1, 2, \cdots, m.$$

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[1] Finsler, *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Comment. Math. Helv. vol. 9 (1937) pp. 188–192. Hestenes and McShane, *A theorem on quadratic forms and its application in the calculus of variations*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 501–512. Dines, *On the mapping of n quadratic forms*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 467–471.

[2] *A note on the maximum principle for elliptic differential equations*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 268–271.

This condition, unlike those occurring in other treatments of the problem, focuses attention on the coefficients of the given forms. Because of its purely algebraic character and its possible usefulness in applications it seems worthy of further study, and that is the purpose of the present note. In Theorem I there is an analysis of possibilities in terms of this relationship, and in Theorem II there is an equivalent statement from a different and possibly interesting point of view.

1. **Orthogonality of quadratic forms as to their coefficients.** The bilinear form consisting of the sum of products of corresponding coefficients of two quadratic forms

$$Q(x) = \sum_{k,l=1}^{n} a_{kl}x_kx_l, \quad B(x) = \sum_{k,l=1}^{n} b_{kl}x_kx_l, \qquad a_{kl} = a_{lk},\ b_{kl} = b_{lk},$$

will be denoted by $(Q\cdot B)$. That is[3]

$$(Q\cdot B) = \sum_{k,l=1}^{n} a_{kl}\cdot b_{kl}.$$

Obviously $(Q\cdot B)=(B\cdot Q)$; and for a set of forms such as those in (1), $(\sum\lambda_iQ_i\cdot B)=\sum\lambda_i(Q_i\ B)$. Also it may be easily verified that $(Q\cdot B)$ is invariant under the group of orthogonal transformations on $x_1, x_2, \cdots, x_n$.

If the two quadratic forms $Q(x)$ and $B(x)$ satisfy the relation $(Q\ B)=0$ they will be said to be *orthogonal as to coefficients* (or *c-orthogonal*).[4]

Relative to a given system of real quadratic forms (1), we now state

THEOREM I. (a) *A necessary and sufficient condition that there exist a definite linear combination $\sum\lambda_iQ_i$ is that every quadratic form c-orthogonal to all the $Q_i$ be indefinite.*

(b) *A necessary and sufficient condition that every linear combination $\sum\lambda_iQ_i$ be indefinite is that there exist a definite quadratic form c-orthogonal to all the $Q_i$.*

(c) *A necessary and sufficient condition that there exist a semi-definite*

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[3] M. R. Hestenes has kindly called attention to the fact that this function of the coefficients of two forms was used by L. Fejér in a paper entitled *Über die Eindeutigkeit der Losung der linearen partiellen Differentialgleichung zweiter Ordnung*, Math. Zeit. vol. 1 (1918) pp. 70–73 Fejér obtained a theorem which may be stated in our notation as follows: *If $Q(x)$ and $B(x)$ are both non-negative, then* $(Q\cdot B)\geqq 0$.

[4] However, to avoid bothersome trivialities, we shall assume that an assertion $(Q\cdot B)=0$ implies *per se* that each of the forms $Q\cdot B$ has at least one nonzero coefficient. In the same spirit, the notation $\sum\lambda_iQ_i$ implies *per se* that at least one $\lambda_i$ is different from zero.

*(but no definite) linear combination* $\sum \lambda_i Q_i$ *is that there exist a semi-definite (but no definite) quadratic form c-orthogonal to all the* $Q_i$.

2. **Proof of Theorem I(a).** In view of the theorem quoted in our introduction, the validity of I(a) is an immediate consequence of the following lemma.

LEMMA. *A necessary and sufficient condition that the quadratic forms* (1) *admit relations* (2) *is that there exist a definite* (*or semi-definite*) *form* $B(x)$ *such that*

$$(Q_i \cdot B) = 0, \qquad i = 1, 2, \cdots, m. \tag{3}$$

The first half of this lemma is due to John, as is the following neat proof of it.

Suppose there exist relations (2). These may be written

$$\sum_{j=1}^{r} \mu_j \sum_{k,l=1}^{n} \overset{i}{a}_{kl} \overset{j}{x}_k \overset{j}{x}_l = 0, \qquad i = 1, 2, \cdots, m,$$

or

$$\sum_{k,l=1}^{n} \overset{i}{a}_{kl} \cdot b_{kl} = 0 \quad \textit{where} \quad b_{kl} = \sum_{j=1}^{r} \mu_j \overset{j}{x}_k \overset{j}{x}_l.$$

Hence the quadratic form $B(x)$ with coefficients $b_{kl}$ is $c$-orthogonal to every $Q_i(x)$.

Furthermore the form $B(x)$ is definite or semi-definite, since

$$B(x) \equiv \sum_{k,l=1}^{n} b_{kl} x_k x_l = \sum_{k,l=1}^{n} \left( \sum_{j=1}^{r} \mu_j \overset{j}{x}_k \overset{j}{x}_l \right) x_k x_l = \sum_{j=1}^{r} \mu_j \left( \sum_{k=1}^{n} \overset{j}{x}_k x_k \right)^2.$$

*Conversely*, suppose there is a definite (or semi-definite) form $B$ satisfying (3). From the assumption of definiteness, it can be expressed in the form

$$B(x) \equiv \sum_{k,l=1}^{n} b_{kl} x_k x_l = \sum_{j=1}^{r} \mu_j \left( \sum_{k=1}^{n} \overset{j}{\alpha}_k x_k \right)^2$$

where the coefficients $\overset{j}{\alpha}_k$ are suitably chosen constants and the $\mu_j$ are positive constants. Hence

$$b_{kl} = \sum_{j=1}^{r} \mu_j \overset{j}{\alpha}_k \overset{j}{\alpha}_l.$$

And in view of (3)

$$\sum_{k,l=1}^{n} \overset{i}{a}_{kl} \sum_{j=1}^{r} \mu_j \overset{j}{\alpha}_k \overset{j}{\alpha}_l = 0, \qquad i = 1, 2, \cdots, m,$$

and equivalently

$$\sum_{j=1}^{r} \mu_j \sum_{k,l=1}^{n} a_{kl}^{i} \alpha_k^j \alpha_l^j = 0, \qquad i = 1, 2, \cdots, m,$$

whence follows (2) with $x_k^j = \alpha_k^j$.

3. **Proof of Theorem I(b).** This proof is facilitated by the following corollary.

COROLLARY OF I(a). *If two quadratic forms are c-orthogonal and one is definite, then the other is indefinite.*

The *sufficiency* part of I(b) follows immediately. For if a definite form $B$ is $c$-orthogonal to every $Q_i$ it is $c$-orthogonal to every linear combination $\sum \lambda_i Q_i$, and hence by the corollary every such linear combination is indefinite.

To prove the *necessity* part of I(b), we assume that every linear combination $\sum \lambda_i Q_i(x)$ changes sign. On account of the homogeneity property $Q(tx) = t^2 Q(x)$, we are indeed justified in assuming that every such linear combination changes sign as $x$ varies on the unit hypersphere $\|x\| = 1$ in the $n$-dimensional $x$-space.

The set of points $\mathfrak{M}_1$ with coordinates represented by

$$\mathfrak{M}_1: \qquad (Q_1(x), Q_2(x), \cdots, Q_m(x)), \qquad \|x\| = 1,$$

in $m$-dimensional space is closed and bounded. Hence, in view of our assumption, its convex extension $C(\mathfrak{M}_1)$ contains the origin of that $m$-dimensional space as an *inner* point.[5] This origin can therefore be the centroid of positive masses at a suitably chosen finite and *truly* $m$-dimensional subset of points[6]

$$\mathfrak{M}_2: \qquad (Q_1(x^p), Q_2(x^p), \cdots, Q_m(x^p)), \qquad p = 1, 2, \cdots, r.$$

To the finite set $\mathfrak{M}_2$ we arbitrarily adjoin the set of $n$ points

$$\mathfrak{M}_3: \qquad (Q_1(x^j), Q_2(x^j), \cdots, Q_m(x^j)), \qquad j = 1, 2, \cdots, n,$$

where each $n$-partite number $x^j = (x_1^j, x_2^j, \cdots, x_n^j)$ has all zero components with the exception that $x_j^j = 1$.

The set $\mathfrak{N} = \mathfrak{M}_2 + \mathfrak{M}_3$ is a truly $m$-dimensional finite set, and its convex extension contains the origin as an inner point. Hence this origin can be the centroid of positive masses at *all*[7] points of $\mathfrak{N}$; and the analytic expression of this fact leads to the system of relations

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[5] Cf. Dines, *Convex extension and linear equalities*, Bull. Amer. Math. Soc. vol. 42 (1936) p. 357 Theorem 3.

[6] Ibid. p. 358.

[7] Ibid. p. 358 Theorem 4.

$$(4)\qquad \sum_{j=1}^{n} \mu_j Q_i(x^j) + \sum_{p=1}^{r} \nu_p Q_i(x^p) = 0, \qquad i = 1, 2, \cdots, m,$$

where each $\mu_j$ and $\nu_p$ is positive.

Applying to the relations (4) precisely the same type of argument as was applied to relations (2) in proving the lemma of §2, we arrive at the existence of a quadratic form $B(x)$ which is $c$-orthogonal to all the $Q_i$, which has the formal expression

$$B(x) = \sum_{j=1}^{n} \mu_j x_j^2 + \sum_{p=1}^{r} \nu_p \left( \sum_{k=1}^{n} x_k^p x_k \right)^2$$

and is therefore certainly definite. This completes the proof of I(b).

4. **Proof of Theorem I(c).** This follows almost immediately from I(a) and I(b). First suppose there is a $\sum \lambda_i Q_i$ which is semi-definite, but none which is definite. Then by I(b) there is no definite form $c$-orthogonal to all $Q_i$, and so by I(a) there must be one which is semi-definite.

Conversely, suppose there exists a semi-definite (but no definite) form $c$-orthogonal to all the $Q_i$. Then by I(a) there exists no definite linear combination $\sum \lambda_i Q_i$, and hence by I(b) there must exist a semi-definite $\sum \lambda_i Q_i$.

5. **The system of equations** $(Q_i \cdot B) = 0$. The substance of Theorem I can be equivalently expressed in terms of the system of equations

$$(5)\qquad (Q_i \cdot B) = 0, \qquad i = 1, 2, \cdots, m,$$

where the $Q_i$ are given quadratic forms and $B$ is a quadratic form restricted by the system of equations.

THEOREM II. *The system of equations* (5)

(a) *can admit only indefinite solutions $B$ if the $Q_i$ admit a definite linear combination,*

(b) *admits a definite solution $B$ if every linear combination of the $Q_i$ is indefinite,*

(c) *admits a semi-definite (but no definite) solution $B$ if the $Q_i$ admit a semi-definite (but no definite) linear combination.*

It will be noted that the system certainly admits some solution $B$ except in the case where there exists a definite linear combination of the $Q_i$. In this latter case there may be indefinite solutions or *no* solution as illustrated in the following two examples.

EXAMPLE 1. The two forms

$$Q_1 = 2x_1^2 - x_2^2, \qquad Q_2 = x_1^2 - 2x_2^2$$

admit the definite linear combination $Q_1 - Q_2 = x_1^2 + x_2^2$, and the corresponding system (5) admits the indefinite solution $B = x_1x_2$.

EXAMPLE 2. The three forms

$$Q_1 = 2x_1^2 - x_2^2, \qquad Q_2 = x_1^2 - 2x_2^2, \qquad Q_3 = x_1x_2$$

admit the definite linear combination $Q_1 - Q_2 - Q_3 = x_1^2 - x_1x_2 + x_2^2$, but the corresponding system (5) admits no solution form $B$.

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# NOTE ON A CONJECTURE DUE TO EULER

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Euler's conjecture (1772) that

$$x_1^n + \cdots + x_t^n = x^n,$$

where $n$ is an integer greater than 3 and $2 < t < n$, has no solution in rational numbers $x_1, \cdots, x_t, x$ all different from zero, is still unsettled even in its first case, $n = 4$, $t = 3$. It may therefore be of some interest to note a solution of this equation for any $n > 3$ and any $t \lessgtr 1$ in terms of (irrational) algebraic numbers, which can be made algebraic integers by suitable choice of a homogeneity parameter, all different from zero, all the numbers being polynomials in numbers of degree $2d$, where $4d \leqq 2n - 5 + (-1)^n$. If solutions differing only by a parameter are not considered distinct, there are at least $d^{t-1}$ sets of solutions $x_1, \cdots, x_t, x$.

The solutions described are

$$x_1 = u, \qquad x_2 = r_{t-1}u, \qquad x = (1 + r_1) \cdots (1 + r_{t-1})u;$$

$$x_j = r_{t-j+1}(1 + r_{t-j+2})(1 + r_{t-j+3}) \cdots (1 + r_{t-1})u, \qquad j = 3, \cdots, t,$$

where $u$ is a parameter and the $r$'s are any roots, the same or different, of any factor $F_n(r)$, irreducible in the field of rational numbers, of

$$f(r) \equiv \sum_{s=1}^{n-1} (n, s) r^{n-s-1},$$

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where $(n, s)$ is the binomial coefficient $n!/s!(n-s)!$. For, $(r\neq 0)$, $f(r)=0$ implies $(1+r)^n=1+r^n$; whence the verification is immediate on successive reduction of $x_1^n+x_2^n$, $x_1^n+x_2^n+x_3^n$, $x_1^n+\cdots+x_t^n$. The remarks on $d$ and the number of sets of solutions then follow since $f(r)=0$ is a reciprocal equation, and $F_n(r)$ has no multiple roots.

With $y=r+r^{-1}$, the first seven $F_n(r)$ are

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\begin{aligned}
n &= 4: && 2y+3;\\
n &= 5: && y+1;\\
n &= 6: && 6y^2+15y+8;\\
n &= 7: && y+1;\\
n &= 8: && 4y^3+14y^2+16y+7;\\
n &= 9: && 3y^3+9y^2+10y+5;\\
n &= 10: && 10y^4+45y^3+80y^2+75y+32.
\end{aligned}
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# SET-COORDINATES FOR LATTICES[1]

ALAN D. CAMPBELL

On p. 26 of Garrett Birkhoff's *Lattice theory* (Amer. Math. Soc. Colloquium Publications, vol. 25, 1939) we find the following theorem.

THEOREM 2.12. *Any partially ordered system has a one-one representation by sets which preserves inclusion and meets.*

In the proof of the above theorem each element of the partially ordered system is represented by its "normal hull." We shall call this the *regular* representation.

DEFINITION 1. *By a representation by sets of the elements of a lattice $L$ we mean any one-one representation by sets which preserves inclusion and carries meets into set-products.*

The above-mentioned regular representation is by no means the most economical. The following illustration bears out this point by using ten elements instead of the twenty-eight required by the regular representation.

ILLUSTRATION 1. In the following manner we can represent the lattice in Figure 5 on p. 49 of Birkhoff (loc. cit.) by suitable set-coordinates (using sets of integers):

$O=(\ )$, $v_3=(1)$, $v_2=(2)$, $v_1=(3)$,
$d_1=(1, 2)$, $d_2=(1, 3)$, $d_3=(2, 3)$,
$a_1=(1, 2, 4)$, $a_2=(1, 3, 5)$, $d=(1, 2, 3)$, $a_3=(2, 3, 6)$,
$x_1=(1, 2, 4, 7)$, $x_2=(1, 3, 5, 8)$, $e_1=(1, 2, 3, 4)$, $e_2=(1, 2, 3, 5)$, $e_3=(1, 2, 3, 6)$, $x_3=(2, 3, 6, 9)$,
$b_1=(1, 2, 3, 4, 7)$, $b_2=(1, 2, 3, 5, 8)$,
$c=(1, 2, 3, 4, 5, 6)$, $b_3=(1, 2, 3, 6, 9)$,
$c_1=(1, 2, 3, 4, 5, 6, 7)$, $c_2=(1, 2, 3, 4, 5, 6, 8)$, $c_3=(1, 2, 3, 4, 5, 6, 9)$,
$u_3=(1, 2, 3, 4, 5, 6, 7, 8)$, $u_2=(1, 2, 3, 4, 5, 6, 7, 9)$, $u_1=(1, 2, 3, 4, 5, 6, 8, 9)$,
$I=(1, 2, 3, 4, 5, 6, 7, 8, 9)$.

We remark that only nine elements (the integers 1, 2, 3, 4, 5, 6, 7, 8, 9) together with certain of the "sums" of these integers are necessary to represent this lattice (instead of the twenty-eight elements in the given Hasse diagram required in the regular representation).

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[1] The author wishes to express his gratitude to Professor Garrett Birkhoff for his advice and interest in the preparation of this paper.

DEFINITION 2. *By a simple join in a lattice $L$ we mean the join of a collection $\gamma$ of elements in $L$ if this join goes by every representation by sets into the sum of the sets corresponding to the elements of $\gamma$.*

ILLUSTRATION 2. Let a general finite lattice $L$ have join-irreducible elements (or atoms) $a_1, a_2, \cdots, a_n$. We define the set-coordinate of any $x \in L$ as the set of integers $i$ such that $a_i \leq x$. Suppose that an atom 2 has above it a chain of two elements that are not joins of atoms, we can label these elements $(2, n+1)$ and $(2, n+1, n+2)$. If the atoms 3, 4, and 5 have a common simple join (that is also the join of each of the pairs 3, 4 and 3, 5 and 4, 5), then the lattice has no elements (3, 4), (3, 5), and (4, 5) but has an element (3, 4, 5).

We define the representation of Illustration 2 as the *economical* representation of $L$, to contrast it with the regular representation. It is an unsolved problem whether or not the economical representation actually involves the fewest possible points as set-coordinates.

Note the fact (compare Birkhoff, loc. cit., p. 26) that there exists no one-one representation of a lattice $L$ by sets which carries all meets of elements into set-products and also all joins into set-unions, unless $L$ is distributive. Also there exist representations of the elements of $L$ by sets which preserve inclusion without sending meets into set-products or joins into set-unions.

We shall denote the product (intersection) of two sets $x$ and $y$ by $x \cdot y$, the union (sum) of two sets $x$ and $y$ by $(x, y)$, and the fact that a set $z$ includes a set $w$ by $z \geq w$. We shall denote the empty set by 0. If two sets $u$ and $v$ have no intersection except the empty set, we shall say that $u$ and $v$ have the product 0 (in algebraic form $u \cdot v = 0$). If the element $u$ of $L$ corresponds to the set-product $x\ y$, we write this fact as $u = x \cdot y$ or $u \longleftrightarrow x \cdot y$. By the expression $x - y$ we mean the set $x$ diminished by the part of the set $y$ that is included in $x$. If $x > y$ we shall say that $x$ is "over" $y$ and that $y$ is "under" $x$.

The least upper bound (or join) of any collection $\alpha$ of elements in a lattice $L$ may include elements which are not in $\alpha$, nor under[1] $\alpha$, nor simple joins of elements in $\alpha$, nor simple joins of elements under $\alpha$. Let $\beta$ denote the set of all such elements. In this case we cannot represent the join of the elements of $\alpha$ by the sum of the sets corresponding to these elements in any representation of $L$ by sets. If in such a representation of $L$ by sets we denote by $\breve{\alpha}$ the sum of the sets corresponding to the elements of the collection $\alpha$ in $L$ (and similarly for $\beta$), we shall put $\breve{\beta} - \breve{\alpha} = a_\alpha$ (hence we have $\alpha\ a_\alpha = 0$), and then we have the following theorem.

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[1] By "under $\alpha$" we mean having $\alpha$ as a set of upper bounds, see Definition 2.

THEOREM 1. *Suppose we have a representation of the elements of a lattice $L$ by sets. Then the meet of a collection $\alpha$ of elements of $L$ will be represented by the product of the sets of the set-union $\breve{\alpha}$ which correspond to the elements of the collection $\alpha$. The join of the elements of $\alpha$ will be represented by the set $(\alpha, a_\alpha)$ where the set $a_\alpha=\breve{\beta}-\breve{\alpha}$ and where the set-union $\breve{\beta}$ corresponds to the totality $\beta$ of elements in $L$ that are included in the join of $\alpha$ and yet are not in $\alpha$, nor under $\alpha$, nor are they simple joins of elements under $\alpha$.*

PROOF. The part of the proof that concerns the meet of $\alpha$ follows directly from the fact that our representation by sets carries meets into set-products. The part of the proof that concerns the join of $\alpha$ follows from the uniqueness of the join of $\alpha$, plus the uniqueness of the set $(\alpha, a_\alpha)$, plus the fact that the set $(\alpha, a_\alpha)$ is the least upper bound of the sets $\alpha$ and $\beta$, and plus the fact that our representation of elements of $L$ by sets is one-one and preserves inclusion.

DEFINITION 3. *By set-coordinates of the elements of any lattice $L$ in any representation of $L$ by sets we mean the sets corresponding to these elements.*

If $x\cup y=(x, y, a_{x,y})$ then from the paragraph on joins we see that $a_{x,y}=a_{y,x}$, also that $a_{x,y}=0$ if $y\leqq x$. Moreover $a_{x,y}=0$ if and only if $x\cup y$ contains no $z$, that have neither $x$ nor $y$ as an upper bound and that are not simple joins of elements under $x$ and $y$. The relation $x\cup(y\cup z)=(x\cup y)\cup z=x\cup y\cup z$ implies the relation $(x, y, z, a_{y,z}, a_{x,y\cup z})=(x, y, z, a_{x,y}, a_{x,y\cup z})=(x, y, z, a_{x,y,z})$ because $x\cup(y\cup z)\longleftrightarrow(x, y, z, a_{y,z}, a_{x,y\cup z})$ and $(x\cup y)\cup z\longleftrightarrow(x, y, z, a_{x\cup y}, a_{x\cup y,z})$ and $x\cup y\cup z\longleftrightarrow(x, y, z, a_{x,y,z})$.

Now we shall prove some more theorems that will be of use in applying these set-coordinates to the study of lattices.

THEOREM 2. *For any three elements $x$, $y$, and $z$ of $L$ we have (in set-coordinates) $x\cdot a_{y,z}\geqq a_{x\cdot y,x\cdot z}$.*

PROOF. From the latter part of Corollary 1 on p. 22 of Birkhoff (loc. cit.) we see that for any such $x$, $y$, $z$ of $L$ we have $x\cap(y\cup z)\geqq(x\cap y)\cup(x\cap z)$. In set-coordinates this inequality becomes $x\ (y, z, a_{y,z})=(x\cdot y, x\cdot z, x\cdot a_{y,z})\geqq(x\cdot y, x\cdot z, a_{x\cdot y,x\cdot z})$ or briefly $x\cdot a_{y,z}\geqq a_{x\cdot y,x\cdot z}$.

THEOREM 3. *The lattice $L$ is a modular lattice if and only if for $x\geqq z$ in $L$ and for $y$ any other element of $L$ we have $x\cdot a_{y,z}=a_{x\cdot y,z}$ (in set-coordinates).*

PROOF. On p. 34 of Birkhoff (loc. cit.) we find that for $L$ to be modular we must have (for $x \geqq z$ and for $y$ any other element in $L$) the equality $x \cap (y \cup z) = (x \cap y) \cup z$. In set-coordinates this equality becomes $x \cdot (y, z, a_{y,z}) = (x \cdot y, x \cdot z, x \cdot a_{y,z}) = (x \cdot y, z, a_{x \cdot y,z})$ or finally (since $x \geqq z$) $x\ a_{y,z} = a_{x \cdot y,z}$.

THEOREM 4. *The lattice $L$ is a distributive lattice if and only if for $y$ and $z$ any two elements of $L$ we have $a_{y,z} = 0$ (in set-coordinates).*

PROOF. On p. 74 of Birkhoff (loc. cit.) we find that one necessary and sufficient condition for $L$ to be a distributive lattice is that for all $x$, $y$, and $z$ in $L$ we have $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$. In set-coordinates this condition is $x \cdot (y, z, a_{y,z}) = (x \cdot y, x \cdot z, x \cdot a_{y,z}) = (x \cdot y, x \cdot z, a_{x \cdot y, x \cdot z})$ for all $x$, $y$, and $z$, or $x \cdot a_{y,z} = a_{x \cdot y, x \cdot z}$ or finally (since $x$, $y$ and $z$ are arbitrary elements of $L$) we have $a_{y,z} = 0$ for all $y$ and $z$ in $L$.

We note that Theorem 4 asserts that a lattice is distributive if and only if every join is a simple join, and that a lattice is distributive if and only if the economical representation is join-true.

ILLUSTRATION 3. We can start in a general lattice from any set of "independent" elements and proceed as in Illustration 2 and thus obtain the part of the lattice "above" these independent elements.

ILLUSTRATION 4. The general complemented distributive lattice with $n$ atoms can readily be represented as follows. Here every join is a simple join. The atoms are $1, 2, \cdots, n$. Above the atoms are the elements $(1, 2)$, $(1, 3)$, $\cdots$, $(n-1, n)$. Above these elements are $(1, 2, 3)$, $\cdots$, and so forth up to $I = (1, 2, 3, \cdots, n)$.

SYRACUSE UNIVERSITY

# ON NON-CUT SETS OF LOCALLY CONNECTED CONTINUA

W. M. KINCAID

W. L. Ayres[1] and H. M. Gehman[2] have proved independently that if a locally connected continuum $S$ contains a non-cut point $p$, there exists an arbitrarily small region $R$ containing $p$ and such that $S-R$ is connected. Our paper is concerned with certain generalizations of this theorem.

We shall consider a space $S$ which is a locally connected continuum and contains a closed set $P$ such that $S-P$ is connected. We show that under these hypotheses $P$ can be enclosed in an open set $R$, the sum of a finite number of regions, whose complement is a locally connected continuum. We show further that if there exists a family of sets $\mathfrak{F}$ no element of which separates $S-P$, then there exist two open sets $R$ and $R'$ (with $R\supset R'\supset P$) of the above type and having the property that no element of $\mathfrak{F}$ contained in $S-R$ separates $S-R'$. When the elements of $\mathfrak{F}$ are single points, it is possible to choose $R'=R$; but this is not possible in the more general case.

We close by showing that if $S$ is not separated by any element of $\mathfrak{F}$ plus any set of $n$ points, and if $Q$ is the sum of $n$ sets of sufficiently small diameter and having sufficiently great mutual distances, then the set $S-Q$ has at most one component whose diameter is greater than a preassigned positive quantity, and this component is not separated by any element of $\mathfrak{F}$ at a sufficiently great distance from $Q$.

We recall some well known results.[3]

Let $M$ be a locally connected continuum. Then:

(1) *$M$ is a metric space having property $S$.*[4]

(2) *$M$ is the sum of a finite number of arbitrarily small connected*

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[1] See W. L. Ayres, *On continua which are disconnected by the omission of a point and some related problems*, Monatshefte für Mathematik und Physik vol. 36 (1929) pp. 135–147. The theorem quoted here corresponds to Theorem 2 p. 149.

[2] See H. M. Gehman, *Concerning certain types of non-cut points, with an application to continuous curves*, Proc. Nat. Acad. Sci. U.S.A. vol. 14 (1928) pp. 431–433. Theorem 4 p. 432 is essentially that quoted here.

[3] See G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28 (1942) p. 20 ff.

[4] A set is said to have property $S$ if for any $\epsilon>0$ it can be expressed as the sum of a finite number of connected sets of diameter less than $\epsilon$. The property was first introduced by W. Sierpinski in his paper *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fund. Math. vol. 1 (1920) pp. 44–60.

*subsets each having property $S$. Furthermore these subsets may be chosen either as open sets or as closed sets.*

(3) *If $N \subset M$ and $N$ has property $S$, then any set $N_0$ such that $N \subset N_0 \subset \overline{N}$ is locally connected.*

From the preceding results follows:

LEMMA 1. *If $N$ is any subset of a locally connected continuum $M$ and $V$ is any $\delta$-neighborhood of $N$, then there exists an open set $U$ that contains $N$, has property $S$, and is such that $\overline{U}$ is the sum of a finite number of locally connected continua contained in $V$.*

It may clearly be supposed that every component of $U$ contains a point of N.

Throughout this paper we shall deal with a compact metric space $S$, which we suppose to be a locally connected continuum. We denote by $\delta(A)$ the diameter of any set $A$, and by $V_\epsilon(A)$ the $\epsilon$-neighborhood of $A$.

We shall require the following lemma, which was pointed out to the author by Dr. D. W. Hall. Its proof follows directly from the definitions involved.

LEMMA 2. *Let $M$ be any locally connected subcontinuum of $S$. Then if $T$ is the sum of any set of components of $S-M$, the set $K=S-T$ is a locally connected continuum.*

Using methods very similar to those of Ayres[1] and Gehman,[2] we obtain the following generalization of their theorem.

THEOREM 1. *Let $P$ be any closed set such that $S-P$ is connected. Then for any $\epsilon>0$ there exists an open set $R$ such that* (1) $P \subset R \subset V_\epsilon(P)$, (2) *the set $R$ is the sum of a finite number of regions, each of which intersects $P$, and* (3) *$S-R$ is a locally connected continuum.*

PROOF. Let $R_1=V_\delta(P)$ and $R_2=V_{\delta'}(P)$, where $0<\delta'<\delta\leqq\epsilon$. Then, since $S$ is locally connected, at most a finite number of components of $S-R_2$ intersect $S-R_1$; let these be $K_1, K_2, \cdots, K_n$, and choose $p_i \in K_i(R_1-R_2)$ for $i=1, \cdots, n$. Since $P$ is closed, $S-P$ is a region, and we can find arcs $p_1p_i \subset (S-P)$ for $i=1, 2, \cdots, n$; thus we construct the connected set $C=\sum_{i=1}^{n}(K_i+p_1p_i)$.

Taking $d=\rho(P, C)$, we conclude from Lemma 1 that there exists a locally connected continuum $V$ contained in $V_d(C)$ and therefore disjoint with $P$. Now let $R$ be the sum of all components of $S-V$ that contain points of $P$. It follows from the local connectivity of $S$ that $R$ is open. To prove the theorem, we must show that $R$ satisfies the conditions (1), (2), and (3).

Obviously $P \subset R$. On the other hand, $R \subset (S-V) \subset (S-C) \subset R_1 \subset V_\epsilon(P)$. Thus $R$ satisfies (1).

To see that (2) holds, we note that $\rho(P, S-R) > 0$, because $R$ is open. It follows from Lemma 1 that there exists an open set $R_2$, the sum of a finite number of regions, such that $P \subset R_2 \subset R$. We see that $R$ is likewise the sum of a finite number of regions, each of which intersects $P$, for every component of $R$ contains a point of $P$ and thus a component of $R_2$.

Applying Lemma 2 with $M = V$ and $T = R$ shows that $S-R$ is a locally connected continuum. Consequently $R$ satisfies (3), and the proof is complete.

The following example shows that Theorem 1 loses its validity if the requirement that $P$ be closed is dropped.

EXAMPLE. Take for $S$ the closed rectangle in the $xy$-plane bounded by the lines $x = \pm 2$, $y = \pm 1$. Divide the rectangle into four rectangles by drawing the lines $x = 0$, $x = \pm 1$. Denote by $P'$ the set consisting of the two end rectangles (open or closed), the segment $x = 0$, $-1 \leqq y \leqq 1$, and the curve $y = \sin(1/x)$, $-1 \leqq x \leqq 1$; thus $P'$ is connected but not locally connected. We see that Theorem 1 does not hold for $P = S - P'$, for no set having the properties of $R$ can be found corresponding to $\epsilon < 1$.

THEOREM 2. *Let $P$ be any closed set such that $S-P$ is connected, and suppose that $\mathfrak{F}$ is a family of subsets of $S$ such that $S-(P+Q)$ is connected for each $Q \in \mathfrak{F}$. Then, given $\epsilon > 0$, there exist open sets $R$ and $R'$, each of which is the sum of a finite number of regions intersecting $P$, such that* (1) $P \subset R' \subset R \subset V_\epsilon(P)$, (2) *the sets $S-R$ and $S-R'$ are locally connected continua, and* (3) *if $Q \in \mathfrak{F}$ and $Q \subset (S-R)$, then $S-(R'+Q)$ is connected.*

PROOF. We first select an open set $R \supset P$ which has the same properties as the $R$ of Theorem 1. Next we choose another open set $R_1 \supset P$, having the same properties as $R$, and such that $\overline{R}_1 \subset R$. Then all components of $R - R_1$ have limit points in both $S-R$ and $\overline{R}_1$, since $S - R_1$ is connected and $R$ is the sum of a finite number of regions, each intersecting $P$ and in one of which any component of $R-R_1$ must lie. It follows from the local connectivity of $S$ that the number of such components is finite.

Now, if any two components of $R-R_1$ lie in some component $C$ of $R-P$, we connect them by a simple arc in $C$; this is possible because $C$ is a region. We define $V_1$ as the sum of $S-R_1$ and all such arcs.

Clearly $V_1$ is connected. Moreover, if $Q \in \mathfrak{F}$ and $Q \subset (S-R)$, then $V_1 - Q$ is connected. For suppose that $x$ and $y$ are points of $V_1 - Q$.

Then $Q$ cannot separate $x$ from $R-R_1$ in $V_1$. This is obvious if $x\in RV_1$; if $x\in(V_1-R)$, we see (since $R-R_1$ separates $x$ from $S-V_1$ in $S$) that $Q$ cannot separate $x$ from $R-R_1$ in $V_1$ without separating them in $S$, contrary to hypothesis. Thus there exists a component $X$ of $R-R_1$ such that $x$ and $X$ lie in the same component of $V_1-Q$. Similarly, there exists a component $Y$ of $R-R_1$ such that $y$ and $Y$ lie in the same component of $V_1-Q$.

If $X$ and $Y$ are in the same component of $R-P$, there exists an arc in $RV_1$ connecting $X$ and $Y$. On the other hand, if $X$ and $Y$ lie in different components of $R-P$, they must lie in the same component of $V_1-Q$. For suppose that $X\subset AB$, where $A$ is a component of $R-P$ and $B$ is a component of $V_1-Q$. Then $A$ is closed in $R-P$, while $B$ is closed in $V_1-Q$. Thus the sets $(A+B)\bar{R}_1=A\bar{R}_1$ and $A+B-R=B-R$ are closed in $S-(P+Q)$. From the construction of $V_1$ we see that $A(R-R_1)\subset B(R-R_1)$, and it follows that $(A+B)(\bar{R}-R_1)$ is closed in $S-(P+Q)$. Hence $A+B$, being the sum of three sets closed in $S-(P+Q)$, is closed in $S-(P+Q)$. Now suppose $Y\not\subset A+B$. Then we can find another set $A'+B'\supset Y$ of the same form as, and disjoint with, $A+B$. In this way it follows that $S-(P+Q)$ is the sum of a finite number of disjoint sets closed in $S-(P+Q)$, which is impossible since $S-(P+Q)$ is connected. Thus $X$ and $Y$, and therefore $x$ and $y$, lie in the same component of $V_1-Q$.

By Lemma 1, there exists a locally connected continuum $V'$ containing $V_1$ and disjoint with $P$. We denote by $R'$ the sum of all components of $S-V'$ that contain points of $P$. It follows from Lemmas 1 and 2, as in the proof of Theorem 1, that $R'$ is the sum of a finite number of regions and that $S-R'$ is a locally connected continuum. Moreover, if $Q\in\mathfrak{F}$ and $Q\subset(S-R)$, we conclude, since $(S-R')-V_1\subset R_1$, that $S-(R'+Q)$ is connected. This completes the proof.

The question naturally arises whether, under the hypotheses of Theorem 2, it is possible to find a single open set $T$ containing $P$ and playing the parts of both $R$ and $R'$ in that theorem. The following example shows that such a set $T$ cannot in general be found.

EXAMPLE. Take for $S$ the plane set consisting of two line segments, $a_0p$ and $b_0p$, together with a sequence of parallel lines $a_0b_0$, $a_1b_1$, $a_2b_2$, $\cdots$, where $a_0$, $a_1$, $a_2$, $\cdots$ and $b_0$, $b_1$, $b_2$, $\cdots$ are sequences of points, converging monotonically to $p$, on the lines $a_0p$ and $b_0p$, respectively. We denote by $c_n$ the midpoint of the segment $a_nb_n$ for $n=0, 1, 2, \cdots$.

Now, take for $P$ the point $p$ and for $\mathfrak{F}$ the family of all pairs of points of $S$ not separating $S-p$. Let $R$ be any region containing $p$ (but disjoint with $a_1b_1$). Then there exists a greatest integer $n\geqq 1$

for which $a_nb_n \subset (S-R)$. It follows that the pair of points $(c_n+a_{n-1})$ separates $S-R$. However, the set $S-(p+c_n+a_{n-1})$ is connected; hence $(c_n+a_{n-1}) \in \mathfrak{F}$. Thus $R$ cannot be taken as $T$.

However, the stronger conclusion can be drawn when $\mathfrak{F}$ is a family of single points, as we now show.

THEOREM 3. *Let $P$ be any closed set such that $S-P$ is connected. Suppose $F$ is a set such that $S-(P+q)$ is connected for $q \in F$. Then for any $\epsilon>0$ there exists an open set $R \supset P$, contained in $V_\epsilon(P)$ and consisting of a finite number of regions, such that $S-R$ is a locally connected continuum and $S-(R+q)$ is connected for $q \in F$.*

PROOF. By Theorem 2, there exist open sets $R_1$ and $R_2$, each consisting of a finite number of regions, such that (1) $P \subset R_2 \subset R_1 \subset V_\epsilon(P)$, (2) the sets $S-R_1$ and $S-R_2$ are locally connected continua, and (3) $S-(R_2+q)$ is connected for $q \in F(S-R_1)$. We write $V_2 = S-R_2$.

Now, for $y \in V_2R_1F$ we define $K_y$ as the set consisting of $y$ plus the component of $V_2-y$ containing $S-R_1$. Then we let

$$V = \prod_{y \in V_2R_1F} K_y, \qquad R = S - V = \sum_{y \in V_2R_1F} (S - K_y).$$

For any $y \in V_2R_1F$, the set $S-K_y$ is the sum of a finite number of regions. For suppose $x \in (S-K_y)$. Since $S-y$ is a region, there exists an arc $xr \subset (S-y)$ for all $r \in R_2$. If there exists a point $x_1 \in K_y$ on $xr$, there exists a first such point $x_2$, since $K_y$ is closed. The arc $xx_2$ is not contained in $V_2$, since $x$ and $x_2$ lie in different components of $V_2-y$; thus there exists a point $x_3 \in (S-V_2) = R_2$ on $xx_2$. In any case, therefore, there exists in $S-K_y$ an arc joining $x$ to some point of $R_2$. It follows that $S-K_y$ is the sum of a finite number of regions, each containing at least one component of $R_2$. Since this is true for all $y \in V_2R_1F$, the same must be true of $R$.

The set $V$ is an $A$-set[6] in $V_2$. For suppose otherwise. Then, since $V$ is closed, there exists an arc $xqy$ in $V_2$ spanning $V$. Since $q \notin V$, there exists a point $z \in V_2R_1F$ such that $q \notin K_z$. But $x+y \subset K_z$. Therefore $z$ must separate both $x$ and $y$ from $q$ in $V_2$, which is impossible. Since $V$ is an $A$-set, it is a locally connected continuum.

Moreover, $V$ has no cut points in $F$. For let $x \in V$, $y \in V$, and $q \in VF$. If $q \in (V-R_1)$, there exists an arc $xy \subset (V_2-q)$, because $q$ is not a cut point of the locally connected continuum $V_2$; since $V$ is an $A$-set in $V_2$, the arc $xy \subset (V-q)$. If $q \in VR_1$, we have $x+y \subset K_q$ and hence there exists an arc $xy \subset (K_q-q)$; again $xy \subset (V-q)$.

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[6] See Kuratowski and Whyburn, *Sur les éléments cycliques et leurs applications*, Fund. Math. vol. 16 (1930) pp. 305–331.

We have now shown that $R$ is the sum of a finite number of regions, that $S-R$ is a locally connected continuum, and that $S-(R+q)$ is connected for any $q\in F$. Thus the proof is complete.

Theorem 4. *Suppose that no $m$ points separate $S$, and that $\mathfrak{F}$ is a family of sets such that $S-(Q+\sum_{i=1}^{m}p_i)$ is connected for any $Q\in\mathfrak{F}$ and any $m$ points $p_1, p_2, \cdots, p_m$ of $S$. Then corresponding to any $\epsilon>0$ there exists a number $\delta>0$ such that if $P_1, P_2, \cdots, P_m$ are $m$ sets contained in $S$, each of diameter less than $\delta$, while $\rho(P_i, P_j)>2\epsilon$ for $0\leqq i\leqq j\leqq m$, the set $S-\sum_{i=1}^{m}P_i$ has at most one component $K$ of diameter greater than $\epsilon$, and (if $K$ exists) $K-Q$ is connected for every $Q\in\mathfrak{F}$ for which $\rho(Q,\sum_{i=1}^{m}P_i)>\epsilon$.*

Proof. Let $\mathfrak{F}^{(1)}$ be the family of sets having as elements all sets of the type $Q+Q_1$, where $Q\in\mathfrak{F}$ and $Q_1$ is any set of at most $m-1$ points. Then if $F\in\mathfrak{F}^{(1)}$, the set $S-(F+p)$ is connected for every $p\in S$.

Using Theorem 2, we obtain for every point $x\in S$ two regions $V_x$ and $W_x$, each of diameter less than $\epsilon$, such that $x\in V_x\subset W_x$, while the sets $S-V_x$ and $S-W_x$ are locally connected continua, and $S-(F+V_x)$ is connected if $F\in\mathfrak{F}^{(1)}$ and $F\subset(S-W_x)$. We then choose a third region $U_x\supset x$ such that $\overline{U}_x\subset V_x$. By the Heine-Borel theorem, there exists a finite subfamily $\{U_1, \cdots, U_{n_1}\}$ of the family $\{U_x\}$ such that $S=\sum_{i=1}^{n_1}U_i$. In each set $U_i$ $(i=1, 2, \cdots, n_1)$ we choose a point $x_i$ for which $U_{x_i}=U_i$ and define $V_i=V_{x_i}$, $W_i=W_{x_i}$. Let $\delta_1=\min_{i=1,\cdots,n_1}\rho(U_i, S-V_i)$.

Now denote by $\mathfrak{F}^{(2)}$ the family of sets having as elements all sets of the type $Q+Q_2$, where $Q\in\mathfrak{F}$ and $Q_2$ is any set of at most $m-2$ points, and for $i=1, 2, \cdots, n_1$ define $\mathfrak{F}_i^{(2)}$ as the largest subfamily of $\mathfrak{F}^{(1)}$ all of whose elements are contained in $S-W_i$. Then if $F\in\mathfrak{F}_i^{(2)}$, we see that $(S-V_i)-(F+p)$ is connected for every $p\in(S-W_i)$.

Applying Theorem 2 to the locally connected continuum $S-V_i$, we obtain for every point $x\in(S-W_i)$ $(i=1, \cdots, n_1)$ three regions $U_{ix}$, $V_{ix}$, and $W_{ix}$, each of diameter less than $\epsilon$, in the locally connected continuum $S-V_i$, such that $x\in U_{ix}\subset\overline{U}_{ix}\subset V_{ix}\subset W_{ix}$ and $S-(V_i+V_{ix})$ is a locally connected continuum, while $(S-V_i)-(F+V_{ix})$ is connected if $F\in\mathfrak{F}_i^{(2)}$ and $F\subset(S-W_{ix})$. Writing

$$T_i = \underset{x}{E}\,[\rho(x, W_i) \geqq \epsilon], \qquad i = 1, 2, \cdots, n_1,$$

we see that if $x\in T_i$, the set $W_{ix}$ is contained in the interior of $S-W_i$ and is therefore a region in $S$. It follows by the Heine-Borel theorem that the family of regions $\{U_{ix}\}$ (for all $x\in T_i$) contains a finite subfamily $\{U_{i1}, U_{i2}, \cdots, U_{in_2(i)}\}$ such that

$$T_i \subset \sum_{j=1}^{n_2(i)} U_{ij} \subset (S - \overline{W}_i), \qquad i = 1, \cdots, n_1.$$

We write for simplicity $n_2 = \max_{i=1,\cdots,n_1}[n_2(i)]$, and by making repetitions if necessary we obtain $n_1$ families, each containing $n_2$ regions and having the above properties. Regions $V_{ij}$ and $W_{ij}$ are then selected for $i=1, \cdots, n_1$, $j=1, \cdots, n_2$ as in the preceding case. We let

$$\delta_2 = \min_{i=1,\cdots,n_1;\, j=1,\cdots,n_2} \rho(U_{ij}, S - V_{ij}).$$

We proceed by induction as follows. Suppose that for some $k<m$ we have found three sets of regions $\{U_{i_1i_2\cdots i_k}\}$, $\{V_{i_1i_2\cdots i_k}\}$, and $\{W_{i_1i_2\cdots i_k}\}$ (where $i_j=1, 2, \cdots, n_j$; $j=1, 2, \cdots, k$), having the following properties:

(1) $\overline{U}_{i_1\cdots i_k}\subset V_{i_1\cdots i_k}\subset W_{i_1\cdots i_k}$;

(2) $\delta(W_{i_1\cdots i_k})<\epsilon$;

(3) $S-\sum_{j=1}^{k} V_{i_1\cdots i_j}$ is a locally connected continuum;

(4) $T_{i_1i_2\cdots i_{k-1}}\subset\sum_{i_k=1}^{n_k} U_{i_1\cdots i_k}\subset(S-\sum_{j=1}^{k-1}\overline{W}_{i_1\cdots i_j})$,

where

$$T_{i_1i_2\cdots i_{k-1}} = \underset{x}{E}\,[\rho(x, W_{i_1\cdots i_j}) \geqq \epsilon \quad \text{for} \quad j = 1, 2, \cdots, k-1];$$

(5) if $(Q+\sum_{i=1}^{m-k}q_i)\subset(S-\sum_{j=1}^{k}W_{i_1\cdots i_j})$, where $Q\in\mathfrak{F}$ and $q_i\in S$ for $i=1, 2, \cdots, m-k$, then the set $(S-\sum_{j=1}^{k}V_{i_1\cdots i_j})-(Q+\sum_{i=1}^{m-k}q_i)$ is connected.

In order to take the next step, we define $\mathfrak{F}^{(k+1)}$ as the family of sets having as elements all sets of the type $Q+Q_{k+1}$, where $Q\in\mathfrak{F}$ and $Q_{k+1}$ is any set of at most $m-(k+1)$ points. Then we denote by $\mathfrak{F}^{(k+1)}_{i_1\cdots i_k}$ ($i_j=1, 2, \cdots, n_j$; $j=1, 2, \cdots, k$) the largest subfamily of $\mathfrak{F}^{(k+1)}$ all of whose elements are contained in $S-\sum_{j=1}^{k}W_{i_1\cdots i_j}$. It follows from (5) that $(S-\sum_{j=1}^{k}V_{i_1\cdots i_j})-(F+p)$ is connected for all $F\in\mathfrak{F}^{(k+1)}_{i_1\cdots i_k}$ and $p\in(S-\sum_{j=1}^{k}W_{i_1\cdots i_j})$.

Applying Theorem 2 to the locally connected continuum $S-\sum_{j=1}^{k}V_{i_1\cdots i_j}$, we obtain for any point $x\in(S-\sum_{j=1}^{k}W_{i_1\cdots i_j})$ three regions $U_{i_1\cdots i_kx}$, $V_{i_1\cdots i_kx}$, and $W_{i_1\cdots i_kx}$, each of diameter less than $\epsilon$, in the locally connected continuum $S-\sum_{j=1}^{k}V_{i_1\cdots i_j}$, such that

$$x \in U_{i_1\cdots i_kx} \subset \overline{U}_{i_1\cdots i_kx} \subset V_{i_1\cdots i_kx} \subset W_{i_1\cdots i_kx}$$

and $S-\sum_{j=1}^{k}V_{i_1\cdots i_j}-V_{i_1\cdots i_kx}$ is a locally connected continuum, while $(S-\sum_{j=1}^{k}V_{i_1\cdots i_j})-(F+V_{i_1\cdots i_kx})$ is connected if $F\in\mathfrak{F}^{(k+1)}_{i_1\cdots i_k}$ and $F\subset(S-\overline{W}_{i_1\cdots i_kx})$. Then, defining $T_{i_1\cdots i_k}$ as in (4), we see as before that $U_{i_1\cdots i_kx}$ is a region in $S$; using the Heine-Borel theorem, we deduce the existence of a finite family of regions $\{U_{i_1\cdots i_k i_{k+1}}\}$ $(i_{k+1}=1, 2, \cdots, n_{k+1})$ such that

$$T_{i_1\cdots i_k} \subset \sum_{i_{k+1}=1}^{n_{k+1}} U_{i_1\cdots i_{k+1}} \subset \left(S - \sum_{j=1}^{k} W_{i_1\cdots i_j}\right).$$

Selecting families of regions $\{V_{i_1\cdots i_{k+1}}\}$ and $\{W_{i_1\cdots i_{k+1}}\}$ as before, we obtain three sets of regions for which (1)–(5) hold with $k$ replaced by $k+1$.

We carry out this construction for $k=1, 2, \cdots, m$, and let

$$\delta_k = \min_{i_j=1,\cdots,n_j;\, j=1,\cdots,k} \rho(U_{i_1\cdots i_k}, S - V_{i_1\cdots i_k}), \qquad k = 1, \cdots, m.$$

We shall now show that the theorem holds with $\delta = \min_{k=1,\cdots,m} \delta_k$. Consider any family of sets $\{P_1, P_2, \cdots, P_m\}$ satisfying the conditions of the theorem. Since $S=\sum_{i=1}^{n_1} U_i$, there exists a positive integer $i_1 \leqq n_1$ such that $P_1 U_{i_1} \neq 0$; then since $\delta(P_1) < \delta \leqq \delta_1$, we have $P_1 \subset V_{i_1}$. Since $\rho(P_1, P_2) > 2\epsilon$, it is clear that $P_2 \subset T_{i_1}$, and hence there exists a positive integer $i_2 \leqq n_2$ such that $P_2 U_{i_1 i_2} \neq 0$; it follows that $P_2 \subset V_{i_1 i_2}$. Now suppose that for $j=1, 2, \cdots, k<m$ there exist numbers $i_j \leqq n_j$ such that $P_j \subset V_{i_1\cdots i_j}$. Since $\rho(P_j, P_{k+1}) > 2\epsilon$ for $j=1, \cdots, k$, we see that $P_{k+1} \subset T_{i_1\cdots i_k}$; thus $P_{k+1} U_{i_1\cdots i_k i_{k+1}} \neq 0$ for some $i_{k+1} \leqq n_{k+1}$, whence $P_{k+1} \subset V_{i_1\cdots i_k i_{k+1}}$. Proceeding in this way, we find positive integers $i_j \leqq n_j$ such that $P_j \subset V_{i_1\cdots i_j}$ for $j=1, 2, \cdots, m$.

We conclude from property (5) above that $S-\sum_{j=1}^{m} V_{i_1\cdots i_j}$ is connected, and hence must be contained in a single component $K$ of $S-\sum_{i=1}^{m} P_i$. Any other component of $S-\sum_{i=1}^{m} P_i$ must therefore be contained in one of the regions $V_{i_1\cdots i_j}$; thus the diameter of such a component must be less than $\epsilon$.

Finally, suppose that $Q \in \mathfrak{F}$ and $\rho(Q, \sum_{i=1}^{m} P_i) > \epsilon$. Then by (2) above, $Q \subset (S-\sum_{j=1}^{m} W_{i_1\cdots i_j})$; by (5), $(S-\sum_{j=1}^{m} V_{i_1\cdots i_j} - Q)$ is connected. It follows that $K-Q$ is connected.

REMARK. If no $n$ $(>m)$ points separate $S$, we may take $\mathfrak{F}$ as the family of all sets of $n-m$ points; then, under the above hypotheses, the component $K$ of $S-P$ (where $P=\sum_{i=1}^{m} P_i$) is not separated by any set of $n-m$ points $q_1, q_2, \cdots, q_{n-m}$ such that $\rho(\sum_{i=1}^{n-m} q_i, P) > \epsilon$.

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# NEW PROOFS OF THE THEOREMS OF BELTRAMI AND KASNER ON LINEAR FAMILIES

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1. **Introduction.** We submit here new proofs, from a uniform point of view, of the theorems of Beltrami and Kasner on linear families of curves. Beltrami's result is that a surface $S$ may be mapped upon a plane $\pi$ so that its geodesics correspond to straight lines if and only if $S$ is of constant gaussian curvature [1].[1] Kasner's result states that a complete system of isogonal trajectories of a simple (that is, one-parameter) family of curves $F$ is linear if and only if $F$ is isothermal [2]. We shall also deduce from our work another theorem of Kasner stating that a surface $S$ can possess exactly $\infty^2$ isothermal families of geodesics (maximum possibility) if and only if $S$ is of constant gaussian curvature [3].

2. **Velocity systems.** For the development of our proofs, it is found necessary to consider certain classes of $\infty^2$ curves, namely, velocity systems, natural families, isogonal systems, $\Gamma_0$ and $\Gamma$ families. In Kasner's study of dynamical trajectories [4], an important class of $\infty^2$ curves was encountered which he termed *velocity systems*. In minimal coordinates ($u=x+iy$, $v=x-iy$), any such system is defined by a second order differential equation of the form

$$v'' = v'(c - dv'), \tag{1}$$

where $c$ and $d$ are arbitrary functions of $(u, v)$.

Special types of velocity systems are natural families and isogonal systems. Any natural family is a velocity system for which $c_v = d_u$, whereas any isogonal system is a velocity system for which $c_v = -d_u$.

A system of $\infty^2$ curves is both natural and isogonal if and only if it is the complete set of isogonal trajectories of an isothermal family. Such set is called a *conformal rectilinear wex*[2] and is denoted by $\Gamma_0$. Any family of this type is conformally equivalent to the $\infty^2$ straight lines of the plane. A system $\Gamma_0$ is a velocity system for which $c_v = d_u = 0$.

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[1] The numbers in brackets refer to the references at the end of the paper.

[2] The set of $\infty^2$ integral curves of any differential equation of second order $y'' = F(x, y, y')$ has been termed a *wex* by Kasner. The transformation theory of $\Gamma_0$ systems has been developed by Kasner and DeCicco, *Transformation theory of isogonal trajectories of isothermal families*, Proc. Nat. Acad. Sci. U.S.A. 1942.

3. **The Γ families.**[3] Any set of $\infty^2$ curves which is conformally equivalent to the set of $\infty^2$ circles orthogonal to a fixed proper circle (or a fixed straight line) is called a Γ *family*. A $\Gamma_0$ system may be considered as the limiting case of a Γ family by letting the radius of the fixed circle approach zero. The Γ families are special types of velocity systems.

THEOREM 1. *A velocity system represents a* Γ *family if and only if the functions* $c$ *and* $d$ *with* $c_v \neq 0$ *and* $d_u \neq 0$ *satisfy any one of the three equivalent systems of partial differential equations of second order*

$$(2.1) \qquad c_{uv} = cc_v, \qquad c_{vv} = dc_v;$$

$$(2.2) \qquad d_{uv} = dd_u, \qquad d_{uu} = cd_u;$$

$$(2.3) \qquad c_v = d_u, \qquad c_{uv} = cc_v, \qquad d_{uv} = dc_v.$$

The last set of equations shows that every Γ family is a natural family. But of course not every natural family is a Γ family.

THEOREM 2. *A* Γ *family (including* $\Gamma_0$*) is a velocity system which contains exactly* $\infty^2$ *isothermal families.*

A velocity system may contain exactly $\infty^2$, $\infty^1$, one, or no isothermal families. There are no other possibilities.

4. **Proof of Kasner's linear characterization of isothermal families.** We shall now state and prove Kasner's first theorem. A proof employing a complicated theorem on general linear families due to Lie and R. Liouville, has been given elsewhere [2], but we shall prove it, beginning with basic principles. The result holds for the plane or any surface.

THEOREM 3. *A simple (that is, one-parameter) family of curves is isothermal if and only if the complete set of its isogonal trajectories is a linear system.*

Under any point transformation

$$(3) \qquad U = \phi(u, v), \qquad V = \psi(u, v),$$

with jacobian $J = \phi_u\psi_v - \phi_v\psi_u \neq 0$, the $\infty^2$ straight lines in the $(U, V)$-

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[3] It has been proved by Kasner that the only systems of $\infty^2$ circles which are given by second order differential equations of the cubic type $y'' = Ay'^3 + By'^2 + Cy' + D$, where $A, B, C, D$, are functions of $(x, y)$, are the Γ and $\Gamma_0$ families. For additional properties of Γ and $\Gamma_0$ families see Kasner and DeCicco, *Geometry of velocity systems*, Bull. Amer. Math. Soc. vols. 48–49 (1942–1943).

plane correspond to the $\infty^3$ curves in the $(u, v)$-plane given by the differential equation of second order

$$Jv'' = Av'^3 + Bv'^2 + Cv' + D, \tag{4}$$

where

$$\begin{aligned} A &= \phi_{vv}\psi_v - \phi_v\psi_{vv}, \qquad D = \phi_{uu}\psi_u - \phi_u\psi_{uu}, \\ B &= (\phi_{vv}\psi_u - \phi_u\psi_{vv}) + 2(\phi_{uv}\psi_v - \phi_v\psi_{uv}), \\ C &= (\phi_{uu}\psi_v - \phi_v\psi_{uu}) + 2(\phi_{uv}\psi_u - \phi_u\psi_{uv}). \end{aligned} \tag{5}$$

Now if (4) represents a velocity system, we must have $A = D = 0$. We assume first *both* $\phi_u$ *and* $\phi_v$ *not zero*, and find

$$\psi = a(u)\phi + b(u), \qquad \psi = \alpha(v)\phi + \beta(v). \tag{6}$$

By substituting the first value of $\psi$ into $B$ and $J$, and then substituting the second value of $\psi$ into $C$ and $J$, we find that our functions $c$ and $d$ defining the velocity system (1) are given by

$$c = \frac{\phi_{uu}}{\phi_u} - \frac{2\phi_u\alpha_v}{\alpha_v\phi + \beta_v}, \qquad d = \frac{\phi_{vv}}{\phi_v} - \frac{2\phi_v a_u}{a_u\phi + b_u}. \tag{7}$$

The equations (6) show that $\phi = (\beta - b)/(a - \alpha)$. Substituting this value of $\phi$ into (7), we find that any linear velocity system must be given by

$$\begin{aligned} c &= \frac{a_{uu}(\beta - b) + b_{uu}(a - \alpha)}{a_u(\beta - b) + b_u(a - \alpha)} - \frac{2(a_u\beta_v - \alpha_v b_u)}{\alpha_v(\beta - b) + \beta_v(a - \alpha)}, \\ d &= \frac{\alpha_{vv}(\beta - b) + \beta_{vv}(a - \alpha)}{\alpha_v(\beta - b) + \beta_v(a - \alpha)} - \frac{2(a_u\beta_v - \alpha_v b_u)}{a_u(\beta - b) + b_u(a - \alpha)}. \end{aligned} \tag{8}$$

Differentiating $c$ with respect to $v$ and $d$ with respect to $u$, and performing certain additions and subtractions, we find

$$\begin{aligned} c_v + 2d_u &= -\frac{3(a_{uu}b_u - a_u b_{uu})\left[\alpha_v(\beta - b) + \beta_v(a - \alpha)\right]}{\left[a_u(\beta - b) + b_u(a - \alpha)\right]^3}; \\ 2c_v + d_u &= \frac{3(\alpha_{vv}\beta_v - \alpha_v\beta_{vv})\left[a_u(\beta - b) + b_u(a - \alpha)\right]}{\left[\alpha_v(\beta - b) + \beta_v(a - \alpha)\right]^3}. \end{aligned} \tag{9}$$

Now impose the condition $c_v = -d_u$ for isogonal systems. The addition of the preceding equations shows that

$$\frac{\alpha_{vv}\beta_v - \alpha_v\beta_{vv}}{\left[\alpha_v(\beta - b) + \beta_v(a - \alpha)\right]^3} = \frac{a_{uu}b_u - a_u b_{uu}}{\left[a_u(\beta - b) + b_u(a - \alpha)\right]^3}. \tag{10}$$

Assuming that the numerators of both of these fractions are not zero, the second order partial derivative with respect to $u$ and $v$ of the logarithm of this equation leads to the same equation but with the sign of one fraction changed. The addition and subtraction of this new equation and (10) prove that the numerators must be zero. This contradiction leads us to the fact that the numerators of both fractions in (10) must be zero.

Since the numerators of (10) are zero, it follows from (9) that $c_v = d_u = 0$. Therefore the only possible linear isogonal systems are the $\Gamma_0$ families. That these are linear follows from the definition of $\Gamma_0$ systems. This completes the proof of Kasner's theorem in the general case. See §6 for special case.

5. **The linear natural families.** Next we shall state and prove the following result. (This could also be deduced from Beltrami's theorem but our purpose here is to obtain it independently.)

THEOREM 4. *A natural family of curves is linear if and only if it is a* $\Gamma$ *or* $\Gamma_0$ *family.*

By Theorem 3, we already know that a $\Gamma_0$ family is a linear natural family. Hence we can exclude this case, and we may assume that at least one of the first factors in the right-hand members of equations (9) is not zero.

By this last remark, it is found by taking the logarithmic derivative of the first of equations (9) with respect to $v$ and of the second with respect to $u$ that

$$(11) \qquad c_{vv} + 2d_{uv} = d(c_v + 2d_u), \qquad 2c_{uv} + d_{uu} = c(2c_v + d_u).$$

Note that if either one of the first factors on the right-hand members of equations (9) is zero, the functions $c$ and $d$ must still satisfy the equations (11).

Now imposing the condition $c_v = d_u$ for a natural family, it is found by (11) and Theorem 1 that our linear natural family must be a $\Gamma$ family. Since $\Gamma$ and $\Gamma_0$ systems are linear natural, it is seen that Theorem 4 is proved in the general case. See §6 for special case.

6. **Proofs of Theorems 3 and 4 in the special cases.** We shall prove our theorems for the case where $\phi_v = 0$ and $\phi_u \neq 0$. The other case where $\phi_u = 0$ and $\phi_v \neq 0$ is of course similar. For our special case, it is seen that in order that the differential equation (4) be a velocity system, the function $\psi$ must be given by

$$(12) \qquad \psi = \alpha(v)\phi + \beta(v).$$

The functions $c$ and $d$ defining the velocity system are then

$$c = \frac{\phi_{uu}}{\phi_u} - \frac{2\alpha_v\phi_u}{\alpha_v\phi + \beta_v}, \qquad d = \frac{\alpha_{vv}\phi + \beta_{vv}}{\alpha_v\phi + \beta_v}. \tag{13}$$

Differentiating the first with respect to $v$ and the second with respect to $u$, we find

$$c_v = \frac{-2\phi_u(\alpha_{vv}\beta_v - \alpha_v\beta_{vv})}{(\alpha_v\phi + \beta_v)^2}, \qquad d_u = \frac{\phi_u(\alpha_{vv}\beta_v - \alpha_v\beta_{vv})}{(\alpha_v\phi + \beta_v)^2}. \tag{14}$$

The condition $c_v = -d_u$ for an isogonal system, or the condition $c_v = d_u$ for a natural system shows that in either case we must have $c_v = d_u = 0$. Therefore in our special case it is found that the only possible linear isogonal or linear natural systems are the $\Gamma_0$ families. This completes the proof of both Theorems 3 and 4 in the special cases.

7. **Kasner's characterization of surfaces of constant curvature by isothermal families.** Before proceeding with the proof of Kasner's second theorem, it is necessary to discuss some preliminary material. A natural family is a conformal image upon a plane of the geodesics of a general surface $S$. If $x = \text{const.}$ and $y = \text{const.}$ represent an isothermal net on $S$, the metric $ds$ for $S$ is given by

$$ds^2 = 2e^{\lambda(u,v)}du\,dv. \tag{15}$$

The natural family is then a velocity system for which

$$c = \lambda_u, \qquad d = \lambda_v. \tag{16}$$

The gaussian curvature $G$ of $S$ is $G = -e^{-\lambda}\lambda_{uv}$.

Now we state and prove the following result.

THEOREM 5. *The conformal map of the geodesics of a surface $S$ upon a plane is a $\Gamma$ or $\Gamma_0$ family if and only if $S$ is of constant gaussian curvature. It is a $\Gamma_0$ family if and only if $S$ is developable.*

The proof of this result is easily obtained by eliminating $c$ and $d$ from equations (2.3) and (16) and deducing from these two new equations that $G$ must be constant. Of course, $S$ is developable if and only if $\lambda$ is a harmonic function, and therefore if and only if a conformal image upon a plane of its geodesics is a $\Gamma_0$ family.

Now we shall state and prove Kasner's characterization of surfaces of constant curvature by isothermal families.

THEOREM 6. *A surface $S$ can possess exactly $\infty^2$ isothermal families of geodesics if and only if $S$ is of constant gaussian curvature.*

For let a surface $S$ possess exactly $\infty^2$ isothermal families of geodesics. Our surface $S$ can be mapped upon a plane so that its geodesics are represented by a natural family which contains exactly $\infty^2$ isothermal families. By Theorem 2, the natural family must be a $\Gamma$ or $\Gamma_0$ system. From Theorem 5, we see that $S$ must be of constant gaussian curvature.

The converse may be proved by the reversal of the argument of the preceding paragraph. Thus Theorem 6 is completely proved.

Kasner showed that the surfaces can be classified into three distinct types with respect to the number of isothermal families of geodesics: (1) the surfaces of constant curvature—exactly $\infty^2$; (2) the surfaces applicable to surfaces of revolution but of variable curvature—exactly one, and (3) the surfaces not applicable to surfaces of revolution—none.

8. **Beltrami's theorem.** We shall now show how Beltrami's theorem may be deduced from our preceding work.

THEOREM 7. *A surface $S$ may be mapped upon a plane so that its geodesics correspond to straight lines if and only if $S$ is of constant gaussian curvature.*

Let a surface $S$ be mapped upon a plane $\pi$ by a point transformation $T_1$ so that its geodesics correspond to straight lines. Now there exists a conformal transformation $T_2$ of $S$ upon $\pi$ so that the geodesics are represented by a natural family. Therefore the product $T_1T_2^{-1}$ carries the natural family into straight lines. By Theorem 4, the natural family must be a $\Gamma$ or $\Gamma_0$ system. Therefore by Theorem 5, our surface $S$ must be of constant gaussian curvature.

Conversely, by the reversal of the steps involved in the preceding argument, it may be shown that a surface of constant curvature may be mapped upon a plane $\pi$ so that its geodesics correspond to straight lines. This completes the proof of Beltrami's Theorem 7.

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# THE PELL EQUATION IN QUADRATIC FIELDS

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Consider the equation

$$\xi^2 - \gamma\eta^2 = 1, \tag{1}$$

where $\gamma$ is a given integer of a quadratic field $F$, and integral solutions $\xi$, $\eta$ are sought in $F$. It has been shown[1] that equation (1) has an infinite number of solutions if and only if $\gamma$ is not totally negative when $F$ is a real field, and $\gamma$ is not the square of an integer of $F$ when $F$ is imaginary. We now obtain the following result:

*Let $\gamma$ be such that equation* (1) *has an infinite number of solutions. If $F$ is a real field it is possible to find a solution $\xi_1$, $\eta_1$ of* (1) *so that every solution is given by the equations*

$$\begin{aligned} \xi &= \{(\xi_1 + \gamma^{1/2}\eta_1)^n + (\xi_1 - \gamma^{1/2}\eta_1)^n\}/2 \\ \eta &= \{(\xi_1 + \gamma^{1/2}\eta_1)^n - (\xi_1 - \gamma^{1/2}\eta_1)^n\}/(2\gamma^{1/2}), \end{aligned} \qquad n = 1, 2, 3, \cdots, \tag{2}$$

*if and only if $\gamma$ is not a totally positive non-square integer of $F$. If $F$ is imaginary it is always possible to find a solution $\xi_1$, $\eta_1$ so that all solutions are given by* (2).

The latter result is known to hold for the Pell equation in the rational field. The expression $\gamma^{1/2}$ is ambiguous, but no confusion will arise provided it consistently has the same value (we shall specify its value in certain cases). We consider the four sets $\pm\xi$, $\pm\eta$ to be a single solution, so that equations (2) give "every solution" in the sense that one of the four is present for some value of $n$.

*Case* 1. *$F$ real, $\gamma$ positive but not totally positive.* It will be convenient to consider $\gamma^{1/2}$, $\xi$ and $\eta$ positive. We now show that there is but a finite number of solutions of (1) with $\xi$ bounded, say $\xi < N$. For suppose we have an infinitude of solutions $\xi_i$, $\eta_i$ with $\xi_i < N$ for $i = 1, 2, 3, \cdots$. Taking conjugates in equation (1) we would have

$$\bar{\xi}_i^2 - \bar{\gamma}\bar{\eta}_i^2 = 1,$$

and since $-\bar{\gamma}$ is positive, this implies that $\bar{\xi}_i \leq 1$ for $i = 1, 2, 3, \cdots$. But it is not possible to have an infinite set of real quadratic integers which, along with their conjugates, are bounded.

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[1] *Quadratic diophantine equations in the rational and quadratic fields*, Trans. Amer. Math. Soc. vol. 52 (1942) p. 2 Theorem 4. We refer to this paper as (Q).

Hence there exists a least $\xi$, say $\xi_1$, among the positive solutions of (1); the corresponding $\eta$, say $\eta_1$, is also a least value. Thus we have, among the $\xi+\gamma^{1/2}\eta$, a minimum value $\xi_1+\gamma^{1/2}\eta_1$; it is unique, since otherwise $\gamma^{1/2}$ would be an element of $F$, contrary to hypothesis. Now any solution $\xi$, $\eta$ must correspond to a rational integer $n$ such that

$$(\xi_1 + \gamma^{1/2}\eta_1)^n \leqq \xi + \gamma^{1/2}\eta < (\xi_1 + \gamma^{1/2}\eta_1)^{n+1}.$$

Multiplying by $(\xi_1-\gamma^{1/2}\eta_1)^n$ we have

$$1 \leqq (\xi + \gamma^{1/2}\eta)(\xi_1 - \gamma^{1/2}\eta_1)^n < \xi_1 + \gamma^{1/2}\eta_1.$$

The central term is of the form $\xi_2+\gamma^{1/2}\eta_2$, $\xi_2$ and $\eta_2$ being integers of $F$. On multiplying this central term by $\xi_2-\gamma^{1/2}\eta_2$ we see that $\xi_2$, $\eta_2$ is a solution of (1); also neither $\xi_2$ nor $\eta_2$ is negative. But since $\xi_1+\gamma^{1/2}\eta_1$ is the least with positive $\xi$ and $\eta$, we must have $\xi_2=1$, $\eta_2=0$ and hence

$$\xi + \gamma^{1/2}\eta = (\xi_1 + \gamma^{1/2}\eta_1)^n.$$

This implies equations (2).

*Case* 2. *$F$ real, $\gamma$ negative but not totally negative.* All solutions of (1) are obtained by taking conjugates of solutions of

$$\xi^2 - \bar{\gamma}\eta^2 = 1, \tag{3}$$

so the problem reduces to Case 1. The particular solution $\xi_1$, $\eta_1$ can be obtained in this case by taking the conjugate of the least positive solution of (3).

*Case* 3. *$F$ real, $\gamma$ a perfect square in $F$ (and hence totally positive).* Let $\gamma=\alpha^2$, where $\alpha$ is a positive integer in $F$. We prove that there is only a finite number of positive solutions of (1) with $\xi+\alpha\eta$ bounded. Any solution gives us two integers of the field $\xi+\alpha\eta$ and $\xi-\alpha\eta$, with product unity. But two integers of a real quadratic field have this property only if one is the conjugate, or the negative of the conjugate, of the other. Hence we can write $\xi+\alpha\eta=\rho$, $\xi-\alpha\eta=\pm\bar{\rho}$. Taking $\xi$, $\eta$ and $\alpha$ positive we have $\rho>1$ and $1>|\bar{\rho}|>0$. But there is only a finite number of integers $\rho$ which exceed 1, are bounded, and possess the property $\rho\bar{\rho}=\pm 1$. Thus among the infinitude of solutions of (1) there exists one having $\xi+\alpha\eta$ a minimum. It is unique; for if $\xi_1+\alpha\eta_1=\xi_2+\alpha\eta_2$ with, say, $\xi_1>\xi_2$ and $\eta_1<\eta_2$, then not both $\xi_1$, $\eta_1$ and $\xi_2$, $\eta_2$ can satisfy (1). Hence we proceed as in Case 1 to derive all solutions from this unique smallest one.

*Case* 4. *$F$ real, $\gamma$ totally positive but not a perfect square in $F$.* We shall show that (1) has infinitely many solutions with $\xi$ bounded. In Lemma 2 in §3 of (Q) it is proved that if $\gamma$ is positive but not totally positive there are infinitely many solutions of (1); we shall make direct use of this proof.

Note first that the infinitude of solutions obtained are such that $\bar{\xi}$ is bounded. To prove this, we observe that the integer $\rho$ introduced in connection with the infinite series (10) in (Q) is bounded, as also is its conjugate; this is seen from inequality (9) and the argument following. And since $\xi$ is obtained in (15) by dividing $\xi_r\xi_s-\gamma\eta_r\eta_s$ by $\rho$, we can obtain $\bar{\xi}$ by multiplying $\bar{\xi}_r\bar{\xi}_s-\bar{\gamma}\bar{\eta}_r\bar{\eta}_s$ by the reciprocal of $\bar{\rho}$, both of which are bounded: the first because of (8), and the second because the reciprocal of $\bar{\rho}$ does not exceed $\rho$, which is bounded. Instead of obtaining the infinitude of solutions as suggested in (Q) (see the sentence immediately preceding Lemma 3), we can get them by compounding the first pair in (11) with all subsequent pairs, and thus $\bar{\xi}$ is bounded for all resulting solutions.

Next we note that the proof of Lemma 2 in (Q) is valid if $\gamma$ is a totally positive non-square integer of $F$. Two remarks should be made in explanation: after inequality (9) the argument used to show that the left side of (7) cannot be zero needs a slight alteration; the result follows from the fact that $\gamma^{1/2}$, or $\delta$, cannot be an element of $F$, by hypothesis; also the inequality just preceding (10) must be changed since $\bar{\gamma}$ is now positive; the essential idea, that $\bar{\xi}^2-\bar{\gamma}\bar{\eta}^2$ is bounded is still correct.

Thus the proof of Lemma 2 of (Q) can be used to obtain the result that our present equation (1) has infinitely many solutions with $\bar{\xi}$ bounded, $\gamma$ being totally positive but not a square. But $\bar{\xi}$ and $\bar{\eta}$ are solutions of

$$\bar{\xi}^2 - \bar{\gamma}\bar{\eta}^2 = 1,$$

$\bar{\gamma}$ also being totally positive but not a square; interchanging $\xi$, $\eta$ and $\gamma$ with their conjugates we have the result that our equation (1) has infinitely many solutions with $\xi$ bounded.

Now if there were some least positive solution $\xi_1$, $\eta_1$, we could proceed as in Case 1 and derive all solutions from it by equations (2). But $\xi$ in equations (2) is bounded for only a finite number of values of $n$. Thus we cannot obtain all solutions by such a scheme.

*Case* 5. *$F$ imaginary, $\gamma$ not a perfect square in $F$.* We first show that, except for trivial solutions $\pm 1, 0$, we cannot have $|\xi+\gamma^{1/2}\eta|=1$. For otherwise $|\xi-\gamma^{1/2}\eta|=1$, and the inequality

$$(4)\qquad |\xi| \leqq (1/2)\,|\xi+\gamma^{1/2}\eta| + (1/2)\,|\xi-\gamma^{1/2}\eta|$$

would give us $|\xi|\leqq 1$, yielding only a trivial solution. Hence for convenience we can choose the sign of $\xi$ (or $\eta$) so that

$$| \xi + \gamma^{1/2}\eta | > 1, \qquad | \xi - \gamma^{1/2}\eta | < 1.$$

Next we show that two essentially different solutions $\xi_1$, $\eta_1$ and $\xi_2$, $\eta_2$ cannot be such that

$$| \xi_1 + \gamma^{1/2}\eta_1 | = | \xi_2 + \gamma^{1/2}\eta_2 |.$$

For if this were the case we would have another solution $\xi_3$, $\eta_3$ defined by

$$\xi_3 = \xi_1\xi_2 - \gamma\eta_1\eta_2, \qquad \eta_3 = \xi_1\eta_2 - \xi_2\eta_1.$$

It would follow that

$$| \xi_3 + \gamma^{1/2}\eta_3 | = | \xi_1 - \gamma^{1/2}\eta_1 | \cdot | \xi_2 + \gamma^{1/2}\eta_2 | = 1,$$

and by the argument above, $\xi_3 = \pm 1$ and $\eta_3 = 0$. These imply that $\xi_1 = \pm \xi_2$, $\eta_1 = \pm \eta_2$.

Now $|\xi + \gamma^{1/2}\eta|$ is less than any given positive value $N > 1$ for but a finite number of solutions of (1); for by (4) if $|\xi + \gamma^{1/2}\eta|$ is bounded so is $|\xi|$, and this cannot be bounded for an infinite set of integers of an imaginary quadratic field. Hence there exists a unique nontrivial solution of (1) having $|\xi + \gamma^{1/2}\eta|$ a minimum, and we can proceed as in Case 1.

PURDUE UNIVERSITY

# FIELD CONCENTRATION NEAR CIRCULAR CONDUCTORS

H. PORITSKY

1. **Introduction.** The problem considered in this note is the calculation of the maximum field concentration which obtains in the field shown in Figure 1a. Here the semicircles represent the upper half of an infinite row of cylindrical metallic conductors at a constant potential, the conductors touching each other. There is a potential difference between these and the upper plane surface, and it is required to calculate the resulting field concentration which obtains at the semicircular conductors. The maximum electric gradient will obviously occur at the midpoints between the points of contact with the adjacent conductors.

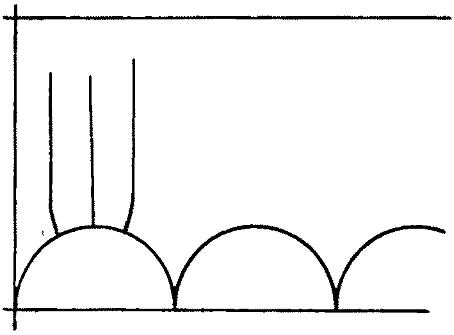

FIG. 1a

It is assumed that the plane surface representing the other electrode is sufficiently far away so that the field is quite uniform near it and the field concentration is not affected by the distance between the circular wires and the plane surface. In fact, throughout the following it will be supposed that this plane is at infinity.

The above problem is of interest in the construction of electric cable where a row of circular wires is often arranged around a circular cylindrical surface and a potential difference maintained between the ions and an outer circular sheath concentric with the axis of the cylinder and external to the wires. To the first approximation the field concentration factor which obtains in Figure 1a may be carried over to the case of a cable by first assuming that the surface made up

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of the semicircles is replaced by a tangent cylinder, calculating the resulting electric gradient at this surface, and then multiplying it by the field concentration factor derived from the plane case.

2. **Solutions by means of the modular function.** The method employed in this note utilizes the modular function, $\lambda(\tau)$, and is based upon the fact that the function just mentioned maps the shaded region of Figure 1b in the $\tau$-plane upon the shaded region of the $\lambda$-plane

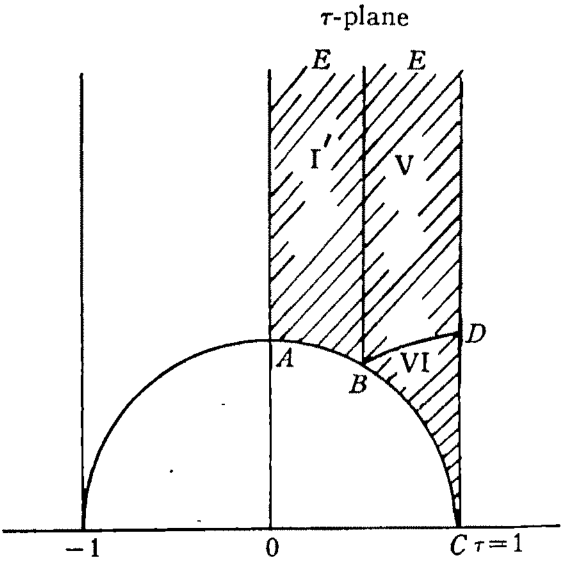

FIG. 1b

shown in Figure 1c (see Burkhardt, *Elliptische Funktionen*, Berlin, 1920, p. 259). Thus the rather complicated boundary consisting partly of circles and partly of straight lines is now replaced by rectilinear boundaries only. Therefore the solution of the problem can be carried out very simply. The analytic expression for $w$ in terms of $\lambda$ is given below.

It will be noticed that the $\tau$-plane is simply related to the original $z$-plane. In fact, this relation is given by

$$z = (\pi/2)\tau. \tag{1}$$

If the region to the left of the pure imaginary axis of Figure 1b, and forming the mirror image of the shaded region, is added to the latter, the combined region which is bounded by the complete semicircle from $\tau = -1$ to $\tau = 1$ and two vertical lines through $\tau = 1$,

$\tau = -1$ goes into a half-plane in Figure 1c, namely the half-plane to the left of the vertical axis through $\lambda = 1/2$. Since the flux lines in Figure 2 all meet the circular boundary normally, it follows that the corresponding flux lines of Figure 1c will all cross the vertical straight line through $\lambda = 1/2$ normally. In Figure 1b the flux lines go off to infinity; in Figure 1c they must converge toward the origin $E$, which point corresponds to the infinite region of Figure 1b, at least for ways of recession to infinity in the direction of the pure imaginary axis in the $\tau$-plane. This means that the flux lines in Figure 1c are simply due to positive and negative point charges placed, respectively, at the origin $\lambda = 0$ and at the point $\lambda = 1$, which is its negative image in the vertical line through $\lambda = 1/2$ (that is, through $A$ of Figure 1c). Therefore one might at once express the flux function in terms of $\lambda$ as follows:

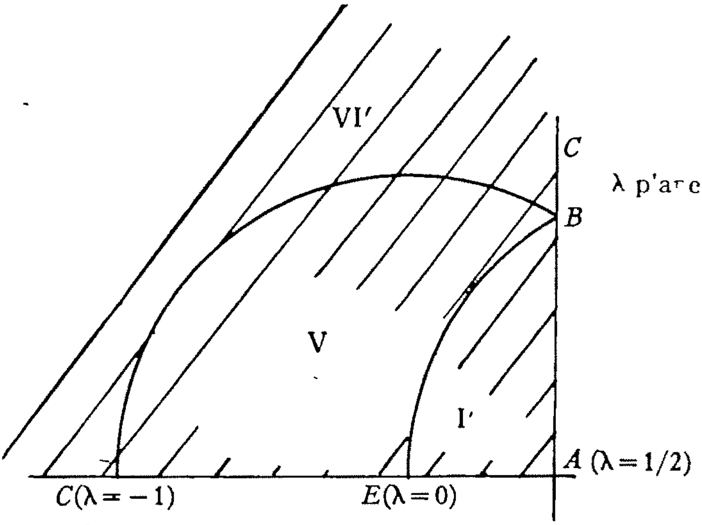

Fig. 1c

$$w = (1/2) \ln \frac{\lambda - 1}{\lambda}. \tag{2}$$

The factor 1/2 is due to the fact that $\pi$ flux lines are supposed to converge toward the origin $\lambda = 0$, since $\pi$ flux lines receded to infinity in the original $z$-plane for each period strip.

3. **Expansions in powers of** $q$. The expression for the modular function is given by

$$\lambda = \lambda(\tau) = \frac{16q(1 + q^{1\cdot 2} + q^{2\cdot 3} + \cdots)^4}{(1 + 2q + 2q^4 + 2q^9 + \cdots)^4} \tag{3}$$

(see Burkhardt, loc. cit. p. 253), where $q$ is given by

$$q = e^{\tau\pi i}. \tag{4}$$

It will be recalled that this modular function is of interest in connection with transformation of periods of an elliptic function and that the expression obtained for it, namely equation (3) above, can be derived from the $\vartheta$-functions, being in fact equal to

$$\lambda(\tau) = \frac{\vartheta_2^4(0)}{\vartheta_3^4(0)}; \tag{5}$$

the $\vartheta$'s are periodic functions of interest in the theory of elliptic functions and involve as a parameter the quantity $q$, though this has not been indicated in the notation of the right-hand side of (5). $q$ itself is given by (4) where $\tau$ is the ratio of the two periods of the elliptic function in question. This connection of the function $\lambda(\tau)$ with elliptic functions is not utilized in the following, and is mentioned here only in passing. The only essential feature is the fact that the function $\lambda(\tau)$ defined by (3) and (4) maps the shaded region of Figure 1b upon that of Figure 1c, and yields a simple expression for $w$ (equation (2)).

To proceed with the calculation of the field it is now necessary to evaluate the expression of the derivative $dw/dz$ whose absolute value gives the field strength at any point. This derivative will be evaluated as a product of four distinct derivatives as follows:

$$\frac{dw}{dz} = \frac{dw}{d\lambda}\,\frac{d\lambda}{dq}\,\frac{dq}{d\tau}\,\frac{d\tau}{dz}. \tag{6}$$

It is clear from (3) that it is easier to differentiate the logarithm of $\lambda$ rather than $\lambda$ itself; hence we shall utilize this derivative:

$$\begin{aligned} \frac{d\log\lambda}{dq} = \frac{1}{\lambda}\,\frac{d\lambda}{dq} = \frac{1}{q} &+ 4\,\frac{1\cdot 2q + 2\cdot 3q^5 + \cdots}{1 + q^{1\cdot 2} + q^{2\cdot 3} + \cdots} \\ &- 4\,\frac{1\cdot 2q + 2\cdot 3q^5 + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots}. \end{aligned} \tag{7}$$

Again, from (4) follows

$$dq/d\tau = \pi i e^{\tau\pi i} = \pi i q \tag{8}$$

while (1) and (2) yield

$$ds/d\tau = 2/\pi, \tag{9}$$

$$\frac{dw}{d\lambda} = \frac{1}{2}\left[\frac{1}{\lambda - 1} - \frac{1}{\lambda}\right]. \tag{10}$$

Substituting from (7)–(10) into (6) there results

$$\begin{aligned}\frac{dw}{ds} &= \frac{1}{2}\left[\frac{1}{\lambda + 1} - \frac{1}{\lambda}\right]\lambda\,\frac{d(\log\lambda)}{dq}\,\pi i q\,\frac{2}{\pi} \\ &= i\left[\frac{1}{\lambda - 1}\right]\left[1 + 4\,\frac{1\cdot 2q^{1\cdot 2} + 2\cdot 3q^{2\cdot 3} + \cdots}{1 + q^{1\cdot 2} + q^{2\cdot 3} + \cdots}\right. \\ &\qquad\qquad \left. - 8\,\frac{q + 4q^4 + 9q^9 + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots}\right].\end{aligned} \tag{11}$$

The values of the various variables corresponding to the point $A$ of Figure 1b are

$$\tau = i, \quad \lambda = 1/2, \quad q = e^{-\pi} = 10^{\bar{2}.635624} = .043214.$$

Substituting in (11) and taking absolute values, there results

$$\left|\frac{dw}{ds}\right| = 2[1 + .0149 - .3184] = 1.3952.$$

This leads to the field concentration factor 1.3952.

The above served to check the field concentration factor 1.4 which was first obtained by means of a free-hand flux plot of this field shown in Figure 2.

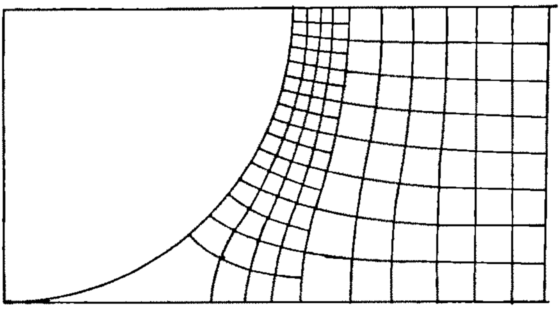

FIG. 2

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# ON THE SIEVE METHOD OF VIGGO BRUN

R. D. JAMES

1. **Introduction.** Numerous improvements have been made in the the sieve method since the appearance of Brun's work.[1] Rademacher,[2] Estermann,[3] and, more recently, Buchstab[4] have introduced new ideas and so obtained more precise results. If their work is examined it will be seen that the various estimates which they use can all be made to depend on the single estimate

$$\sum_{p \leq x} (\log p)/p = \log x + O(1), \tag{1.1}$$

the summation extending over all primes $p \leq x$.

In a previous paper[5] it was shown that Rademacher's results could be extended to any infinite set of primes for which the estimate

$$\sum_{p \leq x}{}' (\log p)/p = r \log x + O(1) \tag{1.2}$$

applies. Here, and subsequently, the dash indicates that the summation extends over all the primes of a given infinite set which do not exceed $x$, and $r$ is a given real number such that $0 < r \leq 1$.

In this paper it will be shown that Buchstab's results can be similarly extended. Application will be made to primes in arithmetic progression since the set of all primes $p \equiv h \pmod{k}$ with $(h, k) = 1$ satisfies (1.2) with $r = 1/\phi(k)$.

2. **Preliminary lemmas.** It is well known[6] that from (1.1) it follows that

$$\sum_{p \leq x} 1/p = \log \log x + C + O(1/\log x)$$

and

$$\prod_{p \leq x} (1 - (2/p)) = D/\log^2 x + O(1/\log^3 x),$$

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[1] *Le crible d'Eratosthène et le théorème de Goldbach*, Christiania, 1920

[2] Abh. Math. Sem. Hamburgischen Univ. vol. 3 (1924) pp. 12–30.

[3] J. Reine Angew. Math. vol. 168 (1932) pp. 106–116.

[4] Rec. Math. (Mat. Sbornik) N.S. vol. 46 (1938) pp. 375–387. C. R. (Doklady) Acad. Sci. URSS. vol. 29 (1940) pp. 544–548.

[5] Trans. Amer. Math. Soc. vol. 43 (1938) pp. 296–302.

[6] Landau, *Primzahlen*, vol. I, pp. 98–102.

where $C$ and $D$ are constants. These two estimates form the basis of Rademacher's results.

In precisely the same way it follows from (1.2) that

$$\sum_{p\leqq x}{}' 1/p = r \log\log x + C(r) + O(1/\log x) \tag{2.1}$$

and

$$\prod_{p\leqq x}{}' (1-(2/p)) = D(r)/\log^{2r} x + O(1/\log^{2r+1} x), \tag{2.2}$$

where $C(r)$ and $D(r)$ depend only on $r$. Another result which Buchstab uses (Lemma 3 of the first paper referred to) and which also follows from (1.1) is

$$\begin{aligned}\sum_{x^{1/v}\leqq p<x^{1/u}} & 1/\{p(\log x-\log p)^2\} \\ &= (1/\log^2 x)\{\log (v-1) - \log (u-1) \\ &\qquad + u/(u-1) - v/(v-1)\} + O(1/\log^3 x),\end{aligned} \tag{2.3}$$

where $u$ and $v$ are real numbers such that $2\leqq u\leqq v$.

This may also be written in the form

$$\begin{aligned}\sum_{x^{1/v}\leqq p<x^{1/u}} & 1/\{p(\log x-\log p)^2\} \\ &= (1/\log^2 x)\int_{u-1}^{v-1} (z+1)z^{-2}dz + O(1/\log^3 x)\end{aligned} \tag{2.4}$$

which may be generalized.

The proof of (2.4) is made in the usual way by partial summation. An outline of the proof follows. Let

$$R(x) = \sum_{p\leqq x} (\log p)/p$$

so that by (1.1)

$$R(x) = \log x + r(x),$$

where $r(x)=O(1)$ and, in particular $r(1)=0$. Then the left side of (2.4) is equal to

$$\begin{aligned}\sum & \{R(n) - R(n-1)\}/\{\log n(\log x - \log n)^2\} \\ &= \sum \{\log n - \log (n-1)\}/\{\log n(\log x - \log n)^2\} \\ &\qquad + \sum \{r(n) - r(n-1)\}/\{\log n(\log x - \log n)^2\},\end{aligned} \tag{2.5}$$

where the summation range is $x^{1/v}\leqq n<x^{1/u}$. Since

$$\log n - \log (n-1) = 1/n + O(1/n^2),$$

the first sum will be

$$\sum 1/\{n \log n(\log x - \log n)^2\} \tag{2.6}$$

plus an error term. The sum (2.6), in turn, is given by

$$\int_{x^{1/v}}^{x^{1/u}} 1/\{t \log t(\log x - \log t)^2\}\,dt$$

plus an error term. The substitution $\log t = (\log x)/(z+1)$ transforms this integral into the integral in (2.4). The proof is then completed by showing that the second sum in (2.5) and the various error terms are all $O(1/\log^3 x)$.

The generalized form of (2.4) is given in the following lemma.

Lemma 1. *Let s be any positive real number and let u and v be positive real numbers such that* $1 \leqq u \leqq v$ *if* $2s < 1$, $1 < u \leqq v$ *if* $2s \geqq 1$.
*Then*

$$\sum_{x^{1/v} \leqq p < x^{1/u}}' 1/\{p(\log x - \log p)^{2s}\}$$
$$= (r/\log^{2s} x) \int_{u-1}^{v-1} (z+1)^{2s-1} z^{-2s}\,dz + O(1/\log^{2s+1} x).$$

The proof is omitted since it follows that given for (2.4) but uses (1.2) instead of (1.1).

3. **The first theorem.** Next, following Rademacher and Buchstab, we define a function $P_\omega(x; y, r) = P(x; y, r; \Delta, a; p_i, a_i, b_i)$. Let $\Delta$ and $a$ be given positive integers and let $p_1, \cdots, p_k$ be the primes not dividing $\Delta$ and less than $y$ of a given infinite set of primes for which (1.2) holds. Let $\omega$ denote a given set of non-negative integers $a_1, b_1; \cdots; a_k, b_k$; $a_i < p_i$, $b_i < p_i$ with $a_i \neq b_i$, $i = 1, \cdots, k$. Then $P_\omega(x; y, r)$ is defined as the number of integers $n$ which satisfy the conditions

$$\begin{gathered} n \leqq x, \qquad n \equiv a \pmod{\Delta}, \\ (n - a_i)(n - b_i) \not\equiv 0 \pmod{p_i}, \qquad i = 1, \cdots, k. \end{gathered} \tag{3.1}$$

Since $P_\omega(x; y, r) = P_\omega(x; x, r)$ for $y \geqq x$ we may allow $y$ to be infinite and consider an infinite set of integers $a_i, b_i$.

It follows as in Rademacher's paper that if $p_k$ is the largest prime of the given set less than $x^{1/\gamma}$ then

$$P_\omega(x; x^{1/\gamma}, r) = P_\omega(x; p_{k+1}, r) > Ex/\Delta - R,$$

where

$$E = \left\{1 - 2 \sum_{1 \leqq a \leqq k}' (1/p_a)\right\}$$
$$+ \sum_{1 \leqq a \leqq k}' \sum_{1 \leqq b \leqq k_1}' (4/p_a p_b)\left\{1 - 2 \sum_{1 \leqq c \leqq b}' (1/p_c)\right\} + \cdots$$

and

$$k \geqq k_1 \geqq \cdots \geqq k_n \geqq 1,$$
$$R = (2k + 1)(2k_1 + 1)^2 \cdots (2k_n + 1)^2.$$

From (2.1) and (2.2) it follows that for every $\epsilon > 0$ there exists a number $w_0 = w_0(\epsilon)$ such that for all $w \geqq w_0$ we have

$$\tag{3.2} \begin{gathered} \sum{}' 1/p < r \log (m + \epsilon), \\ \prod{}' (1 - (2/p)) > (m + \epsilon)^{-2r}, \qquad w \leqq p < w^m. \end{gathered}$$

We now define three positive numbers $k$, $B$, and $B'$ which depend on $r$. Thus

$$k = e^{1/4r} - \epsilon, \qquad B = (22/17)^{1/r} - \epsilon, \qquad B' = (26/23)^{1/r} - \epsilon,$$

where $\epsilon$ is an arbitrarily small positive number. Buchstab uses these numbers with $r = 1$ in the proofs of his first two lemmas which correspond to our Theorem 1.

Following Buchstab we choose $k_1 = k$ where $p_k$ is the largest prime of the given set less than $x^{1/\gamma}$; for $i \geqq 2$ we choose $k_i$ so that $p_{k_i}$ is the largest prime in the given set less than $x^{1/\gamma B k^{i-2}}$. The process is continued until we come to $i = t+1$ where $p_{k_{t+1}}$ is the least prime in the given set for which $p_{k_{t+1}}^{1/k} < w_0 \leqq p_{k_{t+1}}$. For $t+1 < i \leqq n$ we choose $p_{k_i} = p_{k_{t+1}}$.

We denote by $E_v$ $(v = 1, 2, \cdots, n)$ the sum formed by taking from $E$ only those terms for which the subscripts on the $p$'s exceed $k_{v+1}$. We then write

$$E_v = \sum_{i=0}^{2v+1} E_v^{(i)},$$

where

$$E_v^{(i)} = \sum_{s=0}^{i} E_{v-1}^{(s)} S_v^{(i-s)}$$

and $S_v^{(i-s)}$ is the $(i-s)$th elementary symmetric function of

$$2/p_m, \; m = k_{v+1} + 1, k_{v+1} + 2, \cdots, k_v.$$

It follows as in the papers of Rademacher and Buchstab that for

$v \geqq 2$ we have

$$E_v \geqq E_{v-1}P_v - \Phi_v, \tag{3.3}$$

where

$$P_v = \prod\nolimits'_{r_{v+1}<i\leqq r_v} (1-(2/p_i))$$

and

$$\Phi_v = \sum_{s=0}^{2v-1} E_{v-1}^{(s)} S_v^{(2v+1-s)}.$$

For $v=1$ we have

$$E_1 > P_1 - S_1^{(4)} \tag{3.4}$$

and hence using (3.3) and (3:4)

$$\begin{aligned} E = E_n > P_1 \cdots P_n\{1 - P_1^{-1}S_1^{(4)} - P_1^{-1}P_2^{-1}\Phi_2 \\ - P_1^{-1}P_2^{-1}P_3^{-1}\Phi_3 - \cdots\}. \end{aligned} \tag{3.5}$$

From (3.2) and the values for $B$ and $h$ it follows that

$$S_1^{(1)} = 2 \sum\nolimits'_{r_2<s\leqq r_1} 1/p_s < 2r \log(B+\epsilon) = 2\log(22/17),$$
$$S_v^{(1)} = 2 \sum\nolimits'_{r_{v+1}<s\leqq r_v} 1/p_s < 2r \log(h+\epsilon) = 2\log e^{1/4} \doteq 1/2.$$

Similarly from (3.2) we obtain

$$P_1^{-1} < (22/17)^2, \qquad P_v^{-1} < e^{1/2}.$$

These numerical results are exactly the inequalities used by Buchstab to estimate the value of $E$ from (3.5). We may therefore use his calculations and conclude that

$$P_\omega(x;\, x^{1/\gamma},\, r) > 0.98(cx/\log^{2r} x) + O(x/\log^{2r+1} x) + O(x^\delta)$$

where

$$\begin{aligned} c &= c(r) = \gamma^{2r}D(r)/\Delta, \\ \delta &= \{3 + 2h/B(h-1)\}/\gamma. \end{aligned} \tag{3.6}$$

Similarly it can be shown that

$$P_\omega(x;\, x^{1/\gamma},\, r) < 1.016(cx/\log^{2r} x) + O(x/\log^{2r+1} x) + O(x^{\delta'})$$

where

$$\delta' = \{2 + 2h/B'(h-1)\}/\gamma.$$

If we now choose $\gamma = \gamma(r)$ so that $\delta$ and $\delta'$ are each less than 1 we obtain this theorem.

THEOREM 1. *We have*

$$P_\omega(x;\ x^{1/\gamma},\ r) > 0.98\ (cx/\log^{2r} x) + O(x/\log^{2r+1} x),$$
$$P_\omega(x;\ x^{1/\gamma},\ r) < 1.016(cx/\log^{2r} x) + O(x/\log^{2r+1} x)$$

*independently of the set* $\omega$ *of integers* $a_i$, $b_i$.

4. **The second theorem.** Corresponding to Buchstab's Theorems 1 and 2 we have this theorem.

THEOREM 2. *Suppose that for fixed* $r$ *there exist functions* $f_k(\alpha)$ *and* $F_k(\alpha)$ *having a finite number of finite discontinuities in the respective ranges* $0 \leqq \alpha \leqq B_1$, $0 \leqq \alpha \leqq B_2$, *where* $B_1$ *and* $B_2$ *are certain constants such that* $|B_1 - B_2| \leqq 1$. *Suppose further that the inequalities*

$$(4.1)\qquad \begin{aligned} P_\omega(x;\ x^{1/\alpha},\ r) &> f_k(\alpha)(x/\log^{2r} x) + O(x/\log^{2r+1} x),\\ P_\omega(x;\ x^{1/\alpha},\ r) &< F_k(\alpha)(x/\log^{2r} x) + O(x/\log^{2r+1} x) \end{aligned}$$

*hold independently of* $\omega$. *Then the functions* $f_{k+1}(\alpha)$ *and* $F_{k+1}(\alpha)$ *defined by*

$$f_{k+1}(\alpha) = f_k(\alpha), \qquad F_{k+1}(\alpha) = F_k(\alpha), \qquad 0 \leqq \alpha < 1;$$

$$f_{k+1}(\alpha) = \max \left\{ f_k(\alpha),\ f_k(\beta) - 2r\int_{\alpha-1}^{\beta-1} F_k(z)(z+1)^{2r-1}z^{2r}dz \right\},$$
$$1 \leqq \alpha \leqq \beta \leqq B_1;$$

$$F_{k+1}(\alpha) = \min \left\{ F_k(\alpha), F_k(\beta) - 2r\int_{\alpha-1}^{\beta-1} f_k(z)(z+1)^{2r-1}z^{2r}dz \right\},$$
$$1 \leqq \alpha \leqq \beta \leqq B_2;$$

*also satisfy the inequalities* (4.1).

Buchstab's proof for the case $r = 1$ proceeds as follows. The difference between $P_\omega(x;\, p_{k+1},\, 1)$ and $P_\omega(x;\, p_k,\, 1)$ is the number of integers $n$ of the form

$$a_k + mp_k \quad \text{or} \quad b_k + mp_k, \qquad m = 0, 1, 2, \cdots,$$

which satisfy the conditions (3.1) for $i = 1, 2, \cdots, k-1$. This is the same as the number of integers $m$ which satisfy the conditions

$$m \leqq (x - a_k)/p_k, \qquad a_k + mp_k \equiv a \pmod{\Delta},$$
$$(m - a_i')(m - b_i') \not\equiv 0 \pmod{p_i}, \qquad i = 1, 2, \cdots, k-1,$$

plus the number of integers which satisfy a similar set of conditions with $a_k$ replaced by $b_k$. Hence we have

$$\begin{aligned} P_\omega(x; p_{k+1}, 1) = P_\omega(x; p_k, 1) &- P_{\omega_k'}((x - a_k)/p_k; p_k, 1) \\ &- P_{\omega_k''}((x - b_k)/p_k; p_k, 1). \end{aligned}$$

Now, $P_{\omega_k}(x/p_k;\ p_k,\ 1)$ counts at most one more integer than $P_{\omega_k}((x-a_k)/p_k; p_k, 1)$, so that we have the recursion formula

$$\begin{aligned} P_\omega(x; p_{k+1}, 1) = P_\alpha(x; p_k, 1) &- P_{\omega_k}(x/p_k; p_k, 1) \\ &- P_{\omega'_k}(x/p_k; p_k, 1) + \mu_k \end{aligned} \tag{4.2}$$

where $0 \leqq \mu_k \leqq 2$. Suppose that $p_j, \cdots, p_k$ are the primes of the given set between $x^{1/\beta}$ and $x^{1/\alpha}$; that is,

$$p_{j-1} < x^{1/\beta} \leqq p_j < \cdots < p_k < x^{1/\alpha} \leqq p_{k+1}.$$

By repeated application of (4.2) we obtain

$$\begin{aligned} P_\omega(x; x^{1/\alpha}, 1) = P_\omega(x; x^{1/\beta}, 1) &- \sum_s P_{\omega_s}(x/p_s; p_s, 1) \\ &- \sum_s P_{\omega_s'}(x/p_s; p_s, 1) + \sum_s \mu_s, \end{aligned} \tag{4.3}$$

where $\sum \mu_s = O(x^{1/2})$. The two sums on the right side of (4.3) are each divided into partial sums defined by

$$x^{1/(u_{i+1}+1)} \leqq p < x^{1/(u_i+1)},$$

where

$$u_i = \alpha - 1 + i(\beta - \alpha)/n,$$

and $n$ is an integer such that $c_1 \log x \leqq n \leqq c_2 \log x$.

For a typical partial sum $T_i$ we use (4.1) and obtain

$$\begin{aligned} T_i \leqq F_k(u_{i+1}) &\sum x/\{p(\log x - \log p)^2\} \\ &+ O(x/\log x) \sum 1/\{p(\log x - \log p)^2\}. \end{aligned}$$

By Lemma 1 with $s=1$, $u-1=u_i$, $v-1=u_{i+1}$ this becomes

$$\begin{aligned} T_i \leqq F_k(u_{i+1})(x/\log^2 x) &\int_{u_i}^{u_{i+1}} (z+1)z^{-2}dz \\ &+ O(x/\log^3 x) \int_{u_i}^{u_{i+1}} (z+1)z^{-2}dz. \end{aligned}$$

Then we have

$$\sum_{i=0}^{n-1} F_k(u_{i+1}) \int_{u_i}^{u_{i+1}} (z + 1)z^{-2}dz = \int_{\alpha-1}^{\beta-1} F(z)(z+1)z^{-2}dz + O(1/\log x).$$

Putting these results together we find that

$$P_\omega(x;\, x^{1/\alpha},\, 1) > (x/\log^2 x)\left\{f_k(\beta) - 2\int_{\alpha-1}^{\beta-1} F(z)(z+1)z^{-2}dz\right\} + O(x/\log^3 x)$$

which means that

$$f_{k+1}(\alpha) = f_k(\beta) - 2\int_{\alpha-1}^{\beta-1} F(z)(z+1)z^{-2}dz$$

will serve as a new function in (4.1).

The result for $F_{k+1}(\alpha)$ is obtained in exactly the same way. The proof of Theorem 2 is the same except that we use Lemma 1 with $s=r$ instead of $s=1$.

Since $0\leqq P_\omega(x;\, x^{1/\alpha},\, r)\leqq P_\omega(x;\, x^{1/\gamma},\, r)$ for $0\leqq\alpha\leqq\gamma$ it follows from Theorem 1 that we may take

$$f_0(\alpha) = \begin{cases} 0 & \text{for } 0\leqq\alpha<\gamma,\\ 0.98c & \text{for } \alpha=\gamma;\end{cases}$$

$$F_0(\alpha) = 1.016c \quad \text{for } 0\leqq\alpha\leqq\gamma.$$

Then, using Theorem 2 we compute successively $F_1(\alpha)$, $f_1(\alpha)$; $F_2(\alpha)$, $\cdots$. A numerical example to illustrate the process is given in §6.

5. **A generalization of the second theorem.** Let us consider a given infinite set of primes for which

$$\sum_{p\leqq z}'' (\log p)/p = s\log x + O(1), \qquad s>r, \tag{5.1}$$

holds and an infinite subset of this set for which (1.2) is satisfied. We define a function $P_\omega(x;\, z,\, s;\, y,\, r)$ similar to $P_\omega(x;\, y,\, r)$. Let $p_1, \cdots, p_k$ be the primes of the given set not dividing $\Delta$ and not exceeding $z$ for which (5.1) holds; let $p_{k+1}, \cdots, p_l$ be the primes of the subset not dividing $\Delta$, greater than $z$ and not greater than $y$ for which (1.2) holds. Let $\omega$ denote a given set of non-negative integers $a_1, b_1; \cdots; a_lb_l$; $a_i<p_i$, $b_i<p_i$ with $a_i\neq b_i$, $i=1, \cdots, l$. Then $P_\omega(x;\, z,\, s;\, y,\, r)$ is defined as the number of integers $n\leqq x$ satisfying the conditions

$$n\equiv a \pmod{\Delta};\quad (n-a_i)(n-b_i)\not\equiv 0 \pmod{p_i},\qquad i=1, 2, \cdots, l.$$

THEOREM 3. *Suppose that for fixed $r$, $s$, $A$ there exist functions $g_k(\alpha)$, $G_k(\alpha)$ which have a finite number of finite discontinuities in the range $0\leqq\alpha\leqq A$. Suppose further that the inequalities*

$$(5.2)\quad \begin{aligned} &P_\omega(x, x^{1/A}, s; x^{1/\alpha}, r) > g_i(\alpha)\,(x/\log^{2s} x) + O(x/\log^{2s+1} x),\\ &P_\omega(x; x^{1/A}, s; x^{1/\alpha}, r) < G_k(\alpha)(x/\log^{2s} x) + O(x/\log^{2s+1} x),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leqq \alpha \leqq A, \end{aligned}$$

*hold independently of* $\omega$. *Then the functions* $g_{i+1}(\alpha)$ *and* $G_{k+1}(\alpha)$ *defined by*

$$g_{i+1}(\alpha) = g_i(\alpha), \qquad G_{k+1}(\alpha) = G_k(\alpha), \qquad 0 \leqq \alpha < 1,$$

$$g_{i+1}(\alpha) = \max \left\{ g_i(\alpha),\ g_i(\beta) - 2r \int_{\alpha-1}^{\beta-1} G_k(z)(z+1)^{2s-1} z^{-2s} dz \right\},$$

$$G_{k+1}(\alpha) = \min \left\{ G_k(\alpha), G_k(\beta) - 2r \int_{\alpha-1}^{\beta-1} g_i(z)(z+1)^{2s-1} z^{-2s} dz \right\},$$

$$0 \leqq \alpha \leqq A,$$

*are also functions satisfying* (5.2).

The method of proof is again that given by Buchstab but we use Lemma 1 instead of (2.4).

Since $P_\omega(x; x^{1/A}, s; x^{1/A}, r)$ is the same as the function $P_\omega(x; x^{1/A}, s)$ first used, it is clear from Theorem 1 that for $A=\gamma$ we may take

$$g_0(\alpha) = \begin{cases} 0 & \text{for } 0 \leqq \alpha < \gamma, \\ 0.98c & \text{for } \alpha = \gamma, \end{cases}$$

$$G_0(\alpha) = 1.016c \quad \text{for } 0 \leqq \alpha \leqq \gamma,$$

where $c = c(s)$ is now given by

$$c = \gamma^{2s} D(s)/\Delta$$

instead of (3.6).

6. **Some numerical results.** An interesting question is that of determining how small $r$ must be in order that there should exist integers $n \leqq x$, $n \equiv a \pmod{\Delta}$ for which

$$(6.1)\qquad (n - a_i)(n - b_i) \not\equiv 0 \pmod{p_i}$$

for *every* $p_i$ belonging to the given infinite set satisfying (1.2). Clearly, (6.1) holds if $p_i > x$ so that we have to determine values of $r$ for which $P_\omega(x; x, r) > 0$.

We shall show that this is true for all sufficiently large $x$ when $r = 1/5$. For this value of $r$ we find that

$$k = 3.49, \qquad B = 3.62, \qquad B' = 1.84, \qquad \gamma = 3.78.$$

Then by Theorems 1 and 2 with

$$f_0(3.78) = 0.98c, \qquad F_0(3.78) = 1.016c$$

we have

$$f_1(1) = f_0(3.78) - (2/5)F_0(3.78)\int_0^{2.78} z^{-2/5}(z+1)^{-3/5}dz.$$

It is easy to show that

$$\int_0^{2.78} z^{-2/5}(z+1)^{-3/5}dz < 2.36$$

and hence

$$f_1(1) > c(1/5)\{0.98 - (0.4)(1.016)(2.36)\} > 0.$$

It follows that $P_\omega(x; x, 1/5) > 0$ for all sufficiently large $x$.

A deeper problem is that of determining the greatest value of $r$ for which $P_\omega(x; x, r) > 0$ independently of $\omega$. All that we have been able to prove so far, however, is what we have just shown, namely that $r \geqq 1/5$.

As a simple numerical example of the use of Theorem 3 we take $r = 1/2$, $s = 1$. Here we can improve Theorem 1 by using the results in Buchstab's second paper. From them it follows that we may take

$$g_0(\alpha) = \begin{cases} 0 & \text{for } 0 \leqq \alpha < 7, \\ 43.34b & \text{for } \alpha = 7, \end{cases}$$

$$G_0(\alpha) = 56.23b \qquad \text{for } 0 \leqq \alpha \leqq 7,$$

where $b = 0.\cdot 4161 \cdots$. Then using Theorem 3 with $r = 1/2$, $s = 1$ we find successively

$$G_1(6) = 50.75b, \qquad g_1(6) = 32.39b, \qquad G_2(5) = 47.94b,$$

and finally

$$g_2(4) = g_1(6) - G_2(5)\int_3^5 (z+1)z^{-2}dz > 0.$$

We have thus shown that $P_\omega(x; x^{1/7}, 1; x^{1/4}, 1/2) > 0$ for all sufficiently large $x$ independently of $\omega$.

Suppose now that we take the set of all odd primes as our set with $s = 1$ and the set of all primes congruent to 3 (mod 4) as the subset with $r = 1/\phi(4) = 1/2$. Since the function $P_\omega$ does not decrease if we remove the restriction $a_i \neq b_i$ we may choose

$$a_\iota = 0, \qquad b_\iota \equiv x \pmod{p_\iota}, \qquad 0 \leq b_\iota < p_\iota,$$

$$x \equiv 2 \pmod 4, \qquad a = 1, \qquad \Delta = 4.$$

Then $P_\omega(x; x^{1/7}, 1; x^{1/4}, 1/2)$ counts the number of integers $n \leq x$ such that

$$n \equiv 1 \pmod 4, \qquad n(n-x) \not\equiv 0 \pmod{p_\iota}$$

where $p_1, \cdots, p_k$ are the primes $3, 5, \cdots$ up to $x^{1/7}$, and $p_{k+1}, \cdots, p_l$ are the primes from $x^{1/7}$ to $x^{1/4}$ which are congruent to 3 (mod 4). It follows that neither $n$ nor $x-n$ is divisible by any prime up to $x^{1/7}$ nor by any prime congruent to 3 (mod 4) up to $x^{1/4}$.

Now let $M$ and $N$ be the number of primes congruent to 1 and 3 (mod 4), respectively, which divide $n$, prime factors of multiplicity $m$ counting $m$ times. Each prime of the first set exceeds $x^{1/7}$ and each prime of the second exceeds $x^{1/4}$. Since $n \leq x$ the product of primes dividing $n$ cannot exceed $x$. Hence we have

$$M/7 + N/4 < 1. \tag{6.2}$$

Furthermore, $n \equiv 1$ (mod 4) and so $N$ must be even. The only possible integral solutions of (6.2) are then

$$N = 0, \qquad M \leq 6;$$

$$N = 2, \qquad M \leq 3.$$

Since $x \equiv 2$ (mod 4) the same reasoning applies to $x-n$. We have thus shown that every sufficiently large integer $x \equiv 2$ (mod 4) is the sum of two integers which either have at most six prime factors all congruent to 1 (mod 4) or else have exactly two prime factors congruent to 3 (mod 4) and at most three congruent to 1 (mod 4).

It seems likely that a slight improvement in the method of proof would lead to the elimination of the second possibility, but at present there are still difficulties to be overcome.

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# NOTES ON THE BERTINI INVOLUTION

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1. **Introduction.** Given a pencil of plane cubic curves

$$\lambda w(x) + \mu w'(x) = 0 \tag{1}$$

with the vertices of the reference triangle among its base points. Arranged as to (0, 0, 1) the equations may be written

$$w(x) = x_3^2 u_1 + x_3 u_2 + u_3,$$
$$w'(x) = x_3^2 u_1' + x_3 u_2' + u_3',$$

with

$$u_1 = a_1x_1 + a_2x_2, \qquad u_1' = a_1'x_1 + a_2'x_2,$$
$$u_2 = b_1x_1^2 + b_2x_1x_2 + b_3x_2^2, \quad u_2',$$
$$u_3 = c_1x_1^2x_2 + c_2x_1x_2^2, \quad u_3',$$

and $a_i$, $a_i'$, $b_i$, $b_i'$, $c_i$, $c_i'$ generic constants.

A point $y$ of the plane fixes the curve of the pencil (1) passing through it, hence

$$w(x)w'(y) - w'(x)w(y) = 0, \tag{2}$$

which may be written in the form

$$\begin{aligned} W_3(x) = x_3(A_1x_1 + A_2x_2) + x_3(B_1x_1^2 + B_2x_1x_2 + B_3x_2^2) \\ + C_1x_1^2x_2 + C_2x_1x_2^2 = 0 \end{aligned} \tag{3}$$

in which $A_i = a_iw'(y) - a_i'w(y)$, and similarly for $B_i$ and $C_i$. The tangent to $W_3(x) = 0$ at (0, 0, 1) is

$$A_1x_1 + A_2x_2 = 0, \tag{4}$$

which meets the curve again at $R = (r_1, r_2, r_3)$,

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[1] Miss Moody, Ph.D. Cornell University, an instructor in mathematics at Pennsylvania State College, was killed in an automobile accident April 11, 1941. I had suggested that she compare my cumbersome method of derivation of the equations of this transformation (Amer. J. Math. vol. 33 (1911) pp. 327–336) with that of employing a pencil of cubic curves. The following notes were found among her posthumous papers sent me recently. The equations of the Bertini involution are simpler than those previously known, and other properties found may be extended by others.

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$$\rho r_1 = A_2 r_1', \qquad \rho r_2 = -A_1 r_1', \qquad \rho r_3 = A_1 A_2 r_3', \tag{5}$$

wherein

$$r_1' = B_1 A_2^2 - B_2 A_1 A_2 + B_3 A_1^2, \qquad r_3' = A_2 C_1 - A_1 C_2.$$

As $\lambda:\mu$ describes the pencil (3), the locus of $R$ is

$$\gamma_4(x) = x_1 A_1 + x_2 A_2 = 0, \tag{6}$$

a quartic curve having a triple point at (0, 0, 1) and passing simply through the remaining base points. It is completely fixed by these conditions.

2. **The Bertini involution.** The locus $R$ meets every curve of the pencil (3) in one point apart from base points. The line $Ry$ meets $W_3(x) = 0$ in a third point $y'$. It is proposed to study the involutorial transformation $y \sim y'$.

The point $y'$ is associated with the points $y$ and $R$ by equations of the form $y' = ly + mr$.

Substitute $y'$ for $x$ in $W_3(x) = 0$. Since $W_3(y) = 0$ and $W_3(r) = 0$, the equation in $l:m$ is linear. After various manipulations, it is found that $\gamma_4(y)$ appears as a cubic factor, and the equations of the transformation may be written in the form

$$\begin{aligned} y_1' &= \phi_6(y)\,[A_2^2\phi_6(y) + B_3 r_1']_{y_1}, \\ y_2' &= \psi_6(y)\,[A_1^2\psi_6(y) + B_1 r_1']_{y_2}, \\ y_3' &= \psi_6(y)\phi_6(y)C_5. \end{aligned}$$

A subscript applied to a bracket indicates that the quantity enclosed has the subscript as a factor and that the factor has been removed.

$$\begin{aligned} C_5(y) &= A_2[B_1 + (a_1b_1' - a_1'b_1)y_1y_2^2]_{y_2} \\ &\quad + [A_1 - (a_1'b_1 - a_1b_1')\cdot y_2^2y_3]_{y_2}[A_2y_3 + B_3y_2]_{y_1} \\ &\quad + (a_1b_1' - a_1b_1')B_3y_1y_3, \\ \phi_6(y) &= A_1C_2 + y_3C_5(y), \\ \psi_6(y) &= A_2C_1 + y_3C_5(y). \end{aligned}$$

These expressions have no common factor and are all in terms of the coefficients in $W_3(y)$. From the forms of the equations it follows at once that the transformation is of order 17. The point (0, 0, 1) is not a fundamental point. It is a fixed point of the transformation. The other base points of the pencil enter symmetrically. $\psi_6(y) = 0$ is the principal curve for (1, 0, 0) and $\phi_6(y) = 0$ for (0, 1, 0).

From the forms of the equations of these principal curves it is seen that each has a triple point at the fundamental point to which it belongs, and a double point at each of the other seven. There are no other fundamental points, hence

$$C_1 \sim C_{17}: \ 8P_i^6 \qquad P_i \sim p_6: \ P_i^3 7P_k^2, \qquad k \neq i.$$

It is the general Bertini involution.

3. **Fixed points of $B_{17}$.** From the method of defining $y'$ it follows that every cubic curve of the pencil is transformed itself from the central involution from its associated point $R$.

From $R$ can be drawn four tangents to $W_3(y)=0$ apart from the one at $R$. But one of these touches the curve at $(0, 0, 1)$, as $R$ describes $y(x)=0$. The locus of the other points of tangency is obtained by eliminating $\lambda:\mu$ from (1) and the polar conic of $R$ as to $W_3(y)=0$. By using the same procedure to affect reductions, the equation of the locus is found to be

$$K(x) = \phi_6(y)[A_1y_2 + B_1y_1]_{y_2} - \psi_6(y)[A_2y_2 + B_2y_1]_{y_1} = 0. \tag{7}$$

It is of order 9, has each base point of $|W_3(y)|$ except $(0, 0, 1)$ to multiplicity 3, and no other multiple points. It is of genus 4. Although $(0, 0, 1)$ is a fixed point, its coordinates do not satisfy $K(x)=0$.[2]

It is at once confirmed that at the points $(1, 0, 0)$ and $(0, 1, 0)$ the curve $K(x)=0$ has the same tangents as $\psi_6(y)=0$, $\phi_6(y)=0$, respectively. Since these points have no particular role in the determining equations, we infer that $K(y)=0$ has the same tangents at each base point as the principal curve of that point.

4. **Particular pencils.** Suppose all the curves of the pencil (1) have a common tangent at $(0, 0, 1)$. This requires that a second base point approaches $(0, 0, 1)$ in the direction of the fixed tangent. The curve $\gamma_4(x)=0$ now consists of the fixed tangent and the cubic curve of the pencil having a double point at $(0, 0, 1)$. The transformation reduces to the general Geïser involution having the other seven base points of the pencil as fundamental points.

$$C_1 \sim C_8: \ 7P_i^3, \qquad P_i \sim p_3: \ P_i^2 6P_k, \qquad k \neq i,$$

$$K_6: \ 7P_i^2.$$

Conversely, given any seven points in the plane: if the Geiser involution be determined and its points of invariant points be constructed, a tangent to $K=0$ has the role of the remaining (adjacent) base points of the pencil of cubics belonging to the point on K.

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[2] In Hudson, *Cremona transformations*, p. 127, it is erroneously stated that $(0, 0, 1)$ is triple on $K(x)=0$.

Further particularizations of the Bertini involution are thus reduced to special cases of the Geiser type.[3]

5. **Product of two Bertini involutions.** Returning to the general case, consider the sequences of two Bertini involutions, first $T_3$ with the base point (0, 0, 1) of the pencil, followed by $T_2$ with (0, 1, 0) as the special vertex.

By $T_3$

$$C_1 \sim C_{17}\colon\ 8P_i^{6}, \qquad i \neq 3.$$

By $T_2$

$$C_{17}\colon\ 8P_i^{6} \sim C_{37}\colon\ P_3^{18} P_2^{6} 7P_i^{2}, \qquad i \neq 2, 3.$$

This transformation is not periodic. Its inverse is the product of the same components taken in inverse order.

6. **Components of the curve of fixed points of $T_3T_2$ and $T_2T_3$.** Consider the $(x)=0$ belonging to $T_3$ and that belonging to $T_2$. These two curves meet in three points $Q_i$ apart from base points. Since from $Q_i$ the operations $T_3$ and $T_2$ are identical, it follows that the product of the two involutions leaves every point of the cubic of the pencil determined by $Q_i$ fixed, hence:

*The product of $T_2$ and $T_3$ leaves three cubics of the pencil point by point invariant.*

7. **Product of three $T_i$.** By direct application of the formula we obtain $T_3T_2T_1$

$$\begin{aligned} C_1 &\sim C_{53}\colon\ P_1^{12} P_2^{24} P_3^{12} 6P_i^{18}, && i \neq 1, 2, 3,\\ P_1 &\sim p_{12}\colon\ P_1^{3} P_2^{6} P_3^{2} 6P_i^{4}, && \\ P_2 &\sim p_{24}\colon\ P_1^{6} P_2^{11} P_3^{6} 6P_i^{8}, && \\ P_3 &\sim p_{12}\colon\ P_1^{2} P_2^{6} P_3^{3} 6P_i^{4}, && \\ P_i &\sim p_{18}\colon\ P_1^{4} P_2^{8} P_3^{4} P_i^{7} 5P_j^{6} && j \neq 1, 2, 3, i. \end{aligned}$$

If $T_3T_2T_1$ be calculated, its characteristic and that of its inverse are found to be identical hence:

*The product of three different Bertini involutions associated with the same pencil of cubic curves is involutorial.*

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[3] V. Snyder, *Conjugate line congruences contained in a bundle of quadric surfaces*, Trans. Amer. Math. Soc. vol. 11 (1911) pp. 371–387.

# A NOTE ON APPROXIMATION BY RATIONAL FUNCTIONS

H. KOBER

The theory of the approximation by rational functions on point sets $E$ of the $z$-plane ($z=x+iy$) has been summarized by J. L. Walsh[1] who himself has proved a great number of important theorems some of which are fundamental. The results concern both the case when $E$ is bounded and when $E$ extends to infinity.

In the present note a $L_p$-theory ($0<p<\infty$) will be given for the following point sets extending to infinity:

A. The real axis $-\infty<x<\infty$, $y=0$.

B. The half-plane $-\infty<x<\infty$, $0<y<\infty$.

The only poles of the approximating functions are to lie at preassigned points whose number will be required to be as small as possible.[2] We shall make use of the theory of the class $\mathfrak{H}_p$ the fundamental results of which are due to E. Hille and J. D. Tamarkin;[3] $\mathfrak{H}_p$ is the set of functions $F(z)$ which, for $0<y<\infty$, are regular and satisfy the inequality

$$\int_{-\infty}^{\infty} |F(x+iy)|^p dx \leqq M^p \quad \text{or} \quad |F(z)| \leqq M$$

for $0<p<\infty$ or $p=\infty$, respectively, where $M$ depends on $F$ and $p$ only. By $|f(x+iy)|_p$ we denote

$$\left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx\right)^{1/p} \quad \text{or} \quad \operatorname*{ess.\,u.b.}_{-\infty<x<\infty} |f(x+iy)|$$

for $0<p<\infty$ or $p=\infty$, respectively, and by $\alpha$ and $\beta$ two arbitrarily fixed points in the upper or lower half-plane, respectively. We obtain the following results:[4]

THEOREM 1. *Let $0<p<\infty$ and $F(t)\in L_p(-\infty,\infty)$, let $c$ be an integer greater than $p^{-1}$ and $r_k(z)=(\alpha-z)^k(z-\beta)^{-c-k}$ $[k=0, \pm 1, \pm 2, \cdots]$.*

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[1] *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, 1935.

[2] Compare Walsh, loc. cit., for example, approximation by polynomials.

[3] Fund. Math. vol. 25 (1935) pp. 329–352, $1\leqq p<\infty$. For $0<p<1$ see T. Kawata, Jap. J. Math. vol. 13 (1936) pp. 421–430.

[4] The case $p=\infty$ of each of the results is a special case of Theorem 16, J. L. Walsh, chap. 2.

*Then there are finite linear combinations $s_n(z)$ of the $r_k(z)$ such that*

$$\left| F(t) - s_n(t) \right|_p = \left( \int_{-\infty}^{\infty} \left| F(t) - s_n(t) \right|^p dt \right)^{1/p} \to 0 \qquad \text{as } n \to \infty.$$

THEOREM 2. (a) *Let $0<p<\infty$ and $F(t) \in L_p(-\infty, \infty)$. A necessary and sufficient condition for the existence of rational functions $s_n(z)$ such that their only poles lie in a single point of the lower half-plane and that $|F(t)-s_n(t)|_p \to 0$ as $n\to\infty$ is that $F(t)$ is equivalent to the limit-function of an element $F(z)$ of $\mathfrak{H}_p$.*

(b) *When the latter condition is satisfied then there are rational functions $s_n(z)$, with their only poles at $z=\beta$, such that, uniformly in the half-plane $0<y<\infty$,*

$$\left| F(x+iy) - s_n(x+iy) \right|_p \to 0 \qquad \text{as } n \to \infty.$$

By a well known result[5] concerning $\mathfrak{H}_p$, 2(b) is a consequence of 2(a).

We start with giving explicit approximating functions in some special cases of problem (A), taking $\beta = \bar{\alpha}$.

THEOREM 1'. *Let $F(t) \in L_1(-\infty, \infty)$ or $F(t) \in L_2(-\infty, \infty)$, or let $F(t)$ be continuous everywhere, including infinity.[6] Let $c=2, 1, 0$ for $p=1, 2, \infty$, respectively, and let*

$$s_n(z) = \sum_{k=-n}^{n} a_k \frac{(\alpha - z)^k}{(z-\bar{\alpha})^{k+c}}, \qquad a_k = \frac{i(\alpha-\bar{\alpha})}{2\pi} \int_{-\infty}^{\infty} F(t) \frac{(t-\bar{\alpha})^{k+c-1}}{(\alpha - t)^{k+1}} dt.$$

*Then*

$$\left| F(t) - \frac{1}{N+1} \sum_{n=0}^{N} s_n(t) \right|_1 \quad \text{or} \quad \left| F(t) - s_N(t) \right|_2 \quad \text{or}$$

$$\left| F(t) - \frac{1}{N+1} \sum_{n=0}^{N} s_n(t) \right|_\infty = \underset{-\infty<t<\infty}{\text{u.b.}} \left| F(t) - \frac{1}{N+1} \sum_{n=0}^{N} s_n(t) \right|,$$

*respectively, tends to zero as $N\to\infty$. When $F(t)$ is continuous everywhere, including infinity, and of bounded variation in $(-\infty, \infty)$ then the $s_n(t)$ converge to $F(t)$ uniformly in $(-\infty, \infty)$.*

It will suffice to take $\alpha = i$, the general case being deduced from this one by the substitution $t = \Re(\alpha) + t' \Im(\alpha)$. Let $F(t) \in L_2(-\infty, \infty)$, $t = \tan(1/2)\vartheta$ $[-\pi \leqq \vartheta \leqq \pi]$, and $f(\vartheta) = 2(1+e^{i\vartheta})^{-1} F(\tan \vartheta/2)$. Then $F(t) \in L_2(-\infty, \infty)$ implies that $f(\vartheta) \in L_2(-\pi, \pi)$, and vice versa. Now the Fourier series $\sum b_n e^{in\vartheta}$, belonging to $f(\vartheta)$, converges to $f(\vartheta)$ in the mean square over $(-\pi, \pi)$. We have $e^{i\vartheta}=(i-t)(i+t)^{-1}$, $(1/2)(1+e^{i\vartheta})=i(i+t)^{-1}$; taking $a_n=ib_n$, we arrive finally at the required result. In a similar way we prove the remaining assertions of the theorem. We note[7] that the sequence $\{(2i\pi)^{-1/2}(\alpha-\bar{\alpha})^{1/2}(\alpha-t)^n(t-\bar{\alpha})^{-n-1}\}$ $[n=0, \pm 1, \pm 2, \cdots]$ is a complete orthogonal and normal system with respect to $L_p(-\infty, \infty)$ $[1<p<\infty]$.

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[5] Since $s_n(t) \in L_p$, we have $s_n(z) \in \mathfrak{H}_p$, $F(z) - s_n(z) \in \mathfrak{H}_p$, and we can apply the Hille-Tamarkin Theorem 2.1 (iii), part 2, loc. cit.

[6] A function $F(t)$ is said to be continuous at infinity when its limits, as $t \to \pm\infty$, both exist and are finite and equal.

To prove Theorem 1, we have to show that, given $\epsilon>0$, there is a finite linear combination $s_n(z)$ of the $r_k(z)$ such that $|F(t)-s_n(t)|_p<\epsilon$. We can find a positive number $b$ and a function $f(t)$ such that $f(t)$ is zero for $|t|\geqq b$ and continuous for $-b\leqq t\leqq b$, and that

$$\int_{-\infty}^{\infty} | F(t) - f(t) |^p \, dt \leqq \delta, \qquad \delta = \begin{cases} (\epsilon/2)^p & \text{for } p > 1 \\ (1/2)\epsilon^p & \text{for } p \leqq 1. \end{cases}$$

The function $g(t)=(t-\beta)^q f(t)$ is continuous everywhere, including infinity. From results of Walsh[8] we deduce the existence of functions

$$\sigma_n(z) = \sum_{k=-n}^{n} a_{k,n} \left(\frac{\alpha - z}{z - \beta}\right)^k, \qquad n = 0, 1, 2, \cdots,$$

$|g(t)-\sigma_n(t)|_\infty \to 0$ as $n\to\infty$. Taking $s_n(z)=(z-\beta)^{-q}\sigma_n(z)$, we have

$$| f(t) - s_n(t) |_p^p = \left| \frac{g(t) - \sigma_n(t)}{(t - \beta)^q} \right|_p^p \leqq | g(t) - \sigma_n(t) |_\infty^p \int_{-\infty}^{\infty} \frac{dt}{| t - \beta |^{qp}} .$$

The right side tends to zero as $n\to\infty$. Therefore, for some $n$, we have $|f(t)-s_n(t)|_p^p<\delta$, $|F(t)-s_n(t)|_p^p<\epsilon^p$ which completes the proof.

To prove Theorem 2(a), we need some lemmas.

LEMMA 1.[9] *Let $\varphi(w)$ belong to the Riesz class $H_p$ $[0<p<\infty]$, that is to say, let $\varphi(w)$ be regular for $|w|<1$ and satisfy the inequality*

$$\|\varphi(re^{i\vartheta})\|_p = \left( \int_{-\pi}^{\pi} | \varphi(re^{i\vartheta}) |^p d\vartheta \right)^{1/p} \leqq M, \qquad 0 < r < 1,$$

*where $M$ is independent of $r$.*[10] *Then there are polynomials $P_n(w)$ $[n=1, 2, \cdots]$ such that $\|\varphi(re^{i\vartheta})-P_n(re^{i\vartheta})\|_p\to 0$ as $n\to\infty$, uniformly for $0<r\leqq 1$.*

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[7] Cf. H. Kober, a forthcoming paper in Quart. J. Math. Oxford Ser. 1943.

[8] Walsh, loc. cit. chap. 2, Theorem 16. It can also be deduced from Theorem 1' of this paper.

[9] For $p=\infty$ the result holds if and only if $\phi(e^{i\vartheta})$ is continuous for $-\pi\leqq\vartheta\leqq\pi$. Cf. Walsh, loc. cit., and Trans. Amer. Math. Soc. vol. 26 (1924) pp. 155–170.

[10] F. Riesz, Math. Zeit. vol. 18 (1923) pp. 87–95.

By well known properties of the class $H_p$, it will suffice to take $r=1$. Let $\varphi(w)=\sum a_n w^n$. Since, for any fixed $R$ $[0<R<1]$ and uniformly with respect to $\vartheta$ $[-\pi\leqq\vartheta\leqq\pi]$, the series $\sum a_n R^n e^{in\vartheta}$ converges to $\varphi(Re^{i\vartheta})$, the result can be deduced by means of the well known equation $\|\varphi(e^{i\vartheta})-\varphi(re^{i\vartheta})\|_p\rightarrow 0$ $[r\rightarrow 1]$.

LEMMA 2. *Let* $w=(i-z)(i+z)^{-1}$. *The function* $F(z)$ *belongs to* $\mathfrak{H}_p$ *if, and only if, the function* $(1+w)^{-2/p}\varphi(w)$ *belongs to* $H_p$, *where* $\varphi(w)=F(z)$.

Hille and Tamarkin have proved[11] that the condition $\varphi(w)\in H_p$ is necessary. To define the function $(1+w)^{-2/p}$, we cut the $w$-plane along the negative real axis from $w=-1$ to $w=-\infty$. When $F(z)$ belongs to $\mathfrak{H}_p$ then its limit function $F(t)$ $[y\rightarrow 0,\ x=t]$ belongs to $L_p(-\infty,\ \infty)$, therefore $(1+e^{i\vartheta})^{-2/p}\varphi(e^{i\vartheta})$ to $L_p(-\pi,\ \pi)$. Let $\psi(w)=(1+w)^{-2/p}\varphi(w)$, and $0<q<p/3$. By Hölder's theorem, we have

$$\int_{-\pi}^{\pi}|\psi(re^{i\vartheta})|^q d\vartheta\leq\left(\int_{-\pi}^{\pi}|\varphi(re^{i\vartheta})|^p\right)^{q/p}\left(\int_{-\pi}^{\pi}\frac{d\vartheta}{|1+re^{i\vartheta}|^{2q/(p-q)}}\right)^{1-q/p}.$$

The right side is uniformly bounded for $0<r<1$. Hence $\psi(w)\in H_q$; its limit-function $\psi(e^{i\vartheta})$, however, belongs to $L_p(-\pi,\ \pi)$; hence[12] $\psi(w)\in H_p$. Conversely, let $\psi(w)\in H_p$. From a result due to R. M. Gabriel[13] we deduce that

$$\int_C|\psi(w)|^p|dw|\leqq 2\int_{-\pi}^{\pi}|\psi(e^{i\vartheta})|^p d\vartheta,$$

where $C$ is any circle strictly interior to the unit circle $\Gamma$ $[|w|=1]$. By Fatou's theorem, this inequality holds when $C$ is a circle touching $\Gamma$ from within at $w=-1$. Finally, by the transformation $w=(i-z)(i+z)^{-1}$, we deduce that $|F(x+iy)|_p\leqq 2^{2/p}\|\psi(e^{i\vartheta})\|_p$ $[0<y<\infty]$ which proves the lemma. In a similar way we can show that *when* $F(z)\in\mathfrak{H}_p$ *and* $F(t)\in L_q(-\infty,\ \infty)$ $[0<p_q\leqq\infty]$ *then* $F(z)\in\mathfrak{H}_q$.

LEMMA 3. *Let* $0<p\leqq\infty$, *let* $f_n(z)\in\mathfrak{H}_p$ $[n=1,2,\cdots]$, *and let* $f_n(t)$ *be the limit-function of* $f_n(z)$. *Let* $F(t)$ *be defined in* $(-\infty,\ \infty)$ *and* $|F(t)-f_n(t)|_p\rightarrow 0$ *as* $n\rightarrow\infty$. *Then* $F(t)$ *is equivalent to the limit-function of an element* $f(z)$ *of* $\mathfrak{H}_p$.

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[11] Loc. cit. Lemma 2.5.

[12] V. Smirnoff, C. R. Acad. Sci. Paris vol. 188 (1929) pp. 131–133. A. Zygmund, *Trigonometrical series*, Warsaw, 1935, 7 56(iv).

[13] J. London Math. Soc. vol. 5 (1930) pp. 129–131. Cf. Hille-Tamarkin, Lemmas 2.1 and 2.5.

The proof for $0<p<\infty$ is entirely different from that for $1\leqq p\leqq\infty$, given in a former paper.[14] Let $0<p<1$ and $\rho>0$, and let $\phi(z)\in\mathfrak{H}_p$. Then, for $\rho\leqq y<\infty$, we have $|\phi(z)|\leqq((1/2)\pi\rho)^{-1/p}\,|\phi(t)|_p$.[15] Since $|f_n(t)-f_m(t)|_p\rightarrow 0$ $[m>n\rightarrow\infty]$, taking $\phi(z)=f_m(z)-f_n(z)$ we can deduce that the sequence $\{f_n(z)\}$ converges to an analytic function $f(z)$, uniformly for $-\infty<x<\infty$, $\rho<y<\infty$. Since there is a constant $K$, independent of $n$, such that $|f_n(t)|_p\leqq K$, we have $|f_n(x+iy)|_p\leqq K$, and we can deduce that $|f(x+iy)|_p\leqq K$ for any positive $y$. Hence $f(z)\in\mathfrak{H}_p$. We are left to show that $f(t)$, the limit-function of $f(z)$, is equivalent to $F(t)$ in $(-\infty,\ \infty)$. Given $\epsilon>0$, we have $|(f_m(x)-f_N(x)|_p^p<\epsilon/12$ for $m\geqq N$, fixing $N$ in a suitable way, and $|f_N(x+iy)-f_N(x)|_p^p\leqq\epsilon/6$ for $0<y\leqq\delta=\delta(\epsilon,\ N)$. Hence

$$
\begin{aligned}
|f_m(x+iy)-f_m(x)|_p^p &\leqq |f_m(x+iy)-f_N(x+iy)|_p^p \\
&\quad + |f_m(x)-f_N(x)|_p^p + |f_N(x+iy)-f_N(x)|_p^p \leqq \epsilon/3
\end{aligned}
$$

for $m\geqq N$, $0<y\leqq\delta$, since the first term on the right side is not greater than the second term. Given $M>0$, we have

$$
\begin{aligned}
\int_{-M}^{M}|f(x)-f_m(x)|^p dx &\leqq \int_{-M}^{M}|f(x+iy)-f_m(x+iy)|^p dx \\
&\quad + |f(x+iy)-f(x)|_p^p + |f_m(x+iy)-f_m(x)|_p^p.
\end{aligned}
$$

The right side is smaller than $\epsilon$ for $m\geqq m_0(\epsilon)$, as we see fixing a suitable value for $y$. Consequently $f(x)=F(x)$ almost everywhere in any finite interval $(-M,\ M)$ and, therefore, in $(-\infty,\ \infty)$. With a slight alteration, the proof holds for $1\leqq p<\infty$.

By the lemma, the necessity of the condition in Theorem 2(a) is evident. For $s_n(t)$ belongs to $L_p(-\infty,\ \infty)$, therefore $s_n(z)$ to $\mathfrak{H}_p$. To prove its sufficiency, we take first $1<p<\infty$. By Theorem 1, there are rational functions $R_n(z)$ such that their only poles lie at $z=\bar\beta$ and $z=\beta$ and that $|F(t)-R_n(t)|_p\rightarrow 0$ as $n\rightarrow\infty$. Taking $R_n(z)=s_n(z)+\sigma_n(z)$, where the rational functions $s_n$ and $\sigma_n$ vanish at infinity and have no poles other than at $z=\bar\beta$ or $z=\beta$, respectively, we have $s_n(z)\in\mathfrak{H}_p$, $\overline{\sigma_n(\bar z)}\in\mathfrak{H}_p$. Denoting by $\mathfrak{H}f$ the Hilbert operator

$$
\mathfrak{H}f=\frac{1}{\pi}\,PV\int_{-\infty}^{\infty}\frac{f(t)dt}{t-x},
$$

we have $|\mathfrak{H}f|_p\leqq C_p|f|_p$, $\mathfrak{H}F=iF(x)$ and $\mathfrak{H}s_n=is_n(x)$, $\mathfrak{H}\sigma_n=-i\sigma_n(x)$.[14]

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[14] H. Kober, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 421–427.

[15] This can be shown by means of the inequality (73), M. Plancherel and G. Polya, Comment. Math. Helv. vol. 10 (1937–1938) pp. 110–163.

Hence

$$2\,|F(t)-s_n(t)|_p=|iF+\mathfrak{H}F-(iR_n+\mathfrak{H}R_n)|_p$$
$$\leqq|F-R_n|_p+|\mathfrak{H}(F-R_n)|_p\leqq(C_p+1)\,|F-R_n|_p$$

which tends to zero as $n\to\infty$. Hence $|F(t)-s_n(t)|_p\to 0$ as $n\to\infty$.

Let now $0<p\leqq 1$ and $F(z)\in\mathfrak{H}_p$, let $\beta=-i$, $z=i(1-w)(1+w)^{-1}$ and $\varphi(w)=F(z)$. Given $\epsilon>0$, from the Lemmas 2 and 1 we infer the existence of a polynomial $P(z)$ such that

$$\int_{-\pi}^{\pi}|\varphi(e^{i\vartheta})(1+e^{i\vartheta})^{-2/p}-P(e^{i\vartheta})|^p d\vartheta\leqq\epsilon/4.$$

Hence

$$\int_{-\infty}^{\infty}\left|F(t)-(1+e^{i\vartheta})^{2/p}P\left(\frac{i-t}{i+t}\right)\right|^p dt\leqq\epsilon/2,$$

where $t=\tan\vartheta/2$. Let $b$ be an integer, $p^{-1}<b\leqq 1+p^{-1}$. Then the rational function $\chi(z)=(2i)^b(i+z)^{-b}P\{(i-z)(i+z)^{-1}\}$ has no singularity except at $z=-i$. Since $\chi(t)\in L_p(-\infty,\ \infty)$, we have $|\chi(t)|_p=C<\infty$. Now the function $(1+e^{i\vartheta})^{2/p-b}$ can be approximated by polynomials $Q_m(e^{i\vartheta})$ $[m=1,\ 2,\cdots]$, uniformly for $-\pi\leqq\vartheta\leqq\pi$. Hence, for some $m$, we have

$$\int_{-\infty}^{\infty}\left|(1+e^{i\vartheta})^{2/p}P\left(\frac{i-t}{i+t}\right)-\chi(t)Q_m\left(\frac{i-t}{i+t}\right)\right|^p dt<\epsilon/2.$$

Thus $|F(t)-\chi(t)Q_m\{(i-t)(i+t)^{-1}\}|_p<\epsilon^{1/p}$. This completes the proof which, slightly altered, holds for $1<p\leqq 2$.

For $p=1,\ 2,\ \infty$, we obtain explicit approximating functions by Theorem 1′ and by the lemma:

*Let* $1\leqq p\leqq\infty$ *and* $F(z)\in\mathfrak{H}_p$, *let* $a$ *be an integer and* $a\geqq 0$ *for* $p=1$, $a\geqq 2$ *for* $p=\infty$, $a\geqq 1$ *otherwise; then*

$$\int_{-\infty}^{\infty}F(t)\frac{(\alpha-t)^n}{(t-\beta)^{n+a}}dt=0\quad\textit{for } n=0,1,2,\cdots.$$

THEOREM 2′. *Let* $p=2,\ 1,$ *or* $\infty$ *and* $c=1,\ 2,$ *or* $0$, *respectively; let* $F(z)\in\mathfrak{H}_p$ *and* $F(t)$, *the limit-function of* $F(z)$, *be continuous everywhere including infinity when* $p=\infty$. *Let* $s_n(z)$ *be defined by*

$$\sum_{k=0}^{n}a_k\frac{(\bar\beta-z)^k}{(z-\beta)^{k+c}}\ [p=2]\quad\textit{or}\quad\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=0}^{j}a_k\frac{(\bar\beta-z)^k}{(z-\beta)^{k+c}}\begin{bmatrix}p=1,\\ p=\infty\end{bmatrix},$$

*where*

$$a_k = i\,\frac{\bar{\beta} - \beta}{2\pi} \int_{-\infty}^{\infty} F(t)\,\frac{(t - \beta)^{k+\nu-1}}{(\bar{\beta} - t)^{k+1}}\,dt.$$

*Then, uniformly for* $0 \leqq y < \infty$, $\left| F(x+iy) - s_n(x+iy) \right|_\nu \to 0$ *as* $n \to \infty$.

Applying Theorem 2′ to the components of $g(z) = (1/2)s(1-s)\Gamma((1/2)s)\pi^{-s/2}\zeta(s)$,[16] where $\zeta(s)$ is the Riemann zeta-function and $z = i(1-2s)$, we can deduce the following corollary:

*Let* $0 \leqq a < \infty$, $q = i(1-a)$, $r = i(1+a)$, *let*

$$\vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}; \quad b_0 = \vartheta(1)/2 + (1 - a/2)\int_1^{\infty} v^{a/4}\vartheta'(v)\,dv;$$

$$L_n^{(j)}(x) = \sum_{k=0}^{n} C_{n+j,k+j}\,\frac{(-x)^k}{k!};$$

$$b_n = (-1)^n \int_1^{\infty} v^{a/4}\vartheta'(v)\left\{L_n^{(0)}(\log v)/2 - (a/2)L_n^{(-1)}(\log v)/2\right\}dv, \qquad n > 0.$$

*Then the series*

$$\sum_{n=0}^{\infty} b_n\left\{\left(\frac{q - z}{r + z}\right)^n + \left(\frac{q + z}{r - z}\right)^n\right\}$$

*converges to* $g(z)$ *uniformly for* $-\infty < x < \infty$, $-a \leqq y \leqq a$, *while it does not converge whenever* $|y| > a$.

The series takes a simple form for $a = 0$ (critical line).

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[16] In fact to the function $g_1(z - ia) \in \mathfrak{H}_{\infty}$, where $g(z) = g_1(z) + g_1(-z)$, $g_1(z) = ((1+z^2)/16)\int_1^{\infty}\{\vartheta(t) - 1\}t^{(iz-3)/4}dt - (1/4) - (iz/4)\{\vartheta(1) - 1\}$.

# A REMARK ON ALGEBRAS OF MATRICES

WINSTON M. SCOTT

**1. Introduction.** Let $\mathfrak{A}$ denote a matrix algebra, with unit element, over an algebraically closed field $K$. We shall assume that $\mathfrak{A}$ is in reduced form, that is, that $\mathfrak{A}$ is exhibited with only zeros above the main diagonal, with irreducible constituents of $\mathfrak{A}$ in the main diagonal, and that $\mathfrak{A}$ is expressible as the direct sum of its radical and a semisimple subalgebra which latter has nonzero components only in the irreducible constituents of $\mathfrak{A}$:

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{C}_{11} & \cdot & \cdots & \cdot \\ \mathfrak{C}_{21} & \mathfrak{C}_{22} & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mathfrak{C}_{t1} & \mathfrak{C}_{t2} & \cdots & \mathfrak{C}_{tt} \end{pmatrix}, \tag{1}$$

the $\mathfrak{C}_{ii}$ denoting irreducible constituents; further $\mathfrak{A}=\mathfrak{A}^*+\mathfrak{N}$ where $\mathfrak{N}$ is the radical of $\mathfrak{A}$ and

$$\mathfrak{N} = \begin{pmatrix} 0 & \cdot & \cdots & \cdot \\ \mathfrak{C}_{21} & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mathfrak{C}_{t1} & \mathfrak{C}_{t2} & \cdots & 0 \end{pmatrix}, \qquad \mathfrak{A}^* = \begin{pmatrix} \mathfrak{C}_{11} & \cdot & \cdots & \cdot \\ 0 & \mathfrak{C}_{22} & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \mathfrak{C}_{tt} \end{pmatrix}. \tag{2}$$

As a part of $\mathfrak{A}$, $\mathfrak{C}_{ij}$ forms an additive group or module of matrices upon which $\mathfrak{A}$, itself considered as a module, is homomorphically mapped. We shall consider $\mathfrak{C}_{ij}$ as a matrix module with $\mathfrak{A}$ as both left and right operator system. For a matrix $A$ of $\mathfrak{A}$, we shall use the notation $C_{ij}(A)$, $(j \leqq i,\ i=1, 2, \cdots, t)$, to denote the parts of $A$,

$$A = \begin{pmatrix} C_{11}(A) & \cdot & \cdots & 0 \\ C_{21}(A) & C_{22}(A) & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ C_{t1}(A) & \cdot & \cdots & C_{tt}(A) \end{pmatrix}. \tag{3}$$

Let $B$ be any element of $\mathfrak{A}$, and let $B^*$ be the component of $B$ in the semisimple subalgebra $\mathfrak{A}^*$. We define $B$ as a left and as a right operator of $C_{ij}(A)$ by the relations below, using $\circ$ to distinguish this operation from ordinary matrix multiplication

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$$
\begin{aligned}
B \circ C_{ij}(A) &= C_{ii}(B)\cdot C_{ij}(A) = C_{ij}(B^*A),\\
C_{ij}(A) \circ B &= C_{ij}(A)\cdot C_{jj}(B) = C_{ij}(AB^*).
\end{aligned} \tag{4}
$$

We shall indicate that a matrix module has $\mathfrak{A}$ as both right and left operator system by calling the module an $(\mathfrak{A}, \mathfrak{A})$ module. Under (4), $\mathfrak{C}_{ij}$ is a simple $(\mathfrak{A}, \mathfrak{A})$ module. Moreover, each $\mathfrak{C}_{ij}$ is either a 0-part or there exist elements in $\mathfrak{A}$ such that the corresponding $C_{ij}$ have any arbitrary components from $K$. We shall call the $\mathfrak{C}_{ij}$ *simple parts* of $\mathfrak{A}$.[1]

2. **The basic theorem.** Professor R. Brauer, in a recent paper,[2] has proved a theorem which has a great many applications and among other things includes the Jordan-Hölder Theorem as a special case.

BASIC THEOREM. *Let $G$ and $H$ be two groups, with finite composition lengths, for both of which a given set $\theta$ is the operator set. Let*

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_r = (1)$$

*be a composition series of $G$, and*

$$H = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_s = (1)$$

*be a composition series of $H$. If $\theta$ is a homomorphism which maps $H$ upon a normal subgroup $H^*$ of $G$, $H^* \subseteq G$, then one can choose complete residue systems $P_\rho$ of $G_{\rho-1}$ (mod $G_\rho$) and $Q_\sigma$ of $H_{\sigma-1}$ (mod $H_\sigma$), ($\rho = 1, 2, \cdots, r$; $\sigma = 1, 2, \cdots, s$), such that*

(a) *either $\theta$ maps $Q_\sigma$ on a $P_\rho$ in a $(1-1)$ manner and $G_{\rho-1}/G_\rho \cong H_{\sigma-1}/H_\sigma$, or $\theta$ maps $Q_\sigma$ on 1, and*

(b) *each $P_\rho$ is the image of at most one $Q_\sigma$.*

Here, in our application of the basic theorem, $H$ will be mapped on the whole group $G$ and, consequently, statement (b) of the theorem may be sharpened to "*Each $P_\rho$ is the image of exactly one $Q_\sigma$.*"

3. **An application of the basic theorem.** We shall first prove:

THEOREM 1. *A composition series*

$$\mathfrak{A} = \mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_m$$

*of a matrix algebra $\mathfrak{A}$ considered as an $(\mathfrak{A}, \mathfrak{A})$ module is also a composition series of $\mathfrak{A}$ considered as an $(\mathfrak{A}^*, \mathfrak{A}^*)$ module where $\mathfrak{A}$ is considered as the direct sum of $\mathfrak{A}^*$, a semisimple subalgebra, and $\mathfrak{N}$, the radical of $\mathfrak{A}$.*

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[1] For a further study of simple parts, see W. M. Scott, *On matrix algebras over an algebraically closed field*, Ann. of Math. vol. 43 (1942) pp 147–161.

[2] R. Brauer, *On sets of matrices with coefficients in a division ring*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 502–548.

PROOF. In $\mathfrak{A}_{i-1}/\mathfrak{A}_i$ we consider all residue classes $\langle A^{(i-1)}\rangle$. Select the classes different from $\langle 0\rangle$ which have representatives belonging to[3] a highest power of the radical $\mathfrak{N}$, say $\mathfrak{N}^p$. These together with the $\langle 0\rangle$ class form an admissible subgroup with elements of $\mathfrak{A}$ as operators on both left and right (that is, as $(\mathfrak{A}, \mathfrak{A})$ operators), for if we have $\langle A^{(i-1)}\rangle$ such that $A^{(i-1)}$ belongs to $\mathfrak{N}^p$, then

$$A\langle A^{(i-1)}\rangle B = \langle AA^{(i-1)}B\rangle$$

has the representative $AA^{(i-1)}B$ which is contained in $\mathfrak{N}^p$. Then this admissible subgroup must coincide with the whole factor group $\mathfrak{A}_{i-1}/\mathfrak{A}_i$ since $\mathfrak{A}_{i-1}/\mathfrak{A}_i$ is simple. We can conclude, then, that every class in $\mathfrak{A}_{i-1}/\mathfrak{A}_i$ has a representative which belongs to $\mathfrak{N}^p$.

Now let $\langle A^{(i-1)}\rangle$ be a class with $A^{(i-1)}$ belonging to $\mathfrak{N}^p$. We have that for $N\in\mathfrak{N}$

$$N\langle A^{(i-1)}\rangle = \langle NA^{(i-1)}\rangle$$

has a representative in $\mathfrak{N}^{p+1}$. But we have selected all classes different from $\langle 0\rangle$ belonging to a highest power which was $\mathfrak{N}^p$. Then the class $\langle NA^{(i-1)}\rangle$ must be the 0-class. Therefore, we have that for $A\in\mathfrak{A}$, $A=A^*+N$, $A^*\in\mathfrak{A}^*$, and $N\in\mathfrak{N}$,

$$\begin{aligned} A\langle A^{(i-1)}\rangle &= \langle A\cdot A^{(i-1)}\rangle \\ &= \langle A^*\cdot A^{(i-1)} + N\cdot A^{(i-1)}\rangle \\ &= \langle A^*\cdot A^{(i-1)}\rangle \\ &= A^*\langle A^{(i-1)}\rangle, \end{aligned}$$

and our theorem is proved.

Now considering the simple part $\mathfrak{C}_{ij}$ as an $(\mathfrak{A}^*, \mathfrak{A}^*)$ module (as we may, since $\mathfrak{A}^*\subset\mathfrak{A}$), we have that $A\rightarrow C_{ij}(A)$ is an $(\mathfrak{A}^*, \mathfrak{A}^*)$ homomorphism so that by our basic theorem $\mathfrak{C}_{ij}$ is $(\mathfrak{A}^*, \mathfrak{A}^*)$ isomorphic to a composition factor group of $\mathfrak{A}$. But the operators of the system $\mathfrak{A}$ on $\mathfrak{C}_{ij}$ and on the factor group are equivalent by (4) and Theorem 1 to operators of the system $\mathfrak{A}^*$. From this we have that $\mathfrak{C}_{ij}$ is also $(\mathfrak{A}, \mathfrak{A})$ isomorphic to the composition factor group.

Thus we have, by the basic theorem, a proof of the following theorem.

THEOREM 2.[4] *A nonzero simple part $\mathfrak{C}_{ij}$ of $\mathfrak{A}$ is $(\mathfrak{A}, \mathfrak{A})$ isomorphic to a composition factor group of $\mathfrak{A}$ itself considered as an $(\mathfrak{A}, \mathfrak{A})$ module.*

WASHINGTON, D. C.

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[3] For a definition of *belonging to* as used here, see C. Nesbitt, *On the regular representations of algebras*, Ann. of Math. vol. 39 (1938) pp. 634–658.

[4] For a direct proof of this theorem, see W. M. Scott, op. cit.

# THE DISTRIBUTION OF INTEGERS REPRESENTED BY BINARY QUADRATIC FORMS

GORDON PALL

R. D. James[1] has proved the following theorem:

THEOREM 1. *Let $B(x)$ denote the number of positive integers $m \leqq x$ which can be represented by positive, primitive, binary quadratic forms of a given negative discriminant $d$, but are prime to $d$. Then*

$$B(x) = bx/(\log x)^{1/2} + O(x/\log x), \tag{1}$$

*where $b$ is the positive constant given by*

$$\pi b^2 = \prod_{q} (1 - 1/q^2)^{-1} \prod_{p \mid d} (1 - 1/p) \sum_{n=1}^{\infty} (d \mid n) n^{-1}. \tag{2}$$

*Here $q$ runs over all primes such that $(d|q) = -1$; $p$ denotes any prime greater than or equal to 2; and $(d|n)$ is the Kronecker symbol.*

We shall deduce from his result an asymptotic formula with the restriction that $m$ be prime to $d$ removed.

First, let $p$ be a prime dividing $d$ but not satisfying

$$p > 2 \text{ and } p^2 \mid d, \quad \text{or} \quad p = 2 \text{ and } d \equiv 0 \text{ or } 4 \pmod{16}. \tag{3}$$

Then $pn$ is represented by p. p. b. q. forms of discriminant $d$ if and only if $n$ is likewise represented.[2] Hence if $p^r \leqq (\log x)^{1/2}$ then the number of represented integers less than or equal to $x$ of the form $p^r m$ with $m$ prime to $d$, is

$$\frac{bx/p^r}{(\log x/p^r)^{1/2}} + O\left(\frac{x/p^r}{(\log x/p^r)}\right) = \frac{bx/p^r}{(\log x)^{1/2}} + O\left(\frac{x/p^r}{\log x}\right),$$

since $(\log xp^{-r})^{-1/2} - (\log x)^{-1/2} = O((\log p^r)/(\log x)^{3/2})$. Also, if $p^s > (\log x)^{1/2}$,

$$(bx/(\log x)^{1/2})(1/p^s + 1/p^{s+1} + \cdots) = O(x/(\log x)).$$

Hence the number of positive integers less than or equal to $x$, represented by p.p.b.q. forms of discriminant $d$, and prime to $d$ except that they need not be prime to $p$, is given by

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[1] R. D. James, Amer. J. Math. vol. 60 (1938) pp. 737–744.

[2] G. Pall, Math. Zeit. vol. 36 (1933) pp. 321–343, p. 331.

$$\frac{bx}{(\log x)^{1/2}}(1 + 1/p + \cdots + 1/p^r + \cdots) + O(x/\log x).$$

Thus each prime $p$ dividing $d$ but not satisfying (3) can be removed from the set of primes excluded in $m$, by replacing $b$ by $b/(1-1/p)$. We recall that[3]

$$\sum (d \mid n)n^{-1} = 2\pi h(d)/(w(-d)^{1/2}), \tag{4}$$

where $h(d)$ is the number of p.p.b.q. classes of discriminant $d$, and $w=2$, except that $w=4$ if $d=-4$, $w=6$ if $d=-3$. Hence the number of positive integers less than or equal to $x$ which can be represented by p.p.b.q. forms of discriminant $d$, and which have no prime factor $p$ satisfying (3), is

$$b_0x/(\log x)^{1/2} + O(x/\log x),$$

where

$$\begin{aligned} b_0^2 = (2h(d)/w(-d)^{1/2}) \prod_q (1 - 1/q^2)^{-1} \\ \cdot \prod_{p \text{ sat. } (3)} (1 - 1/p) \prod_{p \mid d, \text{ not } (3)} (1 - 1/p)^{-1}. \end{aligned} \tag{5}$$

Second let $p$ satisfy (3). Then[4] $pm$ is not represented by p.p.b.q. forms of discriminant $d$ if $p \nmid m$; and $p^2n$ is represented by such forms if and only if $n$ is represented by p.p.b.q. forms of discriminant $d_1=d/p^2$. If $b_1$ is the value of $b_0$ corresponding to $d_1$, then from (5),

$$\begin{aligned} b_1 &= b_0 && \text{if } d_1/p^2 \text{ is an integer} \equiv 0 \text{ or } 1 \bmod 4, \\ &= b_0(1 - p^{-2})^{-1} && \text{if } (d_1 \mid p) = -1, \\ &= b_0(1 - p^{-1})^{-1} && \text{otherwise.} \end{aligned} \tag{6}$$

Hence if $d=p^{2k}d'$, where $k \geqq 1$ and $p^2 \nmid d'$, we can remove $p$ from the set of excluded primes by replacing $b_0x/(\log x)^{1/2}$ by $(b_0x/(\log x)^{1/2})$ $(1+p^{-2}+ \cdots +p^{-2k+2}+p^{-2k}\lambda)$, where $\lambda = 1/(1-1/p)$ if $(d' \mid p) \neq -1$, $\lambda = 1/(1-1/p^2)$ if $(d' \mid p) = -1$; hence by multiplying $b_0$ by

$$\frac{1 + p^{-2k-1}}{1 - p^{-2}} \text{ if } (d' \mid p) \neq -1, \qquad \frac{1}{1 - p^{-2}} \text{ if } (d' \mid p) = -1.$$

Thus we have finally this theorem:

THEOREM 2. *The number* $C(x)$ *of positive integers* $n \leqq x$ *which can be*

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[3] See, for example, Landau, *Vorlesungen uber Zahlentheorie*, vol. 2, p. 152.

[4] G. Pall, Amer. J. Math. vol. 57 (1935) pp. 789–799, formula (9).

*represented by positive, primitive, binary quadratic forms of a given negative discriminant $d$ is given by*

$$cx/(\log x)^{1/2} + O(x/\log x),$$

*where $c$ is the positive constant defined by*

$$c = b_0 \prod_{p \text{ n.r. } (d)} (1 - p^{-2})^{-1} \cdot \prod (1 + p^{-2k-1}), \tag{7}$$

*where in the last product $d = p^{2k}d'$, where $p^2 \nmid d'$, $k \geqq 1$, and $(d' | p) \neq -1$.*

*Example.* If $d = -3$,

$$c^2 = b_0^2 = 3^{-1} \cdot 3^{-1/2} \cdot \alpha \cdot (3/2), \qquad \alpha = \prod_{q}^{q \equiv 2(3)} (1 - q^{-2})^{-1},$$

$$= \alpha/(2(3)^{1/2}), \qquad c = .64 \text{ approximately.}$$

If $d = -12$, $b_0^2 = (1/(12)^{1/2})\prod'_{q(\neq 2)}(1 - q^{-2})^{-1} \cdot (1/2) \cdot (3/2) = 9\alpha/(32(3)^{1/2})$; and by (7), $c^2 = b_0^2(16/9) = \alpha/(2(3)^{1/2})$. Hence $c$ is the same for $d = -12$ as for $d = -3$. This agrees with the fact that $x^2 + 3y^2$ represents exactly the same numbers as $x^2 + xy + y^2$.

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# THE BETTI GROUPS OF SYMMETRIC AND CYCLIC PRODUCTS

C. E. CLARK

1. **Introduction.** Consider a finite complex $K$ and a group of permutations of $n$ elements $G=\{G_\lambda\}$, $\lambda=1, \cdots, N$. To define *the product* $k^n$ *of* $K$ *with respect to* $G$, $n=2, 3, \cdots$, we consider an ordered set of $n$ complexes $K_1, \cdots, K_n$ each homeomorphic to $K$; here as throughout the paper we do not distinguish between a complex and a geometric realization of the complex. A point $p$ of the topological product $K^n=K_1\times \cdots \times K_n$ can be represented by the sequence of points $p_1, \cdots, p_n$, $p_i\in K_i$. Each function $G_\lambda(p)$, $\lambda=1, \cdots, N$, gives a homeomorphism of $K^n$ upon itself. We identify each point $p\in K^n$ with all its transforms $G_\lambda(p)$, $\lambda=1, \cdots, N$. The resulting continuous image of $K^n$ is $k^n$. If $G$ is the symmetric group or the cyclic group of permutations of $n$ elements, the product $k^n$ is called the *n-fold symmetric product* or the *n-fold cyclic product of* $K$, respectively.

In this paper we study the integral cohomology groups of $k^n$. Our Theorem 1 gives a convenient method for calculating these groups when $G$ is given. The method is used to construct the cohomology groups when $G$ is either symmetric or cyclic.

The method of this paper differs from that of the earlier papers [3] and [5] of the references at the end of this paper in the following way. All treatments consider Richardson's simplicial transformation $\Lambda$ of $K^n$ upon $k^n$. But Richardson and Walker use $\Lambda$ to determine a transformation of cycles of $K^n$ into cycles of $k^n$, while this paper considers the natural transformation of cocycles of $k^n$ into cocycles of $K^n$. The earlier correspondence of cycles is not (1-1), but the present correspondence of cocycles is (1-1). This fact enables us to get new results.

2. **The general theorem.** By definition $k^n$ is obtained by identifying points of $K^n$. This identification gives a continuous transformation $\Lambda$ of $K^n$ upon $k^n$. Richardson has shown[1] that $K^n$ and $k^n$ can be subdivided into simplicial complexes and the simplexes of these complexes so oriented that $\Lambda$ is simplicial, $G_\lambda$ is simplicial, $\lambda=1, \cdots, N$, and for any oriented simplex $x$ of $K^n$

$$(1) \qquad \Lambda x = \Lambda G_\lambda x, \qquad \lambda = 1, \cdots, N.$$

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[1] See [3, §5].

Henceforth $K^*$ and $k^*$ shall denote these subdivisions.

We say that a chain $F$ of $K^*$ is *invariant under* $G$ if $F(x) = F(G_\lambda x)$, $\lambda = 1, \cdots, N$, for all simplexes $x$ of $K^*$ with the same dimension as $F$.

Let $f$ be a chain of $k^*$, and let $\sigma f$ be the chain of $K^*$ defined by $\sigma f(x) = f(\Delta x)$.

THEOREM 1. *The transformation $\sigma$ gives a (1-1) correspondence between the cocycles of $k^*$ and the cocycles of $K^*$ invariant under $G$, and a cocycle of $k^*$ cobounds if and only if its corresponding invariant cocycle of $K^*$ cobounds an invariant chain.*[2]

PROOF. To show that $\sigma f$ is invariant we have using (1) that $\sigma f(x) = f(\Delta x) = f(\Delta G_\lambda x) = \sigma f(G_\lambda x)$.

Next we show that any invariant chain $F$ can be written $\sigma f$. Indeed, because of (1) and the fact that $F$ is invariant we can define a chain $f$ of $k^*$ by the equation $f(\Delta x) = F(x)$. Then $\sigma f(x) = f(\Delta x) = F(x)$.

Since $\Delta K^*$ covers $k^*$, it follows that $\sigma$ is (1-1) between chains of $k^*$ and invariant chains of $K^*$. To complete the proof of Theorem 1 it is sufficient to show that $\dot{f} = z$ implies $(\sigma f)^\cdot = \sigma z$, and conversely; the dot denotes the coboundary operator. It is well known that $\dot{f} = z$ implies $(\sigma f)^\cdot = \sigma z$.[3] Suppose $(\sigma f)^\cdot = \sigma z$. Then $z(\Delta x) = \sigma z(x) = (\sigma f)^\cdot(x) = \sigma \dot{f}(x) = \dot{f}(\Delta x)$.

3. **The topological product $K^n$.** In this section we state some properties of $K^n$ which can be derived when $n > 2$ in the same way that they have been derived when $n = 2$.[4] Let

$$Z_i,\ z_j,\ f_j \qquad i = 1, \cdots, I,\ j = 1, \cdots, J, \tag{2}$$

form a basis for the integral chains of $K^*$ of all dimensions; furthermore, let (2) be such that the $Z_i$ generate the cocycles that are independent of coboundaries, the $Z_i$ and $z_j$ generate the cocycles, and

$$\dot{f}_j = e_j z_j, \qquad j = 1, \cdots, J, \tag{3}$$

are a complete set of coboundary relations for the cocycles of (2).[5] Corresponding to any set of non-negative integers $a_1, \cdots, a_I$, $b_1, \cdots, b_J$, $c_1, \cdots, c_J$ with $\sum a_i + \sum b_j + \sum c_j = n$ we have a chain $A = A(a_1, \cdots, a_I, b_1, \cdots, b_J, c_1, \cdots, c_J)$ defined as follows. Let

$$x_1, \cdots, x_n \tag{4}$$

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[2] This theorem resembles [4, p. 22, line 15].

[3] See, for example, [2, chap. IV, §4].

[4] See, for example, [2] or [1].

[5] See [1, p. 304, §7], which includes a justification of the (1-1)-correspondence between the $z$'s and $f$'s of (2).

be the sequence of elements of (2) with $Z_1$ in the first $a_1$ places, $Z_2$ in the next $a_2$ places, $z_1$ in the $b_1$ places following the $a_I$ elements equal to $Z_I$, $f_1$ in the $c_1$ places following the $b_J$ elements equal to $z_J$, and so on. Then $A=(x_1\times\cdots\times x_n)$. If we denote the dimensions of $Z_i$ and $z_j$ by $r_i$ and $s_j$, respectively, we see from (3) that the dimension of $f_j$ is $s_j-1$. Hence the dimension of $A$ is $\sum a_ir_i+\sum b_js_j+\sum c_j(s_j-1)$.

Let $S=\{S_\lambda\}$, $\lambda=1,\cdots,n!$, be the symmetric group of permutations on $n$ elements. We can apply $S_\lambda$ to the sequence (4) and obtain the sequence which we denote by $y_1,\cdots,y_n$. We define $S_\lambda\{A\}=(y_1\times\cdots\times y_n)$. Then a basis for the chains of $K^*$ is given by the distinct chains of the set $S_\lambda\{A\}$, $\lambda=1,\cdots,n!$, all $A$.

To obtain a basis for the cocycles of all dimensions we consider $\mathfrak{Z}_1=\mathfrak{Z}_1(a_1,\cdots,a_I,b_1,\cdots,b_J)=A(a_1,\cdots,a_I,b_1,\cdots,b_J)$, $\sum a_i+\sum b_j=n$. Also we consider $\mathfrak{Z}_2=\dot{A}/e$, $\sum c_j>1$, where $e$ is the greatest common divisor of all the $e_j$'s that are associated by (3) with the $f_j$'s that correspond to the nonzero $c_j$'s of $A$; the division of $\dot{A}$ by $e$ can be shown to be always possible. Then a basis for the cocycles of $K^*$ is given by the distinct chains of the set $S_\lambda\{\mathfrak{Z}_1\}$ and $S_\lambda\{\mathfrak{Z}_2\}$. $\lambda=1,\cdots,n!$, all $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$.

**4. The integral cohomology groups of the *n*-fold symmetric product.** We can consider the group $S$ as the group $G$ of §§1 and 2. Then any $S_\lambda$ determines a simplicial map of $K^*$ into itself. Under this simplicial map the chain $A$ is mapped into a chain which we denote by $S_\lambda A$. From [3] we obtain the formula

$$S_\lambda A=(-1)^dS_\lambda\{A\} \tag{5}$$

where $d$ is determined as follows. If $S_\lambda$ interchanges two elements and leaves the other $n-2$ invariant, then $d$ is the product of the dimensions of the two elements of (4) that are interchanged by $S_\lambda$. Since any $S_\lambda$ is a product of permutations of the type just considered, the rule just stated determines $d$ for any $S_\lambda$.

We next determine the chains of $K^*$ that are invariant under $S$. First consider an $A$ with at least one of its $a_i$, $b_j$, or $c_j$ having the properties that it is greater than one and that the $Z_i$, $z_j$, or $f_j$ with which it is associated is of odd dimension. Then (5) implies that there is an $S_\lambda$ such that $S_\lambda A=-A$. This implies that any cocycle invariant under $S$ is linearly independent of $A$ and indeed of $S_\lambda A$, $\lambda=1,\cdots,n!$.

Next assume that no $a_i$, $b_j$, or $c_j$ of $A$ has the properties just considered. Then there are $\pi=a_1!a_2!\cdots b_1!\cdots c_1!\cdots$ values of $\lambda$ for which $S_\lambda A=A$. From this fact and the fact that the $S_\lambda A$ are elements of a basis (because of (5) and the fact that the $S_\lambda\{A\}$ form a basis), we see that $\sum_\lambda S_\lambda A$, $\lambda=1, \cdots, n!$, is divisible by $\pi$ but by no integer greater than $\pi$. Finally, we infer that a basis for the chains of $K^n$ invariant under $S$ is given by the distinct chains of the set $(1/\pi)\sum_\lambda S_\lambda A$, $\lambda=1, \cdots, n!$, where $A$ ranges over all $A$ any of whose factors $Z_i$, $z_j$, and $f_j$ is of even dimension if the corresponding $a_i$, $b_j$, or $c_j$ is greater than 1.

In the same way we deduce from the facts of §3 that a basis for the cocycles invariant under $S$ is given as stated in Theorem 2 below.

We next find the coboundaries of chains invariant under $S$ that are linearly dependent on $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$. Suppose for $\mathfrak{Z}_1$ we have $b_1 \neq 0$. Then $\mathfrak{Z}_1$ is a product of $n$ cocycles at least one of which is $z_1$. Replace the first $z_1$ in this product by $f_1$, $\dot{f}_1 = e_1 z_1$. Let $D$ denote the resulting chain. Then $D' = (b_1/\pi)\sum_\lambda S_\lambda D$, $\lambda=1, \cdots, n!$, is invariant under $S$ and is not a proper multiple of any other invariant chain. Since $(x_1 \times \cdots \times x_n)^{\cdot} = \sum_i \pm (x_1 \times \cdots \times \dot{x}_i \times \cdots \times x_n)$, $i=1, \cdots, n$,[4] and since $(S_\lambda F)^{\cdot} = S_\lambda \dot{F}$, we have $\dot{D}' = \pm (b_1 e_1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$. This implies that $(b_1 e_1, \cdots, b_J e_J)(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$ cobounds a chain invariant under $S$; here as elsewhere we understand that the greatest common divisor of zero and a positive integer is that integer. Furthermore, examination of our basis for the chains invariant under $S$ shows that multiples of this coboundary are the only multiples of $\mathfrak{Z}_1$ that can be linearly dependent upon a coboundary of a chain invariant under $S$.

The definition of $\mathfrak{Z}_2$ implies that $(e/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_2$, $\lambda=1, \cdots, n!$, cobounds a chain invariant under $S$. Furthermore, multiples of this coboundary are the only multiples of $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_2$ that are dependent on coboundaries of chains invariant under $S$. We have proved this theorem.

THEOREM 2. *A basis for the cocycles of* $K^n$ *invariant under* $S$ *is given by the distinct chains of the set* $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$ *and* $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_2$, $\lambda=1, \cdots, n!$, *where* $\mathfrak{Z}_1$ *and* $\mathfrak{Z}_2$ *range over all* $\mathfrak{Z}_1$ *and* $\mathfrak{Z}_2$ *any of whose factors* $Z_i$, $z_j$, *and* $f_j$ *has even dimension if the associated* $a_i$, $b_j$, *or* $c_j$ *is greater than* 1; *furthermore, the cocycles invariant under* $S$ *that cobound chains of* $K^n$ *invariant under* $S$ *are generated by* $(b_1 e_1, \cdots, b_J e_J)$ $(1/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_1$ *and* $(e/\pi)\sum_\lambda S_\lambda \mathfrak{Z}_2$.

**5. The integral cohomology groups of the *n*-fold cyclic product.** Let $C=\{C_n^\mu\}$, $\mu=1, \cdots, n$, denote the group of the cyclic permutations of $n$ elements, where $C_n^1$ is the permutation that replaces each element except the first by its predecessor, and $C_n^\mu$ is the $\mu$th power of $C_n^1$. Let $B=q[x_1 \times \cdots \times x_p]$ denote the chain $(x_1 \times \cdots \times x_p \times x_1 \times \cdots \times x_p \times \cdots)$ of $K^{pq}$. Furthermore, whenever a chain of $K^{pq}$ is written in this notation, it is understood that $q$ is maximal.

As in §4 we can consider $C_n^\mu B$ and $C_n^\mu\{B\}$. These chains satisfy (5). In particular, $C_n^p B=\delta C_n^p\{B\}$, where $\delta=-1$ if $q$ is even and $\sum_1^p r_i$ is odd, $r_i$ = dimension of $x_i$, and $\delta=1$ if either $q$ is odd or $\sum_1^p r_i$ is even. This implies $\sum_\mu C_n^\mu B=0$, $\mu=1, \cdots, pq$, if $q$ is even and $\sum_1^p r_i$ is odd, and the same sum is divisible by $q$ if $q$ is odd or $\sum_1^p r_i$ is even.

A basis for the chains of $K^n$ invariant under $C$ is given by the distinct chains of the set $(1/q)\sum_\mu C_n^\mu B$, $\mu=1, \cdots, n$, $pq=n$, $q$ odd or $\sum_1^p r_i$ even, where the $x_i$ range over the elements of the basis (2).

Let $Z_1=(1/q)\sum_\mu C_n^\mu B$, $q$ odd or $\sum_1^p r_i$ even, where the factors of $B$ contain no $f_j$. If the factors of $Z_1$ contain no $z_j$, then $Z_1$ is linearly independent of coboundaries. Suppose the first factor $x_1$ of $B$ is $z_1$, and $\dot{f}_1=e_1z_1$. Let $E$ be the chain of $K^n$ defined by $E=(f_1\times x_2\times \cdots \times x_p\times x_1\times \cdots \times x_p\times x_1\times \cdots \times x_p\times \cdots)$. We have that $E'=\sum_\mu C_n^\mu E$, $\mu=1, \cdots, n$, is a chain invariant under $C$. Furthermore, $E'$ is not divisible by any integer different from $\pm 1$. We compute $\dot{E}'=e_1\sum_\mu C_n^\mu B=e_1 q Z_1$. Let $\epsilon$ be the greatest common divisor of all the $e_j$'s that are associated by (3) with the $z_j$'s that occur among the factors of $B$. We conclude that $\epsilon q Z_1$ is a coboundary.

Let $Z_2=(1/eq)\sum_\mu C_n^\mu \dot{B}$, $\mu=1, \cdots, n$, $q$ odd or $\sum_1^p r_i$ even, where the factors of $B$ contain at least two $f_j$'s (possibly equal), and $e$ is the greatest common divisor of the $e_j$'s associated with the $f_j$'s among these factors. In counting the factors of $B$ we count each factor the number of times it is repeated due to the symmetry of $B$. We can prove this theorem.

THEOREM 3. *A basis for the cocycles of $K^n$ invariant under $C$ is given by the distinct chains $Z_1$ and $Z_2$; furthermore, the cocycles of $K^n$ invariant under $C$ that cobound chains invariant under $C$ are generated by $\epsilon q Z_1$ and $e Z_2$.*

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# GROUPS TRANSITIVE ON THE $n$-DIMENSIONAL TORUS

DEANE MONTGOMERY[1] AND HANS SAMELSON

In this note we denote by $G$ a compact connected Lie group. We shall be interested in the situation where $G$ acts as a topological transformation group[2] on a space $E$. Such a group is called effective if the identity is the only element of $G$ which leaves every point of $E$ fixed. If $G$ is transitive on $E$, that is, for any two points $x$ and $y$ of $E$ there is an element $g$ of $G$ such that $g(x)=y$, then $E$ is called a homogeneous space or a coset space of $G$. Our purpose is to prove the following theorem:

THEOREM. *If a compact connected Lie group $G$ is transitive and effective on a space $E$ homeomorphic with an $n$-dimensional torus (topological product of $n$ circles), then $G$ is isomorphic with the $n$-dimensional toral group $T_n$ (direct product of $n$ circle groups) and no element of $G$ except the identity leaves any point of $E$ fixed.*

We use a method of proof which has some similarity to a method we have used in studying groups transitive on spheres.[3]

Let $H'$ be a compact, connected, simply connected Lie group, let $T_l$ be an $l$-dimensional toral group, and let $N$ be a finite normal subgroup of the direct product $H' \times T_l$ such that $G$ is continuously isomorphic to the factor-group $(H' \times T_l)/N$.[4] Let $H'$ go into $H$ by the homomorphism obtained by factoring with respect to $N$ and let $T_l$ go into $K$. The group $K$ is also an $l$-dimensional toral group, and $H$ and $K$ are subgroups of $G$ which span $G$ or generate $G$. The elements of $H$ commute with the elements of $K$, in fact $K$ is a central subgroup of $G$.

Let $x$ be an arbitrarily chosen point of $E$ and let $H_x$, $K_x$, and $G_x$ be, respectively, the subgroups of $H$, $K$, and $G$ which leave $x$ fixed. Let $K^*$ be the subgroup of $K$ consisting of those elements $k$ such that $k(x)$ is in the orbit $H(x)$. The orbit $K^*(x)$ is the intersection of $H(x)$ and $K(x)$. It can be seen that if $y=g(x)$ then $K_y=gK_xg^{-1}$ and $H_y=gH_xg^{-1}$. Since $K$ is a central subgroup we see that $K_y=K_x$.

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[1] Guggenheim fellow.

[2] For the theory of topological groups and Lie groups needed see Pontrjagin, *Topological groups*, Princeton 1939. For definitions and results concerning topological transformation groups see Zippin, *Transformation groups*, Lectures in Topology, Ann Arbor 1941 pp. 191–221.

[3] See a paper by us which is forthcoming.

[4] For the existence of these groups see Pontrjagin, loc. cit. pp. 282–285. The group $H'$ is the direct product of the simple Lie groups there mentioned.

Because $G$ is transitive the last remark shows us that every element of $K_x$ leaves every point of $E$ fixed, and this together with the fact that $G$ is effective implies that $K_x$ contains only the identity element. The orbit $K(x)$ is therefore homeomorphic to $K$ and consequently $l$, the dimension of $K$, must be less than or equal to $n$, the dimension of $E$.

We denote by $\mathcal{H}_1(S)$ the one-dimensional homology group with rational coefficients of the space $S$. If a compact connected Lie group is mapped in the natural way on one of its orbits (namely by considering the orbit as a coset space of the group) then the one-dimensional homology group of the group manifold is mapped *onto* the one-dimensional homology group of the orbit. This follows from the fact that the mapping can be carried out in two steps, the first being a fibering of the group with respect to a connected subgroup, and the second being a finite covering. In both steps the one-dimensional homology groups (with rational coefficients) are mapped *onto* the one-dimensional homology groups of the respective spaces.

We now apply the above remark to the group $G$ and the orbit $E$. The homology group $\mathcal{H}_1(E)$ is an $n$-dimensional vector space, that is the first Betti number of $E$ is $n$. Therefore the first Betti number of $G$ is at least $n$, and from this we see that the first Betti number of $H' \times T_l$ must be at least $n$. However the first Betti number of $H' \times T_l$ is $l$ and we see that $l$ is greater than or equal to $n$.

The above results show that $l$ equals $n$. It follows that the orbit $K(x)$ is the whole of $E$ which means that $K$ is transitive on $E$. We see therefore that $K^*(x) = H(x)$. Since $K^*(x)$ is homeomorphic to $K^*$ and since $H(x)$ is connected it follows that $K^*$ is connected. Therefore $K^*$ as a connected subgroup of a toral group must itself be a toral group of some dimension greater than zero or it must contain only the identity element. But from the fact that $\mathcal{H}_1(H) = 0$ it follows that $\mathcal{H}_1[H(x)] = 0$. Hence $\mathcal{H}_1(K^*) = 0$ and $K^*$ contains only the identity element, and $H(x) = x$. The point $x$ was chosen arbitrarily so that the equation $H(x) = x$ holds for every $x$ in $E$. Because $G$ is effective this means that $H$ contains only the identity element and that $G = K$ which proves the theorem.

The same proof applies if instead of assuming that $E$ is an $n$-dimensional torus we merely assume that it is an $n$-dimensional space the first Betti number of which is $n$. It then follows in view of the above proof that $E$ is a torus. If we drop the assumption that $G$ is effective the "effective group" will be a toral group of dimension $n$ as before.

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# ON NON-DENUMERABLE GRAPHS

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The present paper consists of two parts. In Part 1 we prove a theorem on the decomposition of a complete graph. This result is then applied in Part 2 to show that the continuum hypothesis is equivalent to the possibility of decomposing the set of all real numbers into a countable number of summands each consisting of rationally independent numbers.

## PART 1

A graph $G$ is complete if every pair of points of $G$ is connected by one and only one segment. $G$ is called a tree if it does not contain any closed polygon.

THEOREM 1. *A complete graph of cardinal number $m$ (that is, the cardinal number of the vertices is $m$) can be split up into a countable number of trees if and only if $m \leqq \aleph_1$.*

PROOF. We shall first prove that every complete graph of power $\aleph_1$ can be split up into the countable sum of trees.[1] Let $G$ be a complete graph of cardinal number $\aleph_1$. Let $\{x_\alpha\}$, $\alpha < \omega_1$, be any well ordered set of power $\aleph_1$. We may assume that $G$ is represented by a system of segments $(x_\alpha, x_\beta)$, $\alpha < \beta < \omega_1$. For any $\beta < \omega_1$ arrange the set of all $\alpha < \beta$ into a sequence $\alpha_{\beta,n}$, $n = 1, 2, \cdots$, and let $G_n$ be the set of all segments $(x_\alpha, x_\beta)$ such that $\alpha = \alpha_{\beta,n}$. It is clear that $G = \bigcup_{n=1}^{\infty} G_n$ and that for each $G_n$, for every $\beta < \omega_1$, there exists one and only one $\alpha$ such that $(x_\alpha, x_\beta) \in G_n$ and $\alpha < \beta$. From this last fact it is clear that $G_n$ does not contain any closed polygon.

Conversely, let us assume that a complete graph $G$ of cardinal number $m$ is split up into a countable number of trees $T_n$; $G = \bigcup_{n=1}^{\infty} T_n$. We shall prove that $m \leqq \aleph_1$. We can again assume that $G$ is represented by a system of segments $(x_\alpha, x_\beta)$, $\alpha < \beta < \phi$, where $\{x_\alpha\}$, $\alpha < \phi$, is a well ordered set of cardinal number $m$.

We shall first decompose each $T_n$ into four parts $T_{n,i}$, $i = 1, 2, 3, 4$, such that $T_{n,1}$ and $T_{n,2}$ satisfy the condition:

(1) Any two consecutive segments of the graphs $T_{n,1}$ and $T_{n,2}$ are of the form: $(x_\alpha, x_\beta)$, $(x_\alpha, x_\gamma)$, $\alpha < \beta$, $\alpha < \gamma$, $\beta \neq \gamma$. And $T_{n,3}$, $T_{n,4}$ satisfy:

(2) Any two consecutive segments of the graphs are of the form: $(x_\beta, x_\alpha)$, $(x_\gamma, x_\alpha)$, $\beta < \alpha$, $\gamma < \alpha$, $\beta \neq \gamma$.

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[1] This result was also obtained by J. Tukey, oral communication.

For this purpose, let $T_n = \bigcup_\lambda T_n^{(\lambda)}$ be a decomposition of each $T$ into its components $T_n^{(\lambda)}$. The number of components $T_n^{(\lambda)}$ is not necessarily countable. Let $x_n^{(\lambda)}$ be an arbitrary point in each $T_n^{(\lambda)}$, called the origin of $T_n^{(\lambda)}$. A segment $(x_\alpha, x_\beta)$, $\alpha<\beta$, of $T_n^{(\lambda)}$ is of 0th degree of the first kind if $x_\alpha = x_n^{(\lambda)}$ and is of 0th degree, of the second kind if $x_\beta = x_n^{(\lambda)}$. Assume that the segments of degree $0, 1, \cdots, m-1$ of the first and second kind are already defined in $T_n^{(\lambda)}$. Then a segment $(x_\alpha, x_\beta)$, $\alpha<\beta$, of $T_n^{(\lambda)}$ is of the $m$th degree, of the first kind if $x_\alpha$ is one of the end points of a segment of degree $m-1$ and if $(x_\alpha, x_\beta)$ is not of degree at most $m-1$. The segments of degree $m$ of the second kind are analogously defined. In this way it is possible to define a degree of first or second kind for every segment of $T_n^{(\lambda)}$. Let $T_{n,m}^{(\lambda)'}$ be the set of all segments of $T_n^{(\lambda)}$ of degree $m$ of the first kind, and let $T_{n,m}^{(\lambda)''}$ be the set of all segments of $T_n^{(\lambda)}$ of degree $m$ and of second kind. It is clear that $T_{n,m}^{(\lambda)'}$ and $T_{n,m}^{(\lambda)''}$ satisfy conditions (1) and (2), respectively. If we put

$$T_{n,1} = \bigcup_\lambda \bigcup_{2m} T_{n,2m}^{(\lambda)'}, \qquad T_{n,2} = \bigcup_\lambda \bigcup_{2m+1} T_{n,2m+1}^{(\lambda)'},$$
$$T_{n,3} = \bigcup_\lambda \bigcup_{2m} T_{n,2m}^{(\lambda)''}, \qquad T_{n,4} = \bigcup_\lambda \bigcup_{2m+1} T_{n,2m+1}^{(\lambda)''},$$

then $T_n = \bigcup_{i=1}^4 T_{n,i}$ is a required decomposition. The conditions (1) and (2) are satisfied by $T_{n,1}$, $T_{n,2}$ and $T_{n,3}$, $T_{n,4}$, respectively, since the summands on the right-hand side are disjoint.

Thus we have obtained a decomposition of $T$ into a countable number of parts $T_{n,i}$, $n=1, 2 \cdots$; $i=1, 2, 3, 4$. We observe that from (1) and (2) follows (3) and (4), respectively:

(3) For any $\alpha<\phi$, there exists at most one $\beta<\alpha$ such that $(x_\beta, x_\alpha)$ belongs to the graph.

(4) For any $\alpha<\phi$ there exists at most one $\beta>\alpha$, $\beta<\phi$, such that $(x_\alpha, x_\beta)$ belongs to the graph.

Let us now put

$$T' = \bigcup_{n=1}^\infty T_{n,1} \cup \bigcup_{n=1}^\infty T_{n,2},$$
$$T'' = \bigcup_{n=1}^\infty T_{n,3} \cup \bigcup_{n=1}^\infty T_{n,4}.$$

Then $T'$ and $T''$ clearly satisfy the following conditions:

(5) For any $\alpha<\phi$ the set of all $\beta<\alpha$ such that $(x_\beta, x_\alpha)$ belongs to the graph is countable.

(6) For any $\alpha<\phi$ the set of all $\beta>\alpha$, $\beta<\phi$ such that $(x_\alpha, x_\beta)$ belongs to the graph is countable.

From this it easily follows by the same argument as in W. Sierpinski[3] that the cardinal number $m$ of all points $x_\alpha$, $\alpha<\phi$, is at most $\aleph_1$.

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[3] W. Sierpinski, *Hypothese du continu*, Proposition $P_1$, p. 9.

Similarly we can prove that the complete graph of power $\aleph_\nu$ is the sum of $\aleph_{\nu-1}$ trees, but not the sum of less than $\aleph_{\nu-1}$ trees. We can put the following problem: Is the complete graph of power $\aleph_\nu$ the sum of less than $\aleph_{\nu-1}$ such graphs which do not contain a quadrilateral? We cannot answer this question, unless we assume the generalized hypothesis of the continuum, in which case the answer is negative. It can be shown that the complete graph of power $2^m$ is the sum of $m$ graphs, which do not contain even closed polygons,[3] but that the complete graph of power greater than $2^m$ is not the sum of $m$ graphs which do not contain triangles.[4]

## PART 2

THEOREM 2. *The continuum hypothesis is equivalent to the following proposition:*

(P) *The set of all real numbers can be decomposed into a countable number of subsets, each consisting only of rationally independent numbers.*

PROOF. We shall first prove that the continuum hypothesis implies proposition (P). Let $\xi_\alpha$, $\alpha<\omega_1$, be a Hamel basis for the set $R$ of all real numbers, well ordered in a transfinite sequence of type $\omega_1$; that is, the $\xi_\alpha$ are rationally independent, and every real number $x(\neq 0)$ can be uniquely expressed in the following form:

$$x_\alpha = \sum_{\nu=1}^{n} r_\nu \xi_{\alpha_\nu}, \qquad \alpha_1 < \alpha_2 < \cdots < \alpha_n < \omega_1, \tag{2.1}$$

where the $r_\nu$ are rational numbers different from 0, and $n$ is a positive integer.

For any finite system of rational numbers $r_1, r_2, \cdots, r_n$ let $R_{r_1,r_2,\cdots,r_n}$ be the set of all $x \in R$ which are expressed in the form (2.1). Then

$$R = (0) \cup \bigcup_{(r_1,r_2,\cdots,r_n)} R_{r_1,r_2,\cdots,r_n} \tag{2.2}$$

is a decomposition of the set $R$ into the set (0) consisting of 0 alone and a countable number of sets $R^{(\alpha)}_{r_1,r_2,\cdots,r_n}$, where $\bigcup_{(r_1,r_2,\cdots,r_n)}$ means the union for all possible ordered systems $r_1, r_2, \cdots, r_n$ of rational numbers. Consequently in order to prove our theorem it suffices to prove it for all the $R_{r_1,r_2,\cdots,r_n}$.

For each ordinal number $\alpha$, $\alpha<\omega_1$, let $R^{(\alpha)}_{r_1,r_2,\cdots,r_n}$ be the subset of $R_{r_1,r_2,\cdots,r_n}$ consisting of all real numbers $x$, such that $\alpha_n = \alpha_n(x) = \alpha$.

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[3] K. Gödel, oral communication.

[4] P Erdös, *On graphs and sets*, to appear in Revista, Matematicas y Fisica Teorica.

Then since $\alpha_1<\alpha_2<\cdots<\alpha_n=\alpha$, each $R^{(\alpha)}_{r_1,r_2,\ldots,r_n}$ is a countable set. Let us arrange the set $R^{(\alpha)}_{r_1,r_2,\ldots,r_n}$ into a sequence $\{x^{(\alpha)}_{r_1,r_2,\ldots,r_n,m}\}$, $m=1,2,\cdots$, and let us put $S_{r_1,r_2,\ldots,r_n,m}=\{x^{(\alpha)}_{r_1,r_2,\ldots,r_n,m},\ \alpha<\omega_1\}$. Clearly $\alpha_n(x)\neq\alpha_n(y)$ for any $x,y\in S_{r_1,r_2,\ldots,r_n,m}$, $x\neq y$, and $R_{r_1,r_2,\ldots,r_n}=\bigcup_{m=1}^{\infty}S_{r_1,r_2,\ldots,r_n,m}$.

We shall prove that each $S_{r_1,r_2,\ldots,r_n,m}$ consists only of rationally independent numbers. In fact, if we have

$$\sum_{i=1}^{k} p_i x_i = 0 \tag{2.3}$$

where $x_i\in S_{r_1,r_2,\ldots,r_n,m}$, $i=1,\cdots,k$; $x_i\neq x_j$ $(i=j)$, and $p_i$ is an integer different from 0, for $i=1,\cdots,k$, then there would exist an integer $i_0$ for which $\alpha^*=\alpha_n(x_{i_0})>\alpha_n(x_i)$ for all $i\neq i_0$. Hence, in the expansion (2.1) of all the $x_i$, $i=1,\cdots,n$, the term $\xi_{\alpha^*}$ would appear only once, which is clearly impossible because of (2.3). Thus we have proved that the continuum hypothesis implies the proposition (P).

Conversely, let us assume that the proposition (P) is true. We shall prove the continuum hypothesis from it. We shall prove this by showing that, under the assumption (P) the complete graph of power $2^{\aleph_0}$ is the sum of a countable number of trees (see Theorem 1).

Let $R=\bigcup_{n=1}^{\infty}M_n$ be a decomposition of the set of all real numbers into a countable sum of sets each consisting of rationally independent numbers. At least one of the $M_n$ must have power $2^{\aleph_0}$. (This follows from a well known theorem of J. König.[5]) We may assume that $M_1$ has power $2^{\aleph_0}$. Let $G$ be the complete graph of cardinal number $2^{\aleph_0}$. We may assume that $G$ is represented by a system of segments $(x, y)$, $x, y\in M_1$, $x<y$. Let $G_n$ be a subgraph of $G$, consisting of all those segments $(x, y)$, $x<y$ for which $y-x\in M_n$. Clearly $G=\bigcup_{n=1}^{\infty}G_n$. We shall prove that each $G_n$ is a tree. In fact assume that $G_n$ contains a closed polygon. Let $x_1,\cdots,x_k$, $x_{k+1}=x_1$ be the vertices of this polygon. Then $0=\sum_{i=1}^{k}(x_i-x_{i+1})=\sum_{i=1}^{k}\pm|x_i-x_{i+1}|$. On the other hand $|x_i-x_{i+1}|\in M_n$ by construction, and since all $x_i\in M_1$ the numbers $|x_i-x_{i+1}|$, $i=1,2,\cdots,k$, are all different. This is however a contradiction since $M_n$ consists of rationally independent numbers. This completes the proof of Theorem 2.

It seems likely that the following stronger theorem also holds: Assume that the continuum hypothesis is false, and let the sets $M_n$ consist of rationally independent numbers. Then $\bigcup_{n=1}^{\infty}M_n$ has inner measure 0.

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[5] J. König, see W. Sierpinski, ibid. p. 6.

We can of course prove the following theorem: The necessary and sufficient condition for the continuum to be of power $\aleph_{\alpha+1}$ is that $R$ shall be the sum of $\aleph_\alpha$ sets consisting of rationally independent numbers, and that $R$ shall not be the sum of less than $\aleph_\alpha$ such sets. The proof is the same as that of Theorem 2.

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# ADDENDUM TO THE PAPER "GENERALIZED FISCHER GROUPS AND ALGEBRAS"

R. H. BRUCK

The author regrets the omission, in a paper which recently appeared,[1] of an important reference to a paper by N. Jacobson.[2] Indeed Lemma IIIa of the author's paper, and its immediate consequence, Theorem I, are rather special cases (albeit independently obtained) of Theorem I of the latter paper. Accordingly Professor Jacobson's name should have appeared in the introduction along with those of M. Schiffer and W. Specht, and chronologically before that of Specht.

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[1] R. H. Bruck, *Generalized Fischer groups and algebras*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 618–626, in particular p. 623.

[2] N. Jacobson, *Normal semi-linear transformations*, Amer. J. Math. vol. 61 (1939) pp. 45–58, in particular p. 49.

# ON THE RADICAL OF A GENERAL RING

JAKOB LEVITZKI

1. **Introduction.** The radical of a ring has been hitherto defined by using either the notion of a nilpotent or a nil-ideal. In the following we shall ascribe the term *specialized radical* to the sum $N_\sigma$ of all two-sided nilpotent ideals of a ring $S$, and the term *generalized radical* to the sum $N_\gamma$ of all two-sided nil-ideals of the ring. In §2 of the present note the notions of a *semi-nilpotent* ideal and its counterpart the *semi-regular* ideal are introduced, and the term *radical* is suggested for the sum $N$ of all two-sided semi-nilpotent ideals of the ring. These notations may be justified by the following considerations:

(a) Each nilpotent ideal is semi-nilpotent, and each semi-nilpotent ideal is a nil-ideal.

(b) The radical $N$ is a two-sided semi-nilpotent ideal which contains also all one-sided semi-nilpotent ideals of the ring.

(c) The radical of $S/N$ is zero.

(d) The radical $N$ contains the specialized radical $N_\sigma$ and is a subset of the generalized radical $N_\gamma$.

(e) In the case of an algebra the notions: nilpotent, semi-nilpotent and nil-ideal are identical, and $N_\sigma = N = N_\gamma$; but if one turns to general rings, and replaces radical $N$ and semi-nilpotent ideals either by specialized radical $N_\sigma$ and nilpotent ideals or generalized radical $N_\gamma$ and nil-ideals, then some restriction has to be imposed on the ring $S$ in order to assure the validity of (b) and (c).[1]

These results are applied in §3 to semi-primary rings (which will be called in short: $A$-rings).

2. **The radical of a general ring.** In this section certain theorems related to the radical of a general ring are proved.

NOTATION. If $r_1, \cdots, r_n$ is a finite set of elements in the ring $S$, then the ring generated by the $r$ will be denoted by $\{r_1, \cdots, r_n\}$.

DEFINITION. *A right ideal is called semi-nilpotent if each ring gener-*

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[1] Thus the specialized radical $N_\sigma$ is nilpotent (and hence the specialized radical of $S/N_\sigma$ is zero) if (as is well known) the maximal condition or (as can be shown) the minimal condition is satisfied by the two-sided ideals of the ring. As to the generalized radical $N_\gamma$, it has been proved (G. Koethe, *Die Struktur der Ringe*, Math. Zeit. vol 32 (1930) pp. 161–186) that if each regular right ideal of $S$ contains a minimal regular right ideal, then $N_\gamma$ contains also all one-sided nil-ideals of the ring.

*ated by a finite set of elements belonging to the ideal is nilpotent. A right ideal which is not semi-nilpotent, is called semi-regular.*

Remarks. Each *regular* right ideal (that is, an ideal which contains elements which are not nilpotent) is evidently semi-regular, and each semi-regular right ideal is *potent* (that is, not nilpotent). Accordingly, each nilpotent right ideal is semi-nilpotent, and each semi-nilpotent right ideal is a right nil-ideal. One can easily find examples of rings containing semi-nilpotent right ideals which are not nilpotent. The problem of constructing a ring with right nil-ideals which are not semi-nilpotent, is equivalent to the construction of a potent nil-ring with a finite number of generators. It can be easily shown that such rings can always be found among the subrings of a ring with a generalized radical $N_\gamma$ (see §1) which does not contain all one-sided nil-ideals of the ring.

Theorem 1. *The sum $R=(R_1, R_2)$ of two semi-nilpotent right ideals $R_1$ and $R_2$ is a semi-nilpotent right ideal.*

Proof. Suppose $R$ is semi-regular, that is, $R$ contains a finite set $r_{01}, \cdots, r_{0n}$ so that the ring $S^*=\{r_{01}, \cdots, r_{0n}\}$ is potent. Write $r_{0i}=r_{\sigma i}+r_{\gamma i}$, where $r_{\sigma i}\in R_1$, $r_{\gamma i}\in R_2$, then evidently also $S^{**}=\{r_{\sigma 1}, \cdots, r_{\sigma n}, r_{\gamma 1}, \cdots, r_{\gamma n}\}$ is potent, since $S^*\subseteq S^{**}$. Define a set $r_1, \cdots, r_m$ as follows: (1) each $r_i$ is either an $r_{\sigma k}$ or an $r_{\gamma j}$, (2) the ring $\{r_1, \cdots, r_m\}$ is potent, (3) each ring generated by a proper subset of the $r_i$ is nilpotent. One obtains such a set by suppressing one way or another the greatest possible number of elements of the set $r_{\sigma 1}, \cdots, r_{\sigma n}, \cdots, r_{\gamma n}$, so that the remaining set still generates a potent ring. The set of the $r_i$ necessarily contains certain $r_{\sigma i}$ as well as certain $r_{\gamma k}$ (since the $r_{\sigma i}$ as well as the $r_{\gamma k}$ taken separately generate nilpotent rings), hence $m\geqq 2$. Now consider the rings $T=\{r_1, r_2, \cdots, r_m\}$ and $U=\{r_2, \cdots, r_m\}$. By the definition of the $r_i$ it follows that $T$ is potent, while $U$ is nilpotent. Denote by $\rho$ the index of the nilpotent element $r_1$ and by $\sigma$ the index of the nilpotent ring $U$. Denote by $w_1, \cdots, w_s$ the finite set of all elements of the form $r_1^\lambda r_{i_1} \cdots r_{i_t}$, where $0<\lambda<\rho; 0<t<\sigma; i_j\neq 1, j=1, \cdots, t$. Now, since $T$ is potent, it follows that for each positive integer $x$, elements $v_1, \cdots, v_x$ can be found so that each $v_i$ is a certain $r_k$, and the product $v_1\cdot v_2 \cdots v_x$ is different from zero. From the definition of $\rho$ and $\sigma$ it follows that if $x>\rho$, $x>\sigma$, then the set $v_1, \cdots, v_x$ necessarily contains the element $r_1$ as well as elements different from $r_1$. Hence, by choosing an arbitrary integer $y$, and fixing $x$ so that $x>(\rho+\sigma)(y+2)$, we have $v_1 \cdots v_x=f\cdot g_1 \cdots g_y h$, where $f$ and $h$ are certain elements of $T$, while the $g_i$ are elements of the set $w_1, \cdots, w_\nu$. Since $g_1 \cdots g_\nu \neq 0$, it follows that the ring $\{w_1, \cdots, w_\nu\}$ is potent. Since either $r_1 \in R_1$ or $r_1 \in R_2$, it follows that all the $w_i$ are either in $R_1$ or in $R_2$, which is a contradiction to the assumption that $R_1$ as well as $R_2$ are both semi-nilpotent.

REMARK. Theorem 1 probably does not hold if semi-nilpotent ideals are replaced by nil-ideals.

THEOREM 2. *The sum $N$ of all semi-nilpotent right ideals is a semi-nilpotent two-sided ideal, which contains also all semi-nilpotent left ideals of the ring.*

PROOF. First note, that by Theorem 1 it follows easily (by inducduction) that also the sum of any finite number of semi-nilpotent right ideals is again a semi-nilpotent right ideal. If now $r_1, \cdots, r_n$ is any finite set in $N$, then for each $r_i$ a finite number of semi-nilpotent right ideals $R_1^i, R_2^i, \cdots, R_{k_i}^i$ can be found so that $r_i \in (R_1^i, R_2^i, \cdots, R_{k_i}^i)$ $i=1, \cdots, n$. Hence the sum $R$ of all the $R_k^i$, which is a semi-nilpotent right ideal, contains all the $r_i$, which implies that the ring $\{r_1, \cdots, r_n\}$ is nilpotent. Since this is true for each set $r_1, \cdots, r_n$, it follows that $N$ is a semi-nilpotent right ideal. To prove that $N$ is a left ideal, and hence a two-sided semi-nilpotent ideal, we show that if $a \in N$, $s \in S$ and $R=(sa, saS)$ then $R \subseteq N$. Indeed, suppose $R$ is semi-regular, that is, elements $r_1, \cdots, r_n$ of $R$ can be found so that the ring $\{r_1, \cdots, r_n\}$ is potent. Each $r_i$ has the form $r_i = sas_i$, where $s_i$ is either an integer or an element of $S$. For an arbitrary positive integer $x$ we can find an element different from zero which has the form $sas_{i_1} \cdots sas_{i_x}$. Hence it follows that the ring $\{as_1s, as_2s, \cdots, as_ns\}$ is potent, which is a contradiction, since $as_is \in N$. The remaining part of the theorem follows from the fact that (as can be similarly proved) also the sum of all semi-nilpotent left ideals is a semi-nilpotent two-sided ideal.

REMARK. Theorem 2 does not hold if semi-nilpotent ideals are replaced by nilpotent ideals and probably is not true if they are replaced by nil-ideals.

DEFINITION OF THE RADICAL. *The sum $N$ of all two-sided semi-nilpotent ideals of the ring (which by Theorem 2 contains also all one-sided semi-nilpotent ideals) is called the radical of the ring.*

THEOREM 3. *If $N$ is the radical of $S$, then the radical of $S/N$ is zero.*

PROOF. We show that if $R[\geq N]$ is a semi-regular right ideal in $S$, then $R/N$ is semi-regular in $S/N$. Indeed, let the elements $r_1, \cdots, r_n$ of $R$ generate the potent ring $T=\{r_1, \cdots, r_n\}$. If now $R/N$ is semi-nilpotent, then the ring $\bar{T}=(T+N)/N$ is nilpotent, that is, for a certain $\lambda$ we have $\bar{T}^\lambda = \bar{0}$ (where $\bar{0}$ is the zero of $S/N$), or $T^\lambda \subseteq N$. We denote by $u_1, \cdots, u_m$ the finite set of all products of the form $r_{i_1}\cdot r_{i_2}\cdots r_{i_\lambda}$, and put $U=\{u_1, \cdots, u_m\}$. Since with $T$ also $T^\lambda$ for each $\lambda$ is potent, and since evidently $T^{\sigma\lambda+1}\subseteq U^\sigma T$, we have $U^\sigma\neq 0$ for each $\sigma$. On the other hand, by the definition of $U$, we have $U\subseteq T^\lambda$; hence $U\subseteq N$, which is a contradiction, since $N$ is semi-nilpotent.

As immediate consequences of the foregoing theorems we have these following theorems.

THEOREM 4. *If in a ring $S$ each right ideal (other than zero) is semi-regular, then also each left ideal (other than zero) is semi-regular.*

THEOREM 5. *An element $s$ of a ring $S$ belongs to the radical $N$ of $S$ if, and only if, $sS\subseteq N$.*

3. **Remarks on $A$-rings.** Using the notions of specialized radical, radical and generalized radical, one may define the following three classes of $A$-rings (that is, rings which have a similar structure as the algebras):

DEFINITION. *The ring $S$ is called a specialized $A$-ring, if the specialized radical $N_s$ of $S$ is nilpotent, and the ring $S/N_s$ is semi-simple.*

DEFINITION. *The ring $S$ is called an $A$-ring, if the ring $S/N$ is semi-simple. Herein $N$ denotes the radical of $S$.*

DEFINITION. *The ring $S$ is called a generalised $A$-ring, if the ring $S/N_\gamma$ is semi-simple. Herein $N_\gamma$ denotes the generalised radical of $S$.*

Using a well known theorem of E. Noether and the evident fact that the specialized radical of a ring is zero if the radical or the generalized radical of the ring is zero, we obtain the following characterization of $A$-rings and generalized $A$-rings:

THEOREM 6. *A ring $S$ is an $A$-ring (generalized $A$-ring) if, and only if, the minimal condition is satisfied by the right ideals which contain the radical $N$ (generalized radical $N_\gamma$).*

REMARKS. The condition of Theorem 6 concerning generalized $A$-rings, and "condition I" formulated by Koethe[1] are of course equivalent. As to the specialized $A$-rings, since the discovery of E. Artin that rings with a maximal and minimal condition for the right ideals are specialized $A$-rings, various attempts have been made to generalize this result. However, a *criterion* for specialized $A$-rings along the line of Theorem 6 has not been found as yet.

DEFINITION. *A ring $P$ with a unit is called a completely primary ring, if each right ideal of $P$ other than $P$ is semi-nilpotent.*

REMARK. Replacing semi-nilpotent ideals by nilpotent(nil-)ideals, one obtains a *specialised completely primary* (*generalised completely primary*) ring.

DEFINITION. *A ring $Q$ with a unit is called a primary ring, if each two-sided ideal of $Q$ other than $Q$ is semi-nilpotent.*

REMARK. Replacing semi-nilpotent ideals by nilpotent(nil-)ideals one obtains a *specialised primary* (*generalised primary*) ring.

THEOREM 7. *A primary $A$-ring is isomorphic to a total matric-ring of a finite degree over a completely primary ring. Conversely, the total matric-ring of a finite degree $n$ over a completely primary ring $P$ is a primary $A$-ring.*

PROOF. The proof of the first part of the theorem can be derived by well known methods and may be omitted. The second part follows by known argument if only the following lemma is established: *If in the matrix $(p_{ik})$, $i, k=1, \cdots, n$, the elements $p_{ik}$ are contained in the radical $N^*$ of $P$, then the matrix $\alpha=(p_{ik})$ belongs to the radical $N$ of $Q$.* To prove the lemma, suppose $\alpha$ is not in $N$, then by Theorem 5 we can find matrices $\alpha_1=(p_{ik}^1)$, $\alpha_2=(p_{ik}^2)$, $\cdots$, $\alpha_t=(p_{ik}^t)$ so that the subring $A=\{\alpha\alpha_1, \alpha\alpha_2, \cdots, \alpha\alpha_t\}$ of $Q$ is potent. Evidently, the subring $A^*=\{\cdots, p_{ik}p_{jl}^r, \cdots\}$ (where $i, k, j, l=1, \cdots, n$; $r=1, \cdots, t$) of $P$ is then also potent. Since $p_{ik}\in N^*$, also $p_{ik}p_{jl}^r\in N^*$ which is a contradiction, since $N^*$ is semi-nilpotent.

REMARK. The second part of Theorem 7 is a generalization of an analogous theorem concerning specialized completely primary rings, since one can easily find examples of completely primary rings which are not specialized completely primary rings. It does not seem probable that an analogous theorem could be proved for generalized completely primary rings. This has been attempted by Koethe under the assumption that the generalized completely primary ring $P$ satisfies the minimal condition for the right ideals; however, this condition implies that $P$ is a specialized completely primary ring.

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# ON THE CROSSING OF EXTREMALS AT FOCAL POINTS

L. J. SAVAGE

Morse and Littauer[1] have proved the following theorem for an analytic Finsler space, where $g$ is an extremal transversal to the (analytic) hypersurface $\Sigma$.

THEOREM. *A necessary and sufficient condition that $p$ on $g$ be a focal point of $\Sigma$ is that the family of extremals cut transversaly by $\Sigma$ near $g$ shall fail to cover the neighborhood of $p$ simply.*

The purpose of the present paper is to prove this theorem on the weaker hypothesis that the Finsler space and $\Sigma$ are of class $C'''$.

As pointed out in M. L. the sufficiency of the condition is trivial, and in proving the necessity there is no loss in generality if we assume $p$ to be a first focal point. It is further clear from M. L. that the theorem is a consequence of the following lemma.

LEMMA I. *If $p$ is a first focal point on $g$ contained in a (simply covering) field $R$ of extremals transversal to $\Sigma$, then there exists a first focal point $q$ covered by $R$ and a subfield $S$ of $R$ covering $q$ and such that the Hilbert integral is independent of path for paths confined to $S$.*

Before proceeding to the proof of Lemma I we will establish a secondary lemma.

LEMMA II. *Let $T$ be a transformation of class $C'$ mapping a closed coordinate neighborhood $A$ into a closed Riemannian manifold $B$, then almost all points of $B$ (in the measure theoretic sense) have finite counter images.*

PROOF. Call the set of points $K \subset A$ at which the Jacobian of $T$ vanishes critical points, then I assert that if the counter image $T^{-1}b$, $b \in B$, is infinite, it contains a critical point. In fact if $b^i$ are the coordinates of such a point $b$, there is a convergent sequence of points of $T^{-1}b$ with coordinates $a^i_\nu$ approaching a point $a_0$ from a definite direction, as is expressed by the following set of equations.

$$\begin{gathered} a^i_\nu \to a^i_0, \quad \xi^i_\nu = (a^i_\nu - a^i_0)/(\Sigma_i (a^i_\nu - a^i_0)^2)^{1/2} \to \xi^i_0, \\ T^i(a^j_\nu) = b^i = T^i(a^j_0). \end{gathered} \tag{1}$$

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[1] Marston Morse and S. B. Littauer, *A characterisation of fields in the calculus of variations*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 724–730. This paper will hereafter be designated by M. L.

By the theorem of the mean we have

$$(2)\quad T^i(a_\nu^j) = T^i(a_0^j) + \Sigma_k \frac{\partial T^i(a_0^j + \theta_\nu^i(a^j - a_\nu^j))}{\partial a^k}(a_\nu^j - a_0^j),\ 0 \leqq \theta_\nu^i \leqq 1,$$

finally

$$(3)\qquad \Sigma_k \frac{\partial T^i(a_0^j)}{\partial a^k}\xi_0^k = 0$$

so $a_0 \in K$.

It remains to be shown that $TK$ is of measure zero. But this follows quite easily from the measure theoretic significance of the Jacobian of $T$.

Proof of lemma I. Let $s=(s_1, \cdots, s_{n-1})$ denote coordinates of points of $\Sigma$ relative to some admissible coordinate system of $\Sigma$ covering the intersection of $g$ and $\Sigma$ at $s_0$, and along any geodesic transversal to $\Sigma$ in this neighborhood let $r$ denote (Finsler) length from $\Sigma$ (considered positive in the direction of $p$). Now according to the hypothesis of the lemma if $r_0$ is the arc length along $g$ from $\Sigma$ to $p$, there is a $\delta$ such that for $|s^i - s_0^i| \leqq \delta$ and $-\delta \leqq r \leqq r_0 + \delta$ the numbers $(r, s)$ constitute a closed coordinate neighborhood $U$ of class $C^0$.

Of course the coordinate neighborhood $U$ is not admissible. In fact if $X$ denotes an admissible coordinate neighborhood of $p$ and $P$ the natural transformation from $U$ to $X$, then the fact that $p$ is a focal point is expressed by the vanishing of the Jacobian $\Delta(r,s)$ of $P$ at $r_0$, $s_0$.

We are assuming that $p$ is a first focal point. According to a technique of Morse[2] the locus of first focal points near $p$ is a hypersurface determined by an equation of the form $r=f(s)$. Now I assert that there is a point $q$ with coordinates $r_1=f(s_1)$, such that, in a neighborhood of $q$, $f$ is of class $C'$, and all focal points in that neighborhood lie on the locus $r=f(s)$. To find such a point choose $s_1$ in such a way as to maximize the rank of the determinant $\Delta(f(s), s)$ in the neighborhood of $s_0$. Call this rank $k$. By slightly sharpening a result of Morse[3] we can find a $(k+1)$ minor $\Delta^*(r, s)$ of $\Delta(r, s)$ such that $\partial\Delta^*(f(s), s)/\partial r \neq 0$. From the assumption that the rank of $\Delta(f(s), s)$ is at most $k$ in the neighborhood of $s_1$ it follows that $\Delta^*(f(s), s) \equiv 0$ in that neighborhood, but now the implicit function theorem applies to show that $f$ is of class $C'$ near $s_1$. Finally we conclude, following Morse,[4] that all focal points in the neighborhood of $q$ lie on the locus $r=f(s)$.

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[2] Cf. Marston Morse, *The calculus of variations in the large*, Amer. Math. Soc. Colloquium Publications vol. 18 (1934) p. 235 Lemma 13.1.

[3] Marston Morse, *The order of vanishing of the determinant of a conjugate base*, Proc. Nat. Acad. Sci. U.S.A. vol. 17 (1931) pp. 319–320.

[4] See *Calculus of variations in the large*, loc. cit.

We are now in a position to construct the following set-up. Let $Y$ be an admissible coordinate neighborhood containing $q$ and the extremal of $R$ joining $q$ to $\Sigma$. Let $S$ be a closed subfield of $R$, contained in $Y$, covering $q$ and such that the only focal points of $\Sigma$ within $S$ satisfy $r=f(s)$. Call this intersection of the focal point locus with $S$ $\pi$. Let $V$ denote the sub-neighborhood of $U$ covered by $S$.

Now we must consider the Hilbert integral in the region covered by $S$. This is a certain line integral $I$ to be evaluated with respect to the coordinate system $Y$. The corresponding integral $I^*$ with respect to $V$ turns out according to M. L. to be independent of path, and we would like to make the same conclusion about $I$. The difficulty lies in the fact that though the natural transformation $Q$ from $V$ to $Y$ is of class $C'$, its inverse is differentiable only on the complement of $Q\pi$. We can get around this by approximating arbitrary curves of class $C'$ in $Y \cap S$ in the derivative by rectilinear polygons (with respect to $Y$) which intersect $Q\pi$ in only a finite number of points. This is easily achieved by means of Lemma II, from which we can conclude that if a point is not on $Q\pi$, then almost all rays (with respect to $Y$) issuing from it intersect $Q\pi$ in only a finite number of points. $Q^{-1}$ carries such polygons into curves differentiable at all but a finite number of points. $I^*$ can be extended to these curves by means of Riemann-Cauchy integration. The proof of the lemma is then easily completed.

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# THE NUMBER OF INDEPENDENT COMPONENTS OF THE TENSORS OF GIVEN SYMMETRY TYPE

RICHARD H. BRUCK AND T. L. WADE

Let $T_{i_1 \cdots i_p}$ be an arbitrary covariant tensor with respect to an $n$-dimensional coordinate system, and let

$$T_{i_1 \cdots i_p} = {}_{[p]}T_{i_1 \cdots i_p} + \cdots + {}_{[\alpha]}T_{i_1 \cdots i_p} + \cdots + {}_{[1^p]}T_{i_1 \cdots i_p} \tag{1}$$

represent the decomposition[1,2] of $T_{i_1 \cdots i_p}$ into tensors of various symmetry types, the tensor ${}_{[\alpha]}T_{i_1 \cdots i_p}$ corresponding to the partition $[\alpha]$ of the indices $i_1 \cdots i_p$. The number of independent (scalar) components of $T_{i_1 \cdots i_p}$ is $n^p$; and if $c_\alpha$ denotes the number of components of ${}_{[\alpha]}T_{i_1 \cdots i_p}$, then

$$n^p = c_{[p]} + \cdots + c_{[\alpha]} + \cdots + c_{[1^p]} = \sum c_\alpha. \tag{2}$$

For $p = 2, 3, 4$, J. A. Schouten[3] has obtained expressions for the $c_\alpha$'s in terms of $n$; but the difficulties of his method become great for larger values of $p$. The purpose of this paper is to present a method of obtaining $c_\alpha$ in terms of $n$ from the character table for the symmetric group on $p$ letters.

Associated with the immanant tensor[2] $I^{(i)}_{(j)} \equiv {}_{[\alpha]}I^{i_1 \cdots i_p}_{j_1 \cdots j_p}$ we have defined the numerical invariant $r = r_\alpha$, the rank[4] of $I^{(i)}_{(j)}$, which is the greatest integer $r$ for which the tensor

$$I^{(i_1) \cdots (i_r)}_{(j_1) \cdots (j_r)} = \begin{vmatrix} I^{(i_1)}_{(j_1)} & \cdots & I^{(i_1)}_{(j_r)} \\ \cdot & \cdots & \cdot \\ I^{(i_r)}_{(j_1)} & \cdots & I^{(i_r)}_{(j_r)} \end{vmatrix} \tag{3}$$

does not vanish; here $(i_\lambda) = i_{\lambda 1} \cdots i_{\lambda p}$. For convenience, let us regard $I^{(i)}_{(j)}$, for each $(i)$, as a vector $V_{(j)}$ in $N = n^r$ dimensions. Then from the above definition, it is clear that exactly $r_\alpha$ of the $N$ vectors $V_{(j)}$ are linearly independent. Since ${}_{[\alpha]}T_{(j)} \equiv {}_{[\alpha]}T_{j_1 \cdots j_p}$ may be defined by

$${}_{[\alpha]}T_{(j)} = {}_{[\alpha]}I^{(l)}_{(j)}T_{(l)}; \tag{4}$$

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[1] H. Weyl, *The classical groups*, Princeton, 1939, chap. IV.

[2] T. L. Wade, *Tensor algebra and Young's symmetry operators*, Amer. J. Math. vol. 63 (1941) pp. 645–657.

[3] J. A. Schouten, *Der Ricci-Kalkul*, Berlin, 1924, chap VII.

[4] Richard H. Bruck and T. L. Wade, *Bisymmetric tensor algebra*, II, Amer. J. Math. vol. 64 (1942) pp. 734–753. We shall refer to this paper as B.T.A.II.

exactly $r_\alpha$ of the components of $T_{(j)}$ are linearly independent; thus $c_\alpha \leq r_\alpha$. But as an alternative way of writing equation (32), B.T.A. II,

$$n^p = r_{[p]} + \cdots + r_{[\alpha]} + \cdots + r_{[1^p]} = \sum r_\alpha. \tag{5}$$

Hence, using (2),

$$n^p = \sum c_\alpha \leq \sum r_\alpha = n^p, \tag{6}$$

and since the numbers $c_\alpha$, $r_\alpha$ are non-negative we conclude this fact.

THEOREM I. $c_\alpha = r_\alpha$.

Combining Theorem I with Theorem V, B.T.A. II, we obtain this theorem.

THEOREM II.

$$c_\alpha = \frac{f_\alpha}{p!} \sum_{(\rho)} \chi_\alpha^{(\rho)} \cdot \nu_\rho n^{k_\rho},$$

*where* $\chi_\alpha^{(\rho)}$ *is the characteristic for class* $(\rho)$ *corresponding to the irreducible representation* $[\alpha]$ *of the symmetric group on* $p$ *letters,*

$f_\alpha$ *is the characteristic corresponding to the class* $(1^p)$,

$\nu_\rho$ *is the order of class* $(\rho)$,

$k_\rho = \rho_1 + \rho_2 + \cdots + \rho_p$, *where* $\rho = (1^{\rho_1}, 2^{\rho_2}, \cdots, p^{\rho_p})$.

Another method of finding $c_\alpha$ is given by G. de B. Robinson[5] in relating $r_\alpha$ to A. Young's substitutional analysis.

For $p = 4$ the character table is,[6] with the additional row of values of $k_\rho$ inserted:

| Class: | $(\rho)$ | $(1^4)$ | $(1^2, 2)$ | $(1, 3)$ | $(4)$ | $(2^2)$ |
|---|---|---|---|---|---|---|
| Order: | $\nu_\rho$ | 1 | 6 | 8 | 6 | 3 |
| | $k_\rho$ | 4 | 3 | 2 | 1 | 2 |
| | $[4]$ | 1 | 1 | 1 | 1 | 1 |
| | $[3, 1]$ | 3 | 1 | 0 | $-1$ | $-1$ |
| | $[2^2]$ | 2 | 0 | $-1$ | 0 | 2 |
| | $[2, 1^2]$ | 3 | $-1$ | 0 | 1 | $-1$ |
| | $[1^4]$ | 1 | $-1$ | 1 | $-1$ | 1 |

From this, using Theorem II, we have, for example

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[5] G. de B. Robinson, *Note on a paper by R. H. Bruck and T. L. Wade*, Amer. J. Math. vol. 64 (1942) p. 753.

[6] D. E. Littlewood, *Theory of group characters and matrix representations of groups*, Oxford, 1940, p. 265.

$$c_{[4]} = (1/4!)\{1\cdot 1\cdot n^4 + 1\cdot 6\cdot n^3 + 1\cdot 8\cdot n^2 + 1\ 6\cdot n + 1\cdot 3\cdot n^2\}$$
$$= C_{n+3,4},$$

and

$$c_{[3,1]} = (3/4!)\{3\cdot 1\cdot n^4 + 1\cdot 6\cdot n^3 + 0\cdot 8\cdot n^2 - 1\cdot 6\cdot n - 1\cdot 3\cdot n^2\}$$
$$= 9C_{n+2,4}.$$

In this manner we obtain the following tables of $c_\alpha$:

*Three-indexed tensors*

| $[\alpha]$ | $[3]$ | $[2, 1]$ | $[1^3]$ |
|---|---|---|---|
| $c_\alpha$ | $C_{n+2,3}$ | $4C_{n+1,3}$ | $C_{n,3}$ |

*Four-indexed tensors*

| $[\alpha]$ | $[4]$ | $[3, 1]$ | $[2^2]$ | $[2, 1^2]$ | $[1^4]$ |
|---|---|---|---|---|---|
| $c_\alpha$ | $C_{n+3,4}$ | $9C_{n+2,4}$ | $nC_{n+1,3}$ | $9C_{n+1,4}$ | $C_{n,4}$ |

*Five-indexed tensors*

| $[\alpha]$ | $[5]$ | $[4, 1]$ | $[3, 2]$ | $[3, 1^2]$ | $[2^2, 1]$ | $[2, 1^3]$ | $[1^5]$ |
|---|---|---|---|---|---|---|---|
| $c_\alpha$ | $C_{n+4,5}$ | $16C_{n+3,5}$ | $5nC_{n+2,4}$ | $36C_{n+2,5}$ | $5nC_{n+1,4}$ | $16C_{n+1,5}$ | $C_{n,5}$ |

*Six-indexed tensors*

| $[\alpha]$ | $[6]$ | $[5, 1]$ | $[4, 2]$ | $[4, 1^2]$ | $[3^2]$ | $[3, 2, 1]$ |
|---|---|---|---|---|---|---|
| $c_\alpha$ | $C_{n+5,6}$ | $25C_{n+4,6}$ | $(27n/2)C_{n+3,5}$ | $100C_{n+3,6}$ | $(5/3)C_{n+2,4}C_{n+2,3}$ | $(128n/3)C_{n+2,5}$ |

| $[\alpha]$ | $[3, 1^3]$ | $[2^3]$ | $[2^2, 1^2]$ | $[2, 1^4]$ | $[1^6]$ |
|---|---|---|---|---|---|
| $c_\alpha$ | $100C_{n+2,6}$ | $(5/3)C_{n+1,4}C_{n,3}$ | $(27n/2)C_{n+1,5}$ | $25C_{n+1,6}$ | $C_{n,6}$ |

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# NATIONAL COMMITTEE ON PHYSICISTS AND MATHEMATICIANS

Recent action by the National Headquarters of the Selective Service System recognizes a critical shortage in the field of mathematics and sets up a new procedure whereby a national point of view may be brought to bear on the decisions of local boards in the cases of registrants who are occupied as mathematicians or are students of mathematics. This recent action of National Headquarters is best understood by reference to the historical background.

On November 7, 1942, National Headquarters announced a policy under which the Director of Selective Service might authorize the appointment of a National Committee in a critical scientific or specialized field, to assist the Selective Service System by reviewing affidavits for occupational classification; at the same time the National Committee on Physicists was established. On April 5, 1943, this policy was amended so as to apply to mathematicians as well as to physicists; *the National Committee on Physicists was replaced by the National Committee on Physicists and Mathematicians*. In addition to the two physicists who, with representatives of the National Roster of Scientific and Specialized Personnel and of the public, constituted the earlier Committee, the new National Committee on Physicists and Mathematicians will include two mathematicians.

This procedure involves no change in the fundamental principles underlying the granting of occupational deferment, which remains, in the first instance, a matter for the decision of each registrant's local board. The creation of the National Committee on Physicists and Mathematicians does, however, provide a means whereby the individual local boards will be advised concerning the national as well as the local needs in the critical fields of both physics and mathematics.

In Memorandum F (see Bulletin of the American Mathematical Society, March, 1943) attention was called to amendments made on December 14, 1942, to Occupational Bulletins No. 10 and No. 23. *The amended Bulletins remain in force and should be consulted in preparing requests for occupational classification of teachers or students of mathematics.*[1]

Mathematicians should note the changes in procedure made necessary by the establishment of the National Committee on Physicists and Mathematicians. *The principal one is that the employer is now to file papers with the National Committee as well as with the employee's local board.* (See 3(b) below.) The following detailed outline, it is hoped, will be of value to mathematicians and their employers as the new procedure goes into effect.

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[1] The substance of the amended Bulletins No. 10 and No. 23 is to be found also in Activity and Occupation Bulletins No. 33-5 and No. 33-6, issued on March 1, 1943.

1. Although full responsibility for presenting occupational deferment requests lies with employers,[1] the mathematician-registrant will be wise to see that his employer is fully informed of the new procedure established by Activity and Occupation Bulletin No. 35 as amended on April 5, 1943.

2. The employer must first decide whether he regards the registrant as a "necessary" man for whom he wishes to request occupational deferment (or continued occupational deferment). In the case of men with dependents, even if deferred at present, it is desirable to file information with the local boards establishing their occupational status as further ground for deferred classification.

3. For every registrant for whom occupational deferment (or continued deferment) is to be requested, the employer will:

(a) Fill out *completely* Selective Service Form 42 or 42A in triplicate. File a copy with the local board with a brief letter stating that the original has been sent to the National Committee on Physicists and Mathematicians as authorized in Selective Service Activity and Occupation Bulletin No. 35, issued March 1, 1943, and amended April 5, 1943.

(b) Send the original and one copy of Form 42 or 42A (see also (c) below) to the:

National Committee on Physicists and Mathematicians
National Roster of Scientific and Specialized Personnel
Washington, D. C.

(c) Send in addition to Form 42 or 42A, a "Report of Status" for the National Committee on Physicists and Mathematicians. Copies of these "Report of Status" blanks may be secured from the National Committee on Physicists and Mathematicians.

(d) For those *not now deferred*, this procedure should be followed at once. For those *now deferred*, this procedure should be followed *at least thirty days before the expiration of the deferment period.*

4. For every registrant for whom deferment is not to be requested

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[1] Throughout this discussion the term "employer" means that person within an organization—industrial, educational, or governmental—who has been assigned the responsibility for handling deferment problems. In educational institutions, the head of the mathematics department must be an active adviser in matters involving both staff and students.

by his employer or college, a "Report of Status" in duplicate should be filled out and sent to the National Committee on Physicists and Mathematicians at once so that efforts may be made to place him where his talents and training will be of best service to the war effort.

The plan under which deferment applications for mathematicians are now to be made is announced in Activity and Occupation Bulletin No. 35, as amended on April 5, 1943. Extracts giving the essential features of this plan are published herewith:

"*Filing Affidavit—Occupational Classification.* When such a National Committee has been appointed for a scientific or specialized field, the employer or recognized university or college desiring occupational classification for a registrant who possesses the training, qualification, or skill in that field, or is in training or preparation therefor, may prepare Affidavit—Occupational Classification (Form 42 or 42A) in duplicate, file the copy with the local board in the usual manner, and forward the original to the National Committee. (Part I, C, 3)

"*Action by the National Committee.* When the National Committee receives a Form 42 or 42A, it will investigate the registrant. If, in its opinion, the registrant possesses the training, qualification, or skill and is a necessary man in an essential activity or a necessary man in training or preparation therefor, the National Committee is authorized to place a stamped endorsement prescribed by the Director of Selective Service on the original Form 42 or 42A and to file the form with the registrant's local board. (Part I, C, 4)

"*Consideration by local board.* When the original Form 42 or 42A, stamped by the National Committee, is received by the local board, it shall be considered as showing that the registrant has been investigated, that in the opinion of the National Committee he possesses the training, qualification, or skill required, and that, as the case may be, he is a necessary man in an essential activity or a necessary man in training or preparation in such scientific and specialized field. (Part I, C, 5)

"*Classification Advice* (*Form 59*). When a Form 42 or 42A, stamped by a National Committee, has been received for a registrant and the local board, nevertheless, classifies the registrant as available for military service or for assignment to work of national importance, the local board shall notify the National Committee, as well as the employer or university or college, by mailing to the National Committee and to the employer or university or college Classification Advice (Form 59). In such case the National Committee, as well as the employer or university or college, may appeal to the board of appeal from classification of the registrant. (Part I, C, 6)

# MATHEMATICAL QUESTIONS IN SEISMOLOGY

C. F. RICHTER

Seismologists, like mathematicians, are accustomed to dividing their subject into a pure and an applied branch. The desirability of such division is underlined by the manner in which non-seismologists frequently approach the subject. It is too often taken for granted that, because an earthquake is a motion of the ground, seismology must be chiefly concerned with the detailed analysis of that motion for its own sake. Such analysis is important in the engineering section of applied seismology; but it is unrepresentative of seismology as a whole, which deals largely with more remote problems: the causative mechanism of earthquakes, the distribution of their origins geographically and in depth, or the structure and physical condition of the interior of the earth.

Engineering applications are out of bounds for the present speaker. Briefly, they refer to the design of earthquake-resistant (not "earthquake-proof") structures. The problems differ from those of most structural engineering in being dynamical, not statical; buildings, bridges, and other structures must be designed with reference to their behavior when in violent vibratory motion. Further, horizontal forces must be considered more explicitly than is done in ordinary design, where the chief load forces, due to gravity, are vertical. Problems of principle are few; but the mechanical characteristics of large structures are only less complex than the motion of the ground in an earthquake. To arrive at solvable propositions it is usually necessary to idealize both sets of data to a degree which seriously limits the practical applicability. Consequently, progress has depended heavily on empirical investigation of the behavior of models on a shaking-table.

A very vocal offshoot of seismology is its application to geophysical prospecting—the investigation of subsurface structures by recording, with specially designed seismometers, the artificial earthquakes produced when explosives are detonated. Numerous books and periodicals deal with the problems of "geophysics"—which, in the vocabulary of the profession, means geophysical prospecting; a rather unwarranted restriction of the meaning of a necessary term, long in use with its proper significance as the physics of the earth. These problems present little of individual interest to the mathematician,

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since nearly all are specializations of problems and methods long familiar in connection with earthquakes. As an example, consider an idealization of "the reflection method." An explosion is set off at or near the surface of the earth, at a carefully recorded time. At unknown depth is a reflecting layer, of irregular form, which may be discontinuous in the mathematical sense. The arrival of elastic waves from the explosion, reflected at the named layer, is timed accurately at a number of points on the surface of the earth. Supposing the velocity known and constant, is the determination of arrival times at all points of a limited area (taken as plane) sufficient to determine the depth and form of the reflecting layer? More practically, what is the maximum information derivable from a limited number of observations? In practice, the explosions may be repeated at several different points.

An important division of seismology which as yet offers little hold to the mathematician is that dealing with the fundamental mechanisms of earthquakes—the possible physical origin of stresses giving rise to earthquakes, the mechanism of the actual process of fracturing. These problems belong to the field which has lately taken the name of "tectonophysics"; a section of the American Geophysical Union with that title was organized in 1939. This marks a movement to give greater precision, from the point of view of mathematical physics, to many of the concepts used in geology to describe the behavior of great masses of matter. The subject is being furthered by increasing experimental data on the physical properties of rocks and other material under great pressure. A detailed discussion would involve almost the whole of geophysics, of which seismology is only a small part. Reference may be made to the general manuals in the bibliography, and to the Transactions of the American Geophysical Union.

Theoretical seismology, like other branches of physics, is readily characterized mathematically by the occurrence of certain fundamental differential equations, the theory of which is involved in nearly every problem. These occur in three principal sets:

I. equations of motion of the recording instruments,
II. equations of motion of elastic solid bodies,
III. equations of propagation of elastic waves in the interior of the earth (usually assumed as spherical).

The first are the simplest. The motion of a seismometer is a particular case of the forced oscillations of vibrating systems, the theory of which is classical and well worked out. In its simplest form, the seismometer equation is a linear ordinary differential equation of the second order, with constant coefficients. The nonhomogeneous part represents the disturbance due to an earthquake. The homogeneous term of zero-order determines the free period of oscillation of the seismometer pendulum, and that of first order then gives the damping, or rate of decrease of the free oscillation. Further complication occurs when the seismometer displacement, instead of being recorded directly, operates an electromagnetic system which actuates a recording galvanometer. The free period and damping of the galvanometer then enter into the equations of motion of the coupled system, which in complete form are of fourth order. Since the coefficients are still constant, the problem is theoretically solved; but the practical application may become difficult and laborious if the damping coefficients are large enough to approach their critical values. Devices for clarifying the problem or shortening the computations are much to be desired.

Seismic waves are propagated through the whole earth; systematic investigation employs the data of many hundreds of earthquakes, as recorded at seismological stations in all parts of the formerly civilized world. The observations are affected by all ordinary causes of error, so that statistical treatment is necessary to ensure valid conclusions. Harold Jeffreys, at Cambridge, has devoted much of his distinguished ability for many years to the mathematical problems of geophysics and seismology; the professional mathematician will find no better introduction to these problems than his volume *The earth* [2]. Jeffreys' interest in the theory and technique of statistics as such has also appeared in many publications; in a recent volume [11] he has presented a detailed account of his somewhat unorthodox views. Seismologists have observed with much interest the application to their specialty of these highly refined techniques, which are for the most part beyond the range of their critical judgment. Jeffreys' specific results are mostly in excellent agreement with those of others who have not used his methods. He has sometimes claimed a surprisingly high degree of precision; but any difference of opinion on such points refers, not to the method of handling the data, but to the reliability of the observations as such. Others have attempted to reach the same objective by working so far as possible with original seismograms in preference to readings taken from them, and by selecting the data used for further conclusions through detailed consideration of each item, applying individual criteria not easily derived from a purely statistical treatment. Questions arise belonging not merely to statistics, but to the whole theory of scientific method. Reference may be made to Jeffreys' smaller volume [12], and to the excellent summary in Blake's paper [7].

Seismological data have at least one statistical idiosyncrasy; the errors are not symmetrical. The fundamental datum is the time of arrival of the first of a long succession of elastic waves at the recording station. If this first motion is small it may be missed, and a later time used instead. Early times, on the other hand, can arise only from the normally fluctuating causes of error, except occasionally when a seismologist, anxious not to overlook the true beginning, has measured a small extraneous disturbance preceding it.

The equations of set II are:

$$\rho \frac{\partial^2(u, v, w)}{\partial t^2} = (\lambda + \mu) \operatorname{grad} \Theta + \mu \nabla^2(u, v, w), \tag{1}$$

$$X_x = \lambda\Theta + 2\mu \frac{\partial u}{\partial x}, \quad Y_y = \lambda\Theta + 2\mu \frac{\partial v}{\partial y}, \quad Z_z = \lambda\Theta + 2\mu \frac{\partial w}{\partial z}, \tag{2}$$

$$X_y = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad Y_z = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad Z_x = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial \mu}{\partial z}\right). \tag{3}$$

$u$, $v$, $w$ are the cartesian components of displacement of any elementary particle from its equilibrium position. $\rho$ is the density; $\lambda$ and $\mu$ are the Lamé constants of elasticity. $\Theta$, the dilatation, is an abbreviation for the divergence of the vector $(u, v, w)$. Equation (1), which stands for three, is thus a vector equation; it suffices for the theory of motion of a homogeneous isotropic elastic solid of infinite extent, without boundaries or discontinuities. The remaining equations give the components of the stress tensor in terms of the derivatives of the displacements. These are called for whenever it is necessary to apply boundary conditions.

Consider first the consequences of these classical equations. Generalizations and modifications will be cited later. The usual process of separating the displacement vector into two parts, one of which is irrotational, involves taking the divergence and the curl of equation (1); this yields the two wave equations:

$$\frac{\partial^2\Theta}{\partial t^2} = \frac{\lambda + 2\mu}{\rho} \nabla^2\Theta, \tag{4}$$

$$\frac{\partial^2(\xi, \eta, \zeta)}{\partial t^2} = \frac{\mu}{\rho} \nabla^2(\xi, \eta, \zeta). \tag{5}$$

Here $2(\xi,\eta,\zeta) = \operatorname{curl}\,(u,v,w)$.

This process proves the possible existence of two types of elastic waves. In the first, displacements are purely compressional or dilatational, the displacement vector being in the direction of propagation of the wave. These are physically identical with sound waves. In the second type, displacement may be described as purely shearing, irrotational, or equivoluminal, being transverse to the direction of propagation and consequently exhibiting the phenomena of polarization as in optics. Whether these transverse waves should be called sound waves is a matter of terminology. Both longitudinal and transverse waves are observed in the earth. These are the only types of waves which can exist in an unlimited homogeneous isotropic elastic medium subject to equation (1). As soon as boundaries exist, however, other waves occur.

The simplest waves due to a boundary are the surface waves found by Lord Rayleigh [13]. Let the medium be semi-infinite with a free bounding surface at the plane $z=0$. Then a solution of (1) exists which contains a factor $\exp(az+ibx-ict)$ where $a$, $b$, $c$ are real. This represents a wave propagated in the direction of $x$ with velocity $c/b$; its amplitude decreases exponentially with increasing $z$ if $a$ is negative, so that on the positive side of the boundary the disturbance is practically confined to the vicinity of the free surface. The velocity $c/b$ is determined by requiring the stress components (2) and (3) to vanish at the boundary $z=0$. This leads to a cubic equation in $c^2/b^2$. This equation is obviously characteristically connected with the partial differential equation (1); an exact analysis of this type of relationship might be of much interest.

Since the cubic equation is reached by squaring, some of its roots may be extraneous to the problem. One of the roots is always appropriate, and leads to the solution given by Rayleigh. The case when the other two roots are equal falls well within the admissible numerical range of the physical constants, so that with slightly different possible values these roots may be either both real or both complex. When both are complex, they represent a valid solution of the given equations and boundary conditions only if the amplitude becomes infinite for infinite $z$; this is not of physical interest. When both roots are real, $a$ must be taken as imaginary; the solution is periodic and does not vanish at infinity. The velocities come out greater than those of the normal longitudinal and transverse waves represented by (4) and (5). This at first appears physically improbable. However, it readily appears that these solutions represent trains of plane longitudinal and transverse waves of the normal type and with the normal velocities, approaching the surface at specified angles; the high apparent velocity in the $x$ direction is due to these angles of emergence. The result then appears trivial; it seems that the search for solutions satisfying given conditions has led incidentally to a special case of another known solution. However, there is still a possible question: Does this mean physically that a disturbance below the plane free surface will actually produce, not merely the Rayleigh surface wave and the ordinary body waves with spherical wave front, but also these particular body waves with a different orientation? The same result was obtained by Somigliana [14] less directly, and has been applied by Caloi [15, 16] to explain certain observations of anomalously high velocities. To the present speaker it appears more probable that these observations have another explanation, and that the solutions in question correspond to no new physical phenomenon. The same case arises in the work of Jeans [26] to be cited presently.

The Rayleigh wave is only the simplest type of wave dependent on a discontinuity. Most seismograms show large waves of the type whose theory is due to A. E. H. Love [17]. These exist primarily in the space between two discontinuities; in the most important case the upper of these is the surface of the earth, while the lower is the base of the continental rocks. Similar waves may exist without a discontinuity, if the physical constants change rapidly with depth [18, 19]. This shows that a complete theory, now lacking, of the propagation of elastic waves in a nonhomogeneous but isotropic medium, would include several types which do not occur when the medium is homogeneous and unbounded.

More general boundary conditions have been applied. Stoneley and others [20–25] have worked on the theory of waves associated with an internal interface, and on the effect of multiple layering. Jeans [26] has considered the general problem of free oscillations of a spherically symmetrical earth. Surface waves of Rayleigh and Love types, as well as others, appear as particular solutions; the general solution is not worked through. Lamb, Sezawa, Nakano and others [27–41] have developed the theory of generation of bodily and surface waves from an initial disturbance of given form; this bears directly on the problem of earthquake causation.

Earthquake surface waves show considerable dispersion. The simple Rayleigh theory gives no dispersion, the velocity being fixed; however, the velocity depends on the frequency for Love waves [17, 18] and for Rayleigh waves on the surface of a heterogeneous medium [42–49]. The observed dispersion differs regionally, and is evidence for regional differences in crustal structure [4, chap. XII].

A quite different problem of much importance in the interpretation of seismograms is the following: Plane waves, either longitudinal or transverse, are incident at a given angle on the plane boundary between two media which differ in elastic constants and density. There are four derived plane waves; reflected longitudinal and transverse waves in the first medium, refracted longitudinal and transverse waves in the second medium. What is the ratio of the amplitude of each to that of the incident wave?

This problem was solved by Knott [50], and, a little more directly, by Zoeppritz [51]. The solution takes the form of four linear homogeneous algebraic equations in the amplitudes of the five waves, there being two cases according as the incident wave is longitudinal or transverse. Unfortunately, the coefficients are complicated, containing physical constants and trigonometric functions of the angles. Computation is laborious, and has been carried out only for a limited range of assumptions which approximate the actual conditions met with in seismology [52–54]. Our information is incomplete as to maxima and minima of the various amplitudes as functions of the physical parameters, and even as to the occurrence of zeros and singularities. It would appear that some of these questions might be dealt with directly from the equations of motion and the boundary conditions in differential form; but this has not been done.

On seismograms of earthquakes at distances between 100 and 1000 kilometers, the direct longitudinal wave, $\overline{P}$, is preceded by a longitudinal wave $P_n$, refracted through the material beneath the continental layers in which most earthquakes originate. At that depth it travels so fast that it arrives ahead of the direct wave. Analogous waves refracted horizontally through subsurface layers are observed in geophysical prospecting. Simplified forms of the theory usually indicate that such a horizontally refracted wave should exist as a type of surface wave in the lower medium, but that it should carry only a small part of the incident energy. While the direct wave is in fact larger, the observed refracted wave is usually not so small as the simple theory suggests. More precise discussion, such as that of Jeffreys [55, 56; 57, pp. 219–220] and Muskat [58] still does not satisfactorily represent the observations. Alternative explanations have been looked for. Recently O. von Schmidt [59, 60] has succeeded in producing essentially the same phenomenon in the laboratory. He emphasizes the analogy with the "head wave" in ballistics. A theory has been given by Joos and Teltow [61], but the subject is still incompletely investigated.

In practice, the equations of our set III are identical with the fundamental equations of geometrical optics. The dimensions of the earth and its larger structures are so vast compared with the wave lengths of bodily seismic waves (mostly of the order of ten kilometers) that the observations can be discussed without much attention to phenomena of interference or diffraction. The chief difference from geometrical optics lies in the data, the more significant of which are determinations of time, whereas intensities and frequencies, though observed and used, are of only supplementary importance.

The customary general statement is Fermat's "principle of least time"

$$\delta \int ds/v = 0 \tag{6}$$

where the end points of the line integral are fixed during the variation. $ds$ denotes the element of path, and $v$ the velocity of propagation given as a function of position. The name is slightly misleading. Thus, when a shock originates within the earth and sends waves to a distant surface point, there are two principal paths of singly reflected waves satisfying the variation principle. Of these the more obvious and familiar, reflected nearly midway, corresponds to a maximum of the integral. The other, representing a minimum time, is reflected comparatively near the source; it is observed chiefly in earthquakes originating at great depth. Jeffreys [73, p. 560] has pointed out that in general seismic waves corresponding to maximum times are less sharp and well defined on the seismograms than those with minimum times.

Attention is called to the differential equation associated with the variation principle, and named by Bruns the equation of the "eikonal." For a spherically symmetrical earth it takes the form

$$(\partial t/\partial r)^2 + (1/r^2)(\partial t/\partial \theta)^2 = 1/v^2 \tag{7}$$

where $r$, $\theta$, $\phi$ are polar coordinates with origin at the center and polar axis through the source of the earthquake. Since the velocity $v$ is independent of $\phi$, that coordinate drops out. $t$ is the time of propagation, not along a fixed ray, but always along that ray which forms the path of least time from the source to the point $(r, \theta, \phi)$.

This general equation can be made the basis for a complete discussion of the geometrical optics of seismology. More customarily, the wanted information is derived from a special consequence, namely Snell's law of refraction in its polar form, which states that along a given ray

$$(r \sin i)/v = \text{const.} = r_s/v_s. \tag{8}$$

Here $i$ is the angle between the radius vector and the tangent to the ray. Except at the boundary of the core of the earth, velocity increases with depth (since the elasticity increases more rapidly than the density, which alone would give the opposite effect). Seismic rays are convex toward the center of the earth, and each ray has a minimum radius vector $r_s$ corresponding to $i=90°$; the velocity at this point on the ray is $v_s$.

The initial problem is to determine the velocity distribution within the earth. The observed times at seismological stations at varying angular distances from the source give $t$ as a function of $\theta$ for $r=R$, the radius of the earth. The assumption of spherical symmetry is justified by the satisfactory agreement with all observations of tables giving $t$ as a function of $\theta$ alone. The determination of $v(r)$ from $t(\theta)$ involves a Fredholm integral equation, of the type first solved by Abel [62, 63]. The method developed by Wiechert [64] and Herglotz [65], improved by Bateman [66], leads to integrals which are readily evaluated graphically or with a computing machine. The essential equations are:

$$\cosh q(x, \theta) = t'(x)/t'(\theta), \tag{9}$$

$$\log \frac{R}{r_s} = \frac{1}{\pi}\int_0^{\theta} q(x, \theta)dx, \tag{10}$$

$$r_s = v_s t'(\theta). \tag{11}$$

From the given $t(\theta)$ one proceeds to $t'(\theta)$ and thence to $v_s$ as a function of $r_s$, identical with $v(r)$. General relations between the derivatives and singularities of $v(r)$ and $t(\theta)$ are of importance; few have been worked out.[1] A peculiar singularity is involved, which occurs in all similar problems. To state it simply, consider a source of disturbance at $x=0$, $y=0$, $z=-h$, the velocity $v$ being a function of $z$ only. Let the arrival time on the $x$-axis be given as $t(x)$. Then $t'(x)=0$ for $x=0$, so long as $h$ is different from zero; but if $h=0$ then $t'(x)=1/v$ for $x=0$. Again, if $h$ is not zero, $t''(x)=0$ for $x=0$; but it is not easily seen what the general behavior of the second derivative will be if $h=0$. This apparently artificial problem took on a physical significance in the following way [9, IV; 67, 68]: There is a well observed type of seismic wave which originates as a transverse wave, continues along a short path in the presumably liquid core of the earth as a longitudinal wave, and then emerges from the core as a transverse wave, in which form it reaches the recording stations. Shortening distance between earthquake source and observing station shortens the path in the core, until it finally vanishes and the wave in question ceases to exist. Let $t(\theta)$ now represent the times of arrival at the earth's surface; what should be the behavior of $t''(\theta)$ at the minimum distance? The question bears on the expected amplitude of the given wave at this minimum distance. Considering the path in the core leads to the same singularity as in the simple plane case just described. Dix [69] has shown that the second derivative should vanish except for certain highly singular velocity distributions.

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[1] The speaker gave the result by Gutenberg and Richter [9, II] for $dv_s/dr_s$ in terms of $t'$ and $q$. Subsequent discussion has made possible an adequate discussion of the higher derivatives, to be published in the near future. Correction to [9, II, p. 304]: $dv_s/dr_s=0$ for $I=\pi$, not for $I$ infinite.

Some large scale diffraction phenomena occur. The core of the earth, which has a relatively sharp boundary at a depth of 2900 kilometers, does not transmit transverse waves; and at the boundary the velocity of longitudinal waves drops from about 13 km/sec outside to about 8 km/sec inside; this is the only noteworthy exception to the rule of increasing velocity with depth. There results a shadow zone for both types of waves, within which disturbances diffracted around the core are found on the seismograms of sensitive instruments recording great earthquakes. Further, there is an angle of minimum deviation for the principal longitudinal waves refracted through the core; this angle determines the outer limit of the shadow zone, which accordingly exhibits the phenomenon known in optics as a caustic, with a focussing effect producing large amplitudes in a small range of $\theta$. Waves apparently continuous with the others are observed irregularly within the outer limit. For years these were attributed to diffraction associated with the caustic—more precisely, with its continuation as a caustic surface within the earth; but no exact theory was attempted until quite recently [70]. Meanwhile, it has appeared that the observations are better explained by the existence of an "inner" core, the boundary of which need not be very sharp [71–73].

Many generalizations have been aimed at extending the applicability of the theory. Thus, we have handled "optical" seismology assuming a spherical earth. The effects of ellipticity are definitely observable; indeed, it has been suggested [74, p. 231] that seismological data would suffice for a rough determination of the oblateness of the terrestrial spheroid. A good beginning has been made in these problems [75–79], which fall into three evident stages: a figure-of-the-earth problem to indicate the probable variation in ellipticity with depth for the homogeneous shells constituting the earth; a solution for the propagation of elastic waves through such a body; and a detailed examination of this solution for new phenomena of reflection and refraction not possible with spherical symmetry.

Love [17] and later writers have investigated the effect of gravity on the propagation of seismic waves; this proves to be entirely negligible.

A difficult series of papers by Uller [80] offers a generalization of the wave concept as such, with seismological applications. Further critical discussion is needed. Summaries have been given by Macelwane [3, pp. 110–114) and Gutenberg [1, vol. 4, pp. 136–146].

Some extension of theory is required to account for the frequent observation that the prevailing periods of all types of seismic waves increase with distance. This cannot wholly be due to the normal spreading of an impulsive disturbance with filtering out of the shorter periods. For surface waves, where very long paths are observable, often up to several complete circuits of the earth, such an explanation can be ruled out. Surface waves show dispersion, the theory showing that velocity should vary with period; but this cannot account for the increase with distance of the period of a clearly identifiable wave group. For body waves the corresponding evidence is less conclusive, but convincing to anyone familiar with the data. The required effect will be given by some type of viscosity, or internal friction. Sezawa [81–83] has set up a theory which partially covers the case; the problem has been treated by Jeffreys [84, 85; 2, pp. 265–266], and by Gutenberg and Schlechtweg [86].

Confusion, especially when there is insistence on carrying on the discussion in apparently non-mathematical terms, arises from the use of such terms as "viscosity" and "strength" in different senses. Definition of viscosity for solids is difficult, depending on the form of the hypothetically modified equations of elasticity. The property most closely corresponding to the well known and well defined viscosity of fluids is commonly termed internal friction; the entirely different quantity which measures the slow yielding of solids under long-continued pressure is more usually termed viscosity. For discussion see Gutenberg [4, chap. XV].

The classical theory of elasticity requires the components of stress to be linear functions of those of strain. This first-order approximation is being extended by second-order theories. To this end Murnaghan's theory of finite strain [87] may be applied, assuming "constants" to be modified in a medium under strain, the case of most importance being that of hydrostatic pressure. On this basis Birch [89] has found close agreement between observed and calculated seismic wave velocities in the earth, and has shown [88] that initial hydrostatic pressure does not affect the laws of elastic wave propagation. Biot [90] has developed a general second-order theory, and has recently [91] applied it to the general problem of propagation of elastic waves when there is initial stress; hydrostatic pressure produces no effect, but other types of initial stress may lead, among other effects, to a coupling between longitudinal and transverse waves. The non-effectiveness of hydrostatic pressure restricts the applicability of the old results of Rudzki [92], who assumed the earth to be aeolotropic, with different wave velocities horizontally and vertically. He arrived at the same coupling of longitudinal and transverse waves; but his theory gave no ready means of estimating the magnitude of the supposed seismological effect.

Mathematical seismology is much in need of synthesis. Techniques to solve particular problems have been imported from other applications or from pure mathematics, without much examination of their relation to each other and to the general fields in which they originate. Hence the implications of such special results as the cubic equation characterizing the Rayleigh wave, or the Abelian integral equation in the velocity problem, are often imperfectly understood by those who are constantly working with them. Not rarely this leads to difficulties when the problems are to be generalized. Much spade work remains in exploration of the various subjects suggested by the natural articulation of the data; but the ultimate outcome should be a comprehensive treatment akin to the great treatises on optics and acoustics. To repeat—the contents of those treatises have only a limited application in seismology, not merely because of the complication in combining the mathematical characteristics of optics and acoustics, but largely because the principal observed data are not frequencies and intensities, but actual times of propagation of seismic waves.

It would be ungracious to close without reference to the interesting address by Professor Cairns [6], who has since been a welcome and stimulating visitor at the Pasadena laboratory.

In preparing for publication, much use has been made of the valuable summary by Blake [7]; his discussion and references cover a number of points not included here. The paper by Byerly [8] contains further supplementary material.

The following bibliography is selected for general usefulness or for mathematical interest. Many important papers, particularly those reporting observations, are omitted. Less than justice is done to the contributions by Jeffreys; more will be found in the Geophysical Supplement to the Monthly Notices of the Royal Astronomical Society. (Each volume of the Supplement covers a number of years; dates given are those of publication.) Further papers by Sezawa or by Sezawa and Kanai, as well as by other Japanese authors, will be found in the Bulletin of the Earthquake Research Institute of Tokyo Imperial University. Japanese papers often are of mathematical interest only, by reason either of initial assumptions too remote from physical fact, or of failure to carry the calculations far enough for comparison with observations. All those cited here are in English. Throughout the bibliography, the language of the title as given is that of the original paper.

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# SOME SIMPLE DIFFERENTIAL DIFFERENCE EQUATIONS AND THE RELATED FUNCTIONS

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1. **Introduction.** The work of Brook Taylor, Jean Bernoulli and Daniel Bernoulli led to the formulation of differential difference equations which are all included in the equation

$$\begin{aligned} Md^2u_n/dt^2 &+ 2Kdu_n/dt + Su_n \\ &= (na + a + b)(u_{n+1} - u_n) - (na + c)(u_n - u_{n-1}) \end{aligned} \tag{1.1}$$

in which $M$, $K$, $S$, $a$, $b$, $c$ are constants. This equation may be treated in at least six different ways, which may be described briefly as follows:

(1) The method of simple solutions in which the aim is to express the desired solution as a sum of simple solutions of type

$$u_n(t) = U_n(p) \exp (ipt), \tag{1.2}$$

in which $p$ is a constant which may assume a certain set of values. This method was used with great success by the writers just named and was much improved by L. Euler, J. le Rond d'Alembert, J. L. Lagrange and J. Fourier. The method has been greatly developed during the last one hundred fifty years. Some idea of this development may be derived from the excellent report of H. Burkhardt[1] on expansions in series of oscillating functions.

(2) The method of generating functions in which a differential equation is formed for the generating function

$$G(z, t) = \sum z^n u_n(t).$$

This method may, perhaps, be associated with the names of Lagrange and Laplace as these writers developed a theory of generating functions. The important developments for the differential difference equation came quite late and began, perhaps, with the work of Koppe[2] on the function which I shall call the *influence function* for

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[1] H. Burkhardt, *Entwicklungen nach oscillirenden Funktionen und Integration der Differentialgleichungen der mathematischen Physik*, Jber. Deutschen Math. Verein. vol. 10 (1908) pp. 1–804.

[2] M. Koppe, *Die Ausbreitung einer Erschütterung an der Wellenmaschine durch einen neuen Grenzfall der Besselschen Functionen*, Program Andreas-Real-gymnasium, Berlin, No. 96 (1901) 28 pp. See also T. H. Havelock, *On the Instantaneous propagation of disturbance in a dispersive medium*, Philosophical Magazine (6) vol. 19 (1910) pp. 160–168; E. Schrodinger, *Dynamik elastischer gekoppelte Punktsysteme*, Annalen

the equation

$$d^2u_n/dt^2 = u_{n+1} + u_{n-1} - 2u_n. \tag{1.3}$$

(3) The method of definite integrals in which the aim is to express $u_n(t)$ in the form of a definite integral such as

$$u_n(t) = \int z^n f(z, t)dz.$$

(4) The operational method which depends upon the formation of a difference equation satisfied by the integral

$$U_n(x) = \int_0^\infty e^{-xt}u_n(t)dt.$$

Integrals of this type were used by L. Euler[3] and P. S. Laplace[4] to solve differential equations and difference equations by means of definite integrals. The operational method is really the inverse of the transformation of Euler and Laplace since the equation satisfied by $u_n(t)$ is known, while in the method of transformation the equation satisfied by $u_n(t)$ is derived from an equation satisfied by $U_n(x)$. A form of the operational method was used by Poisson[5] in the solution of two differential difference equations occurring in the theory of the conduction of heat in a bar.

In the case of equation (1.1) the equation for $U_n(x)$ is

$$\begin{aligned}(na + a + b)(U_{n+1} - U_n) - (na + c)(U_n - U_{n-1})\\ + Mu_n'(0) + (2K - Mx)u_n(0) = (Mx^2 - 2Kx + S)U_n.\end{aligned} \tag{1.4}$$

This is a difference equation. It is homogeneous for $n \neq m$ if the initial conditions are of type

$$u_n(0) = 0, \quad n \neq m, \qquad u_m(0) = 1, \qquad u_n'(0) = 0.$$

For $n = m$ the equation is not homogeneous. The solution of the set

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der Physik (4) vol. 44 (1914) pp. 916–934; F. Pollaczek, *Über die Fortpflanzung mechanischer Vorgange in einen linearer Gitter,* Annalen der Physik (5) vol. 2 (1929) pp. 991–1011.

[3] L. Euler, *De constructione aequationum*, Akademiia Nauk, Novi Commentarii vol. 9 (1737) 1744 pp. 85–97; *Integralrechnung* vol. 2 section II, chaps. 10, 11.

[4] P. S. Laplace, *Mémoire sur les approximations des formules qui sont fonctions de très grands nombres*, Mémoires de l'Académie Royale des Sciences de Paris (1782; 1785); *Oeuvres*, vol. 10.

[5] S. D. Poisson, *Mémoire sur la distribution de la chaleur dans les corps solides*, J. École Polytech. vol. 12 (1823) pp. 1–144.

of equations may be made unique by adding suitable end conditions. In some cases the condition that $U_n(x)$ should be finite for $n = \pm\infty$ is appropriate. Thus for the equation corresponding to (1.3) the *influence function* is

$$U_{m,n}(x) = (1+x^2)^{-1/2}[(1+x^2)^{1/2} - x]^{2n-2m}, \qquad n > m.$$

For $n < m$ the index is $2m-2n$. This function $U_{m,n}$ corresponds to Koppe's influence function

$$u_{m,n}(t) = J_{2(n-m)}(2t),$$

and it seems desirable to use the same name in both cases. Recurrence relations which are all of the same type except for one or two values of $n$ are of frequent occurrence in the theory of structures. A case which bears some resemblance to the present one occurs in the work of E. H. Bateman[6] on the stability of tall building frames. The framework considered by this writer is described as a two-column bent of $N$ stories rigidly connected together, the beams being connected to the columns by joints of variable rigidity. This framework is acted on by a horizontal load at any panel point. The cases of vertical loading on the beams is also considered, the two cases of horizontal and vertical loading being discussed separately.

An influence function for equation (1.1) may be found by the method of the generating function. When the differential difference equation is of the first order the requirement that $u_n(0)=0$, $n=m$, $u_m(0)=1$, implies that $G(z, 0)=z^m$ and so the arbitrary factor in the solution of the differential equation for $G$ is determined. The influence function for some simple cases is given in the following table:

1. $du_n/dt = u_{n+1} - u_n$, $t^{m-n}e^{-t}/\Gamma(m-n+1)$,
2. $du_n/dt = u_{n-1} - u_n$, $t^{n-m}e^{-t}/\Gamma(n-m+1)$,
3. $2du_n/dt = u_{n+1} - u_{n-1}$, $J_{m-n}(t)$,
4. $2du_n/dt = u_{n+1} + u_{n-1} - 2u_n$, $e^{-t}I_{m-n}(t)$,
5. $2du_n/dt = u_n - 2u_{n+1} + u_{n+2}$, $e^{t/2}(it^{1/2})^{m-n}H_{m-n}(it^{1/2})/\Gamma(m-n+1)$,

where in the last case

$$\exp(zt - t^2/2) = \sum_{n=0}^{\infty} t^n H_n(z)/n!.$$

$J_n(z)$ denotes the Bessel function of order $n$, $I_n(x)$ is the Bessel function of imaginary argument multiplied by the proper factor, $H_n(z)$ is the polynomial of Hermite and $\Gamma(z)$ is the Gamma function.

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[6] E. H. Bateman, *The stability of tall building frames*, Institute of Civil Engineers, London, Selected Paper No. 167, 1934, 49 pp.

When only the second derivative appears on the left-hand side of the equation the requirement chosen for the definition of the influence function is $u_n'(0)=0$, $u_n(0)=0$, $n\neq m$, $u_m(0)=1$, and two additional cases of interest are:

6. $d^2u_n/dt^2=u_{n+1}-u_n$, $(t/2)^{m-n+1/2}J_{m-n-1/2}(t)/\Gamma(m-n+1)$,
7. $d^2u_n/dt^2=u_{n-1}-u_n$, $(t/2)^{n-m+1/2}J_{n-m-1/2}(t)/\Gamma(n-m+1)$.

(5) The method of power series. In the case of equation 2, for instance, there is a solution of type

$$\begin{aligned} u_n(t) &= f(n) + (t-t_0)[f(n-1)-f(n)] + \cdots \\ &= \sum_{m=0}^{\infty} (t-t_0)^m \Delta^m f(n)/m!, \qquad \Delta f(n) = f(n-1) - f(n), \end{aligned}$$

as noted by G. Doetsch.[7] The function $f(n)$ is given by the equation $f(n)=u_n(t_0)$, and $\Delta^m f(n)=u_n^{(m)}(t_0)$. Doetsch remarks that when $u_n(t_0)=\psi_0(n, t)=e^{-t_0}t_0^n/n!$ we have $u_n^{(m)}(t_0)=(d^m/dt_0^m)\psi_0(n, t_0)=\psi_m(n, t_0)=p_m(n, t_0)\psi_0(n, t_0)$, where $p_m(x, t)$ is Charlier's polynomial. In this case $u_n(t)=\psi_0(n, t)$, and $u_n^{(m)}(t)=\psi_m(n, t)$ is also a solution.

(6) The method of successive approximations. In this method the equation is divided into two parts. One of these is multiplied by a parameter $k$ and a solution is sought which can be expressed as a power series in $k$. The coefficients are determined by a succession of equations which in many cases can be solved very easily. The parameter $k$ is then set equal to 1 after all the operations. In the case of equation 1, for instance, we use the form $u_{n+1}=k(1+d/dt)u_n$ and find that the power series for $u_n(t)$ reduces to a single term $u_n(t)=k^n(1+d/dt)^n f(t)$, where $f(t)$ is an arbitrary function which can be differentiated an unlimited number of times. In the case of equation 6 there is a corresponding solution $u_n(t)=(1+D^2)^n f(t)$ where $D=d/dt$. By choosing $f(t)=t^{-1}\cos t$ we obtain the solution $u_n(t)=-(1/2)\Gamma(n+1)\Gamma(1/2)(t/2)^{-n-1/2}Y_{n+1/2}(t)$, and by choosing $f(t)=t^{-1}\sin t$ we get the solution $u_n(t)=(1/2)\Gamma(n+1)\Gamma(1/2)(t/2)^{-n-1/2}J_{n+1/2}(t)$, the expressions for the Bessel functions being essentially those of Hargrave and Macdonald.[8]

2. **Applications of differential difference equations.** The polynomial

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[7] G. Doetsch, *Die in der Statistik seltener Ereignisse auftretenden Charlierschen Polynome und eine damit zusammenhängende Differentialgleichung*, Math. Ann. vol. 109 (1934) pp. 257–266.

[8] See E. T. Whittaker and G. N. Watson, *Modern analysis*, chap. 17, Ex. 30.

of Jacques Bernoulli $B_n(x)$ satisfies the simple equation

$$(2.1)\qquad d/dx\, B_n(x) = nB_{n-1}(x)$$

which is satisfied also by all polynomials of Appell's type. When $B_n(x)/n!$ is denoted by $u_n(x)$ the equation is

$$(2.2)\qquad du_n/dx = u_{n-1}.$$

This is a particular case of the equation

$$(2.3)\qquad du_n/dt = k_{n-1}u_{n-1} - k_n u_n$$

which occurs in the theory of probability and in the theory of the chain processes of chemistry and radioactivity. The literature on this branch of the subject is now quite large. Equation 2 of §1 is also a special case of equation (2.3). Equation 1 may be regarded as the adjoint of equation 2 and it should be noticed that the influence function for this equation is derived from the influence function for equation 2 by interchanging $m$ and $n$. This property is analogous to one which occurs in the theory of the Green's function for a linear differential equation and in the theory of the influence functions of the theory of structures.

In mechanics and in the theory of elasticity the differential difference equations of chief importance are those in which central differences occur. In his interesting book *Weather prediction by numerical process*, L. F. Richardson emphasizes the advantage of using central differences when difference equations are to be used to furnish approximate solutions of differential equations occurring in the solution of mechanical problems. He was led to this conclusion by some preliminary work on the elastic problem of the masonry dam. An account of this work is given in his paper *The approximate arithmetical solution by finite differences of physical problems*, Transactions of the Royal Society of London vol. 210A (1910) p. 307.

Following Richardson's lead an attempt will be made here to use the method of divided differences in the solution of various elastic problems. For brevity differential difference equations may be called $D\Delta$-equations. This notation, however, may be thought to denote simply the equations of mixed differences studied by Poisson, Biot and others. A simple case of such an equation

$$u'(x) = au(x - \theta) - bu(x)$$

is included in an equation which has become important in economics and has been discussed in several of the recent volumes of the publication Econometrica.

**3. DΔ-equations in the theory of the compound pendulum.** In the case of a pendulum consisting of a light string with equal masses equally spaced and a large mass at the end, the equation of small vibrations is of type

$$
\begin{aligned}
mx_n + Sx = {} & (M + nm)g(x_{n+1} - x_n) \\
& - (M + nm - m)g(x_n - x_{n-1}).
\end{aligned} \tag{3.1}
$$

For greater generality a spring has been added so that each mass when displaced is acted upon by a restoring force proportional to the displacement $x_n$.

In the case of a periodic displacement $x_n(t) = X_n \cos(pt)$ the difference equation satisfied by $X_n$ is of type

$$
(n + s + 1)X_{n+1} + (n + s)X_{n-1} = (2n + 1 + r - kp^2)X, \tag{3.2}
$$

where $r$ and $k$ are positive constants.

The equation of motion of the mass $M$ is simply

$$
Mx_0 + Sx_0 = Mg(x_1 - x_0)
$$

and so if $x_0 = X_0 \cos(pt)$ the equation connecting $X_1$ and $X_0$ is

$$
(s + 1)X_1 = (1 + r - s - kp^2)X_0.
$$

If $kp^2 = z$ the quantity $X_n$ is a polynomial of degree $n$ in $z$.

In the more general case in which the spring attached to the mass $M$ is of different strength $S'$, the equation connecting $X_1$ and $X_0$ is of type

$$
(s + 1)X_1 = (q - kp^2)X_0, \tag{3.3}
$$

where $q$ is a constant and $X_n$ is again a polynomial of degree $n$ in $z$.

When the string is regarded as of infinite length, the solution of the initial value problem may be solved by finding a weight function $w(z)$ such that

$$
\int_0^\infty w(z)X_n(z)X_{n'}(z)dz = \begin{cases} 0, & n' \neq n, \\ 1, & n' = n, \end{cases} \tag{3.4}
$$

for then if $x_n(0) = 0$, $n \neq n'$, $x_{n'}(0) = 1$, $x_n'(0) = 0$, we have the formula for the influence function[9]

$$
x_{n',n}(t) = \int_0^\infty w(z)X_n(z)X_{n'}(z) \cos(pt)dz = x_{n,n'}(t). \tag{3.5}
$$

---

[9] The function $x_{n',n}(t)$ has also been called a *selective function*, see H. Bateman, *Selective functions and operations*, Amer. Math. Monthly vol. 41 (1934) pp. 556–562.

The function $w(s)$ often may be found by solving the problem of the moments. Since $X_0$ is a constant which may be taken to be unity the first equation is

$$\int_0^\infty w(s)ds = 1.$$

The recurrence relation may be used to find $X_1, X_2, \cdots$, successively, and as these are polynomials in $s$ the orthogonal relation, with $n'=0$, will give the values of the integral

$$\int_0^\infty s^n w(s)ds, \qquad n = 1, 2, 3, \cdots.$$

In most cases of physical interest the function $w(s)$ is positive for values of $s$ exceeding a certain number $a$ which is either zero or a positive quantity. In the simple case of a pendulum without springs and without the mass $M$ at the end, the function $w(s)$ is $\exp(-s)$ and $X_n = L_n(s)$ where $L_n(s)$ is the polynomial of Daniel Bernoulli,[10] J. L. Lagrange and E. Laguerre. When the pendulum is suspended from a point at which a mass would ordinarily be concentrated the number $p$ is given by an equation of type $L_N(s)=0$, and the initial value problem may be solved by a method indicated by J. L. Lagrange[11] and Bottema[12] in which the first essential step is the derivation of an orthogonal relation of type

$$\sum_{n=0}^{N} L_n(s)L_n(s') = 0, \qquad s \neq s', \tag{3.6}$$

where $s$ and $s'$ are different roots of the equation $L_N(s)=0$. The displacement $u_n(t)$ is then expressed in the form

$$u_n(t) = \sum_{m=1}^{N} [c_m L_n(s_m) \cos (p_m t) + b_m L_n(s_m) \sin (p_m t)], \tag{3.7}$$

the coefficients $b_m$ and $c_m$ being derived by means of the orthogonal relation. The initial value problem may be solved also by means of the method used for the infinite string if the differential equation is used to continue the values of the initial displacements and velocities to the fictitious masses beyond the point of suspension of the string. This method is generally laborious and may not give simple expressions for the displacement and velocity of the fictitious particles far away from the point of suspension.

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[10] D. Bernoulli, Akademiia Nauk, Novi Commentarii vol. 6 (1734) p. 108, vol. 7 (1935) p. 162.

[11] J. L. Lagrange, *Méchanique analitique*, Part 2, Paris, 1788; *Oeuvres*, vol. 11, Paris, 1888, section 6. See also *Miscellanea Taurinensis*, vol. 3 (1762–1765), *Oeuvres*, vol. 1, p. 534. Lord Rayleigh, *Theory of sound*, vol. 1 (1877) p. 129.

[12] O. Bottema, *Die Schwingungen eines zusammengesetzten Pendels*, Jber. Deutschen Math. Verein. vol. 42 (1933) pp. 42–60.

It should be remarked that when the solution for initial velocities without initial displacement has been found, the corresponding solution for initial displacements without initial velocities may be derived by differentiation with respect to $t$ in the manner described by the rule of Stokes for the initial value problem for a differential equation of the second order with respect to the time. d'Alembert's solution of the equation $d^2u/dt^2 = d^2u/dx^2$, $u = 2^{-1}f(x+t) + 2^{-1}f(x-t) + (1/2)\int_{x-t}^{x+t} g(z)dz$, furnishes a good example for the explanation of the rule. The integral represents the solution of the initial value problem $u=0$, $du/dt = g(x)$ and when it is differentiated with respect to $t$ it gives the function $2^{-1}g(x+t)+2^{-1}g(x-t)$, which is the solution of the initial value problem $u = g(x)$, $du/dt = 0$. Replacing $g$ by $f$ in this expression and adding to the integral we have the solution of the general initial value problem $u=f$, $du/dt = g$.

4. **Derivation of asymptotic forms.** The operational method is often useful for the derivation of an asymptotic form of the influence function by means of a method developed by A. Haar,[13] G. Doetsch[14] and others. This method is the analogue for integrals of the method of Darboux for finding an estimate for large values of $n$ of the coefficient of $z^n$ in the expansion of a function in a power series with a finite radius of convergence. The method of Darboux is based on a study of the behavior of the function around the singularities on the circle of convergence. In the case of an integral of Laplace's type a straight line parallel to the imaginary axis is the boundary of the region of convergence and it is necessary to consider the form of the function represented at the singularities which lie on this line. For example, in the case of the integral

$$(4.1) \qquad \int_0^\infty e^{-xt} J_n(t)dt = (1+x^2)^{-1/2}[(1+x^2)^{1/2} - x]^n, \quad R(x) \geqq 0,$$

there are singularities at $x = \pm i$ on the line $R(x) = 0$ and it is readily found that if $s = 2n+1$

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[13] A. Haar, *Über asymptotische Entwicklungen von Funktionen*, Math. Ann. vol. 96 (1926) pp. 69–107.

[14] G. Doetsch, *Ein allgemeines Prinzip der asymptotischen Entwicklung*, J. Reine Angew. Math. vol. 167 (1932) pp. 274–293.

$$J_n(t) \sim (2\pi t)^{-1/2}(e^{(it-in\pi/4)} + e^{(in\pi/4-it)})$$
$$= (2/\pi t)^{1/2} \cos\left[t - (n + 1/2)\pi/2\right].$$

Similarly the relation

$$\int_0^\infty e^{-xt}dt \int_0^t J_n(z)dz = x^{-1}(1 + x^2)^{-1/2}[(1 + x^2)^{1/2} - x]^n, \quad R(x) \geqq 0,$$

gives the estimate

$$\int_0^t J_n(z)dz \sim 1 + (2\pi t)^{-1/2} \sin\left[t - (n + 1/2)\pi/2\right].$$

These estimates are useful in the discussion of the solution of equation (1.3) which is given by the series

$$(4.2) \quad u_n(t) = \sum_{m=-\infty}^{\infty} J_{2(n-m)}(2t)u_m(0) + \sum_{m=-\infty}^{\infty} u_m'(0) \int_0^t J_{2(n-m)}(2s)ds.$$

When $t$ is large each term in the second series gives a finite residual disturbance. These residual disturbances would be eliminated eventually in the case of a finite string by reflections at the ends. The existence of a residual displacement is worth noting, however, as in this respect equation (1.3) differs from its generalization to 2 or more variables of type $n$.

When only one particle has an initial displacement or velocity it is necessary, as Koppe points out, to find an estimate of $J_{2n}(2t)$ and its integral for values of $2n$ and $2t$, which are of about the same order of magnitude. Some results of this nature were already known but Koppe indicated the advantage of making a substitution of type $x-n=v(n/6)^{1/3}$ when discussing the form of $J_n(x)$ for large values of $n$ and $x$.

The investigations relating to the asymptotic form of $J_n(x)$ began when Lagrange published his solution of Kepler's problem in the form of a sine series with coefficients of the form $J_n(ne)$ and analogous series were obtained by Bessel, Poisson, Hansen and others. Asymptotic forms of the coefficients were needed to decide the question of convergence and notable work was done by Laplace, Poisson, Carlini, Jacobi, Cauchy and Hansen. A second period of investigation began when Debye, Watson and others used the method of steepest descent to estimate the values of definite integrals representing Bessel functions, and methods based on the transformation of differential equations and integrals were employed by Nicholson, Langer, Van Veen, Fock and other writers. Estimates of the integral of the function

$J_n(x)$ have been given by Pollaczek. Some of the results of these investigations are summarized below, the chief terms of the estimates are the only ones given.

$$x = n \operatorname{sech} a, \qquad a > 0,$$

$$J_n(x) \sim (2\pi n \tanh a)^{-1/2} e^{-n(a-\tanh a)},$$

$$\int_0^x J_{n-1}(w)dw \sim (2\pi n \tanh a)^{-1/2}(\coth a + 1)e^{-n(a-\tanh a)},$$

$$x = n \sec b, \qquad 0 \leqq b \leqq \pi/4,$$

$$J_n(x) \sim (1/2)\pi n \tan b)^{-1/2} \cos \left[n(\tan b - b) - \pi/4\right],$$

$$\int_0^x J_{n-1}(w)dw \sim 1 + ((1/2)\pi n \tan b)^{-1/2} \operatorname{cosec} b$$

$$\cdot \cos \left[n(\tan b - b) + b - 3\pi/4\right],$$

$$J_n(x) \sim (1 - b \cot b)\left[\cos (\pi/6)J_{1/3}(n \tan b - nb)\right.$$

$$\left. - \sin (\pi/6)Y_{1/3}(n \tan b - nb)\right].$$

If $|x-n| \ll n^{1/3}$ Nicholson's estimate is

$$J_n(x) \sim (6/x)^{1/3}Ai\left[(6/x)^{1/3}(x - n)\right], \qquad Ai(w) = \int_0^\infty \cos (t^3 - wt)dt,$$

and when $x-n=v(n/6)^{1/3}$ Pollaczek's estimate is

$$\int_0^x J_{n-1}(w)dw \sim (1/2\pi i) \int_C e^{-t^3+vte^{\pi i/3}}dt/t.$$

For these results reference may be made to Burkhardt's report,[1] Watson's book,[15] Pollaczek's paper[3] and the papers of Langer[16] and Fock.[17]

The following table shows that the maximum of $J_{2n}(2t)$ occurs at a time slightly greater than $t=n$.

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[15] G. N. Watson, *Bessel functions*, Cambridge University Press, 1922.

[16] R. E. Langer, *On the asymptotic solutions of ordinary differential equations with an application to the Bessel functions of large order*, Trans. Amer. Math Soc. vol. 33 (1931) pp. 23–64; *On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large complex order*, Trans. Amer. Math. Soc. vol. 34 (1932) pp. 447–480.

[17] V. Fock, *Neuer asymptotischer Ausdruck fur Besselsche Funktionen*, C. R. (Doklady) Acad. Sci. U.R.S.S. (2) vol. 1 (1934) pp. 97–102, see also A. Svetlov, *Ueber die asymptotischen Ausdrucke für Besselsche Funktionen bei grossen Indexen*, ibid. (2) vol. 2 (1934) p. 448.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $n$ | | | | | | | |
| 0 | 1 | .2239 | −.3971 | | | | |
| 1 | 0 | .3528 | .3641 | −.2429 | −.1130 | | |
| 2 | | .0340 | .2811 | .3576 | −.1054 | | |
| 3 | | | .0491 | .2458 | .3376 | −.0145 | |
| 4 | | | | .0565 | .2235 | .3179 | |
| 5 | | | | | .0608 | .2075 | .3005 |

5. **Surges in springs and connected systems of springs.** The use of concentrated masses has been found advantageous in the study of surges in springs such as those of spiral or helical type. Reference may be made to the bibliography in an admirable paper on the subject by K. J. De Juhasz.[18]

An important equation which is a simple generalization of (1.3) is obtained when the concentrated masses on a light string are mounted on springs arranged either along a straight line or on the circumference of a circle or helix. The simplest form of this equation is

$$d^2u_n/dt^2 = u_{n+1} + u_{n-1} - 2(1 + k)u_n, \tag{5.1}$$

where $k$ is a positive constant. If $G(z,t) = \sum_{n=-\infty}^{\infty} z^n u_n(t)$ is the generating function $d^2G/dt^2 = (z^{-1}+z-2k-2)G$ and if $u_n(0)=0$, $n \neq 0$, $u_0(0)=1$, $u_n'(0)=0$, an appropriate solution is

$$G(z, t) = ch\left[t(z^{-1} + z - 2k - 2)^{1/2}\right]. \tag{5.2}$$

It is readily seen that $\int_0^\infty \exp(-xt)G(z, t)dt = x(x^2+2k+2-z-z^{-1})^{-1}$. For sufficiently large values of $R(x)$

$$\int_0^\infty e^{-xt}u_n(t)dt = x\left[(x^2 + 2k + 2)^2 - 4\right]^{-1/2}\left[k + 1 + x^2/2 - y\right]^n, \tag{5.3}$$

where $y^2 = (k+1+x^2/2)^2 - 1$.

An application of Haar's method indicates that for large positive values of $t$

$$\begin{aligned} u_n(t) \sim (2\pi t)^{-1/2}\Big[&(2k)^{1/4}\cos\left\{t(2k)^{1/2} + \pi/4\right\} \\ &+ (2k + 4)^{1/4}\cos\left\{t(2k + 4)^{1/2} - \pi/4 - n\pi\right\}\Big]. \end{aligned} \tag{5.4}$$

It should be noticed that we also have the equation

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[18] K. J. De Juhasz, *Graphical analysis of surges in mechanical springs*, Journal of the Franklin Institute vol. 226 (1938) pp. 505–526, 631–644. The free vibrations of helical springs are treated by the same author in vol. 227 (1939) pp. 647–672.

$$(5.5)\qquad \int_0^\infty e^{-ut^2}u_n(t)dt = (1/2)(\pi/u)\exp(-(k+1)/2u)I_n(1/2u),$$

but this expression for the Laplace integral has an essential singularity at the point $u=0$ and is not so convenient for the deduction of the asymptotic form of $u_n(t)$. An extension of Haar's method is needed which will make the deduction easy in such a case. In applying Haar's method to equation (5.3) it is advisable to approach each of the four singularities, $x=i(2k+4)^{1/2}$, $x=i(2k)^{1/2}$, $x=-i(2k)^{1/2}$, $x=-i\cdot(2k+4)^{1/2}$, from the right along a line parallel to the real axis. The limiting forms of the arguments of the factors of $[(x^2+2k+2)^2-4]^{-1/2}$ may then be written down for each line of approach without any error.

It is important to notice that in the present case we find by putting $x=0$ in equation (5.3) that

$$(5.6)\qquad \int_0^\infty u_n(t)dt = 0.$$

Hence there is no residual displacement when all the masses are originally in the equilibrium position and the mass numbered 0 is given an initial velocity. The displacement $v_n(t)$ at time $t$ is in fact $v_n(t)=\int_0^t u_n(s)ds$ and this tends to zero as $t\to\infty$. This result is quite different from that in Koppe's case when $\int_0^\infty J_{2n}(2s)ds=1/2$.

6. **The equations of Born and Kármán.** In their work on crystal lattices in vibration Born and Kármán[19] considered equation (1.3) and also a pair of equations which may be written in the form

$$(6.1)\qquad \begin{aligned} d^2x_{2n}/dt^2 &= a^2(x_{2n+1}+x_{2n-1}-2x_{2n}),\\ d^2x_{2n+1}/dt^2 &= b^2(x_{2n+2}+x_{2n}-2x_{2n+1}). \end{aligned}$$

If the initial conditions are $x_{2n}(0)=0$, $n\neq m$, $x_{2n+1}(0)=1$, $x_{2n+1}(0)=0$, $x_n'(0)=0$, there are two generating functions,

$$(6.2)\qquad E(z,t)=\sum_{n=-\infty}^{\infty}x_{2n}(t)z^{2n},\qquad O(z,t)=\sum_{n=-\infty}^{\infty}x_{2n+1}(t)z^{2n+1},$$

which satisfy the equations

$$(6.3)\qquad \begin{aligned} d^2E/dt^2 &= a^2(z+z^{-1})O-2a^2E,\\ d^2O/dt^2 &= b^2(z+z^{-1})E-2b^2O. \end{aligned}$$

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[19] M. Born and Th. v. Kármán, *Ueber Schwingungen in Raumgittern*, Physikalische Zeitschrift vol. 13 (1912) pp. 297–309.

If $m=0$,

$$d^2 = a^2b^2(z+z^{-1})^2 + (a^2-b^2)^2,$$

$$d-(a^2+b^2) = c^2, \qquad d+a^2+b^2 = k^2,$$

$$E(z, t) = [(d+b^2-a^2)/2d]ch(ct) + [(d+a^2-b^2)/2d]\cos(kt),$$

$$O(z, t) = [b^2(z+z^{-1})/ad][ch(ct) - \cos(kt)].$$

It is advantageous to write

$$\begin{aligned} x_{2n}(t) &= u_{2n}(t, a, b) + (b^2-a^2)v_{2n}(t, a, b), \\ x_{2n+1}(t) &= b^2v_{2n}(t, a, b) + b^2v_{2n+2}(t, a, b), \end{aligned} \tag{6.4}$$

where $u_{2n}$ and $v_{2n}$ are symmetric functions of $a$ and $b$. The generating functions of $u_{2n}(t)$ and $v_{2n}(t)$ are respectively

$$\begin{aligned} U(z, t) &= \sum_{n=-\infty}^{\infty} z^{2n}u_{2n}(t, a, b) = (1/2)ch(ct) + (1/2)\cos(kt), \\ V(z, t) &= \sum z^{2n}v_{2n}(t, a, b) = (1/2d)[ch(ct) - \cos(kt)], \end{aligned} \tag{6.5}$$

and if $U_n(u, a, b) = \int_0^\infty e^{-ut}u_n(t, a, b)dt$, $V_n(u, a, b) = \int_0^\infty e^{-ut}v_n(t, a, b)dt$,

$$U_n(u, a, b) = (u^2+a^2+b^2)D^nE, \qquad V_n(u, a, b) = D^nE, \tag{6.6}$$

where $2abD = (u^2+2a^2)^{1/2}(u^2+2b^2)^{1/2} - (u^4+2a^2u^2+2b^2u^2)^{1/2}$, $E = [(u^2+2a^2)(u^2+2b^2)(u^2+2a^2+2b^2)]^{-1/2}$. There is also a relation

$$\int_0^\infty e^{-ut}u_n(t, a, b)dt/t = D^n/2n. \tag{6.7}$$

Use of the multiplication theorem for integrals of Laplace's type gives the addition formula

$$\int_0^t v_m(t-s, a, b)u_n(s, a, b)ds/s = v_{m+n}(t, a, b)/2n.$$

The differential equations satisfied by the functions $u_{2n}(t, a, b)$, $v_{2n}(t, a, b)$ are

$$\begin{aligned} d^2v_{2n}/dt^2 &= u_{2n} - (a^2+b^2)v_{2n}, \\ d^2u_{2n}/dt^2 + (a^2+b^2)d^2v_{2n}/dt^2 &= a^2b^2(v_{2n+2} + v_{2n-2} - 2v_{2n}). \end{aligned} \tag{6.8}$$

Elimination of $u$ or $v$ gives the result that $u$ and $v$ both satisfy the differential difference equation

$$d^4u_{2n}/dt^4 + 2(a^2+b^2)d^2u_{2n}/dt^2 = a^2b^2(u_{2n+2} + u_{2n-2} - 2u_{2n}).$$

The behavior of $u_{2n}$ and $v_{2n}$ for large values of $t$ may be found by

Haar's method. The result is that if $b>a>0$, $m=0$,

$$
\begin{aligned}
u_{2n}(a, b, t) = {} & 2^{-3/4}(\pi t)^{-1/2}\big[(b^2-a^2)^{1/2}\big\{a^{-1/2}b^{-1} \\
& \cdot\cos\,[at2^{1/2}-(2n+1/2)\pi/2]+a^{-1}b^{-1/2} \\
& \cdot\cos\,(bt2^{1/2}-(2n-1/2)\pi/2)\big\}+((a^2+b^2)^{3/4}/ab) \\
& \cdot\cos\,\{t(2a^2+2b^2)^{1/2}-\pi/4\}\big], \\
v_{2n}(a, b, t) \sim {} & 2^{-3/4}(\pi t)^{-1/2}\big[(1/ab(a^2+b^2)^{1/4}) \\
& \cdot\cos\,\{t(2a^2+2b^2)^{1/2}-5\pi/4\}+(1/ab^{1/2}(b^2-a^2)^{1/2}) \\
& \cdot\cos\,\{t(2b^2)^{1/2}-3\pi/4-n\pi\}+(1/b(a)^{1/2}(b^2-a^2)^{1/2}) \\
& \cdot\cos\,\{t(2a^2)^{1/2}-\pi/4-n\pi\}\big].
\end{aligned}
\tag{6.9}
$$

Asymptotic forms for $x_{2n}(t)$ and $x_{2n+1}(t)$ were worked out by one of my students, C. C. Lin, for the case $b>a$, with the aid of expressions for $\int_0^\infty e^{-ut}x_{2n}(t)dt$, $\int_0^\infty e^{-ut}x_{2n+1}(t)dt$. The results agree with those derived from the foregoing formulae. The case $a>b>0$ seemed more difficult but the difficulties are avoided in the present method, as use is made of the symmetry of $u_{2n}$ and $v_{2n}$ in $a$ and $b$. Lin's result is that if $b^2>a^2$, $c=[2(a^2+b^2)]^{1/2}$,

$$
\begin{aligned}
x_{2n}(t) &\sim (2/\pi t)^{1/2}\big[(a/b(c)^{1/2})\cos\,(ct-\pi/4) \\
&\qquad + ((b^2-a^2)^{1/2}/b(a(2)^{1/2})^{1/2})\cos\,(a(t)^{1/2}-\pi/4-n\pi+m\pi)\big], \\
x_{2n+1}(t) &= (2b/a)(2/\pi ct)^{1/2}\cos\,(ct-\pi/4).
\end{aligned}
$$

Asymptotic forms are needed also for the case in which $ct$ and $2n$ are both large and of the same order of magnitude. Differential equations for $u_{2n}(t)$ and $v_{2n}(t)$ may be derived from those satisfied by their Laplace transforms, but the equations are of high degree and the work of deriving the asymptotic forms from the differential equations has not yet been attempted.

**7. Derivation of the solution of an initial value problem with the aid of Poisson's formula for a solution of the wave-equation.** Poisson's formula[20] for a function $w(x, y, z, t)$ which satisfies the wave equation $\partial^2w/\partial t^2=k^2(\partial^2w/\partial x^2+\partial^2w/\partial y^2+\partial^2w/\partial z^2)$ is

$$
4\pi w(x, y, z, t) = d/dt\left[t\int_0^\pi\int_0^{2\pi} dSf(kt)\right] + t\int_0^\pi\int_0^{2\pi} dSg(kt), \tag{7.1}
$$

---

[20] S. D. Poisson, *Mémoire pour la propagation du mouvement dans les milieux élastiques*, Mémoires de l'Académie des sciences vol. 10 (1823).

where $f(kt)$ is an abbreviation for the result of replacing $a$, $b$, $c$ by $kt \sin\theta\cos\phi$, $kt\sin\theta\sin\phi$, $kt\cos\theta$ respectively in the function

$$f(x+a,\ y+b,\ z+c) \tag{7.2}$$

and $g(kt)$ has a similar meaning. The symbol $dS$ stands for the elementary solid angle $\sin\theta d\theta d\phi$.

It should be noticed that the function (7.2) is a solution of the 3 partial differential equations

$$\partial^2 f/\partial x^2 = \partial^2 f/\partial a^2, \qquad \partial^2 f/\partial y^2 = \partial^2 f/\partial b^2, \qquad \partial^2 f/\partial z^2 = \partial^2 f/\partial c^2. \tag{7.3}$$

We now add subscripts $p$, $q$, $r$ and assume the function $f$ to be a solution of the additional equations

$$\begin{aligned} \partial^2 f_p/\partial a^2 &= f_{p+1} + f_{p-1} - 2f_p, \\ \partial^2 f_q/\partial b^2 &= f_{q+1} + f_{q-1} - 2f_q, \\ \partial^2 f_r/\partial c^2 &= f_{r+1} + f_{r-1} - 2f_r. \end{aligned} \tag{7.4}$$

The function $w(x, y, z, t)$ then becomes $w_{p,q,r}(x, y, z, t)$ and is a solution of the differential difference equation

$$\begin{aligned} d^2 w_{p,q,r}/dt^2 = k^2(&w_{p+1,q,r} + w_{p-1,q,r} + w_{p,q+1,r} + w_{p,q-1,r} \\ &+ w_{p,q,r+1} + w_{p,q,r-1} - 6w_{p,q,r}). \end{aligned} \tag{7.5}$$

The proof is quite simple because, on account of the equations (7.4), (7.3) satisfied by the function $f_{p,q,r}(x+a, y+b, z+c)$, the right-hand side of (7.5) becomes, by virtue of (7.1), the sum of two integrals the first of which has in the integrand

$$k^2(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2) \qquad \text{(subscripts } p,\ q,\ r \text{ omitted)},$$

while the second contains a similar expression derived from the function $g_{p,q,r}(x+a, y+b, z+c)$, which is supposed to satisfy the same two sets of equations as $f_{p,q,r}$. When, however, $a$, $b$, and $c$ have the assigned expressions involving $t$, $\theta$, and $\phi$ it is a simple consequence of the properties of Poisson's integral that the resulting expression is equal to $\partial^2 w_{p,q,r}/\partial t^2$.

In the particular case in which

$$\begin{aligned} &f_{p,q,r}(x+a,\ y+b,\ z+c) \\ &\qquad = J_{2(p-l)}(2x+2a)J_{2(q-m)}(2y+2b)J_{2(r-n)}(2z+2c) \end{aligned}$$

and $g_{p,q,r}(x+a, y+b, z+c)$ has a similar form we obtain a solution which furnishes a solution of the initial value problem for the equation (7.5). Putting $x=0$, $y=0$, $z=0$ we write

$$
(7.6)\qquad \begin{aligned} 4\pi I_{p,q,r;l,m,n}(t) = \frac{d}{dt}\biggl[t\int_0^\pi\int_0^{2\pi} &\sin\theta\,d\theta d\phi J_{2(p-l)}(2kt\sin\theta\cos\phi)\\ &\cdot J_{2(q-m)}(2kt\sin\theta\sin\phi)J_{2(r-n)}(2kt\cos\theta)\biggr]. \end{aligned}
$$

In the two-dimensional case in which $w$ is independent of $r$ the equation for $w_{p,q}$ is

$$
(7.7)\qquad d^2w_{p,q}/dt^2 = k^2(w_{p+1,q} + w_{p-1,q} + w_{p,q+1} + w_{p,q-1} - 4w_{p,q}),
$$

and the influence function is given by

$$
(7.8)\qquad \begin{aligned} 4\pi I_{p,q;l,m}(t) = \frac{d}{dt}\biggl[t\int_0^\pi\int_0^{2\pi} &\sin\theta\,d\theta d\phi J_{2(p-l)}(2kt\sin\theta\cos\phi)\cdot\\ &\cdot J_{2(q-m)}(2kt\sin\theta\sin\phi)\biggr]. \end{aligned}
$$

A change of axes indicates that there is a second formula

$$
\begin{aligned} 4\pi I_{p,q;l,m}(t) = d/dt\biggl[t\int_0^\pi\int_0^{2\pi} &\sin\theta\,d\theta d\phi J_{2p-2l}(2kt\sin\theta\cos\phi)\\ &\cdot J_{2q-2m}(2kt\cos\theta)\biggr]. \end{aligned}
$$

The integration with respect to $\phi$ can be effected and we get

$$
(7.9)\quad I_{p,q;l,m}(t) = \frac{1}{2}\frac{d}{dt}\left[t\int_0^\pi \sin\theta\,d\theta J^2_{p-l}(kt\sin\theta)J_{2q-2m}(2kt\cos\theta)\right].
$$

This is the simplest expression that has been obtained for the function $I$. The method of the generating function indicates that

$$
(7.91)\quad x^ly^m ch\left[kt(x + x^{-1} + y + y^{-1} - 4)^{1/2}\right] = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} I_{p,q;l,m}(t)x^py^q.
$$

If we write

$$
(x + x^{-1} + y + y^{-1} - 4) = (x^{1/2} - x^{-1/2})^2 + (y^{1/2} - y^{-1/2})^2,
$$

a power series for $I$ may be obtained. On the other hand, if we multiply the generating function by exp $(-st^2)$ and integrate with respect to $t$ from 0 to $\infty$, the left-hand side of equation (7.91) becomes

$$
(1/2)(\pi/s)^{1/2} \exp k^2(x + x^{-1} + y + y^{-1} - 4)/4s,
$$

and so

$$(7.92)\quad 2\int_0^\infty e^{-zt^2} I_{p,q;l,m}(t)dt = (\pi/z)^{1/2} e^{-k^2/z} I_{p-l}(k^2/2z) I_{q-m}(k^2/2z).$$

When $u$ is large and positive $I_n(u) \sim (2\pi u)^{-1/2} e^u$ and so when $z \to 0$ we obtain the equation

$$(7.93)\quad \int_0^\infty I_{p,q;l,m}(t)dt = 0.$$

Now the solution of the initial value problem is

$$(7.94)\quad w_{p,q}(t) = \sum I_{p,q;l,m}(t) w_{l,m}(0) + \sum \int_0^t ds I_{p,q;l,m}(s) ds w'_{l,m}(0),$$

consequently there is no residue (as $t \to \infty$) arising from an initial velocity imparted to one of the particles, and the associated displacement of each particle ultimately approaches zero.

It may be noted that the function $I_{p,q}(t) \equiv I_{p,q;0,0}(t)$ satisfies the partial difference equation

$$(7.95)\quad q(I_{p+1,q} - I_{p-1,q}) = p(I_{p,q+1} - I_{p,q-1}),$$

which is the analogue of the equation $y\partial F/\partial x = x\partial F/\partial y$ which expresses that $F$ depends only on $x^2 + y^2$.

When it is desired to extend the present analysis to the equation $\partial^2 w/\partial t^2 + \partial^4 w/\partial x^4 + 2\partial^4 w/\partial x^2 \partial y^2 + \partial^4 w/\partial y^4 = 0$, which occurs in the theory of a vibrating plate, use may be made of the approximations to partial derivatives of high order given in the paper of E. Pflanz.[21] The expression which replaces the derivatives of the fourth order then consists of 25 terms. References are given to the work of other writers on this subject.

8. **Elastic waves in a lattice.** Taking first the case of motion in two dimensions we replace the usual equations of motion by differential difference equations. Let $u_{p,q}$, $v_{p,q}$ be the components of displacement at the lattice point $(p, q)$ and let $P$, $Q$ be operators such that

$$(8.1)\quad Pu_{p,q} = u_{p+1/2,q} - u_{p-1/2,q}, \qquad Qu_{p,q} = u_{p,q+1/2} - u_{p,q-1/2};$$

then if $\lambda$, $\mu$ are the elastic constants of Lamé the components of stress may be defined to be

$$(8.2)\quad \begin{aligned} \overline{xx}_{p,q} &= 2\mu P u_{p,q} + \lambda(Pu_{p,q} + Qv_{p,q}), \\ \overline{yy}_{p,q} &= 2\mu Q v_{p,q} + \lambda(Pu_{p,q} + Qv_{p,q}), \\ \overline{xy}_{p,q} &= \overline{yx}_{p,q} = \mu(Pv_{p,q} + Qu_{p,q}), \end{aligned}$$

---

[21] E. Pflanz, *Über die Annäherung linearer partieller Differentialausdrucke durch finite Ausdrucke*, Jber. Deutschen Math. Verein. vol. 48 (1938) pp. 41–48.

and the equations of motion are

$$(8.3)\quad \begin{aligned} \rho d^2u_{p,q}/dt^2 &= \mu P^2u_{p,q} + (\lambda+\mu)(P^2u_{p,q} + PQv_{p,q}) + \mu Q^2u_{p,q}, \\ \rho d^2v_{p,q}/dt^2 &= \mu P^2v_{p,q} + (\lambda+\mu)(PQu_{p,q} + Q^2v_{p,q}) + \mu Q^2{}_{p,q}. \end{aligned}$$

These may be satisfied by writing

$$(8.4)\quad \begin{aligned} u_{p,q} &= A\alpha a^p b^q e^{wt} + B\gamma a^p c^q e^{wt}, \\ v_{p,q} &= A\beta a^p b^q e^{wt} - B\alpha a^p c^q e^{wt}, \end{aligned}$$

where $A$, $B$, $a$, $b$, $c$, $w$ are arbitrary constants and $\alpha = a^{1/2} - a^{-1/2}$, $\beta = b^{1/2} - b^{-1/2}$, $\gamma = c^{1/2} - c^{-1/2}$. Substitution in the equations of motion indicates, however, that these will be satisfied when $a$, $b$, $c$, $w$ are related to $\rho$, $\lambda$, $\mu$ in such a way that

$$(8.5)\quad -\rho w^2 = (\lambda + 2\mu)(\alpha^2 + \beta^2), \qquad -\rho w^2 = \mu(\alpha^2 + \gamma^2).$$

The boundary condition will be taken to be $(\overline{yy}_{p,q}) = 0$, $\overline{xy}_{p,q} = 0$, for $q = 0$. This gives the two conditions $A[2\mu\beta^2 + \lambda(\alpha^2+\beta^2)] = 2\mu B\alpha\gamma$, $2A\alpha\beta = B(\alpha^2 - \gamma^2)$. Elimination of $A$ and $B$ gives the equation

$$[2\mu\beta^2 + \lambda(\alpha^2 + \beta^2)](\alpha^2 - \gamma^2) = 4\mu\alpha^2\beta\gamma.$$

When the values of $\beta$ and $\gamma$ are inserted a relevant solution of this equation is

$$(8.6)\quad w = -iV_R\alpha,$$

where $V_R$ is the velocity of the Rayleigh wave. If $a = e^{is}$, $b = e^{-r}$, $c = e^{-s}$ we have

$$\beta = 2sh(r/2), \quad \gamma = 2sh(s/2), \quad \alpha = 2i\sin(s/2), \quad w = \pm 2V_R \sin(s/2).$$

Also if $V_L$ is the velocity of propagation of longitudinal waves, $V_T$ the velocity of propagation of transverse waves,

$$(8.7)\quad \begin{aligned} w^2/V_L^2 &= 4\sin^2(s/2) - 4sh^2(r/2), \\ w^2/V_T^2 &= 4\sin^2(s/2) - 4sh^2(s/2). \end{aligned}$$

The expressions for the displacements now take the form

$$\beta = -2sh(r/2), \gamma = -2sh(s/2), \alpha = 2i\sin(s/2), w = \pm 2V_R\sin(s/2)$$

$$(8.8)\quad \begin{aligned} u_{p,q} &= C(s)(2 - V_R^2/V_T^2)e^{i(ps \pm 2V_R t \sin(s/2)) - qr} \\ &\quad - (1/2)C(s)(2 - V_R^2/V_L^2)^2 e^{i(ps \pm 2V_R t \sin(s/2)) - qs}, \\ v_{p,q} &= + iC(s)(2 - V_R^2/V_T^2)(1 - V_R^2/V_L^2)^{1/2} e^{i(ps \pm 2V_R t \sin(s/2)) - qr} \\ &\quad + 2iC(s)(1 - V_R^2/V_L^2)^{1/2} e^{i(ps \pm 2V_R t \sin(s/2)) - qs}, \end{aligned}$$

where $C(z)$ is an arbitrary function of $z$. Taking this to be unity and integrating with respect to $z$ between 0 and $2\pi$ we obtain a type of influence displacement, the values of $u_{p,0}$ and $v_{p,0}$ being

$$(8.9)\quad \begin{aligned} u_{p,0} &= (1/2)\,(V_R^2/V_T^2)\,(2 - V_R^2/V_T^2)\,[\mathfrak{J}_{2p}(2V_Rt) + i\mathfrak{E}_{2p}(2V_Rt)],\\ v_{p,0} &= (V_R^2/V_T^2)(1 - V_R^2/V_L^2)^{1/2}[\mathfrak{E}_{2p}(2V_Rt) - i\mathfrak{J}_{2p}(2V_Rt)], \end{aligned}$$

where

$$\mathfrak{J}_{2p}(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos\,(p\theta \pm x\sin\theta/2)d\theta = J_{2p}(\mp x),$$

$$\mathfrak{E}_{2p}(x) = \frac{1}{2\pi}\int_0^{2\pi}\sin\,(p\theta \pm x\sin\theta/2)d\theta = E_{2p}(\mp x).$$

$J_v(x)$ is the Bessel function of order $v$ when $v$ is an integer and $E_v(x)$ is Weber's function. When $x$ is large and positive the estimates given in Watson's *Bessel functions*[15] are

$$J_v(x) \sim (\pi x/2)^{-1/2}\cos\left(x - \frac{2v+1}{4}\pi\right),$$

$$E_v(x) \sim (\pi x/2)^{-1/2}\sin\left(x - \frac{2v+1}{4}\pi\right).$$

It should be noticed that $\mathfrak{E}_{2p}(0)=0$, $\mathfrak{J}_{2p}(0)=0$, $p\neq 0$, $\mathfrak{J}_0(0)=1$. An initial displacement of the lattice point (0, 0) leaves no residue at any surface point as $t\to\infty$.

A method of finding an estimate of $E_v(x)$ when both $x$ and $v$ are large is indicated in Watson's *Bessel functions*.

California Institute of Technology

# THE APRIL MEETING IN NEW YORK

The three hundred ninety-fifth meeting of the American Mathematical Society was held at Hunter College on Saturday, April 24, 1943. The attendance included the following one hundred forty-two members of the Society:

R. L. Anderson, T. W. Anderson, R. G. Archibald, L. A. Aroian, A. V. Baez, Y. K. Bal, J. L. Barnes, A. E. Basch, P. T. Bateman, E. G. Begle, Stefan Bergman, Felix Bernstein, Garrett Birkhoff, Gertrude Blanch, Henry Blumberg, S. G. Bourne, C. B. Boyer, A. T. Brauer, A. B. Brown, R. C. Buck, Hobart Bushey, J. H. Bushey, R. H. Cameron, B. E. Carlson, T. F. Cope, A. P. Cowgill, H. B. Curry, J. H. Curtiss, M. D. Darkow, A. S. Day, R. P. Dilworth, Jesse Douglas, R. J. Duffin, Nelson Dunford, Jacques Dutka, J. E. Eaton, M. L. Elveback, Benjamin Epstein, Paul Erdös, J. M. Feld, A. D. Fialkow, W. B. Fite, Edward Fleisher, R. M. Foster, G. A. Foyle, J. S. Frame, K. O. Friedrichs, D. L. Fuller, R. E. Fullerton, H. P. Geiringer, B. P. Gill, Leonard Gillman, Wulf Gotz, Bernard Greenspan, Lewis Greenwald, C. C. Grove, W. W. Gutzman, Jacques Hadamard, B. L. Hagen, D. W. Hall, F. C. Hall, N. A. Hall, Hans-Karl Hammer, Einar Hille, Banesh Hoffmann, T. R. Hollcroft, M. W. Hopkins, E. M. Hull, Witold Hurewicz, L. C. Hutchinson, Walter Jennings, R. A. Johnson, F. E. Johnston, A. W. Jones, F. B. Jones, H. A. Jordan, Irving Kaplansky, Edward Kasner, D. E. Kibbey, Mark Kormes, Arthur Korn, B. A. Lengyel, Marie Litzinger, L. H. Loomis, E. R. Lorch, A. N. Lowan, R. K. Luneburg, L. A. MacColl, C. C. MacDuffee, H. M. MacNeille, H. F. MacNeish, D. S. Miller, L. W. Miller, E. C. Molina, C. N. Moore, F. J. Murray, C. O. Oakley, J. C. Oxtoby, M. K. Peabody, E. L. Post, M. H. Protter, A. L. Putnam, R. G. Putnam, L. L. Rauch, Mina Rees, F. G. Reynolds, Moses Richardson, R. G. D. Richardson, C. E. Rickart, J. F. Ritt, S. L. Robinson, P. C. Rosenbloom, Arthur Sard, Max Sasuly, A. T. Schafer, R. D. Schafer, I. E. Segal, James Singer, P. A. Smith, W. J. Strange, E. G. Straus, W. C. Strodt, Otto Szász, J. D. Tamarkin, J. I. Tracey, Peter Treuenfels, H. F. Tuan, Bryant Tuckerman, J. W. Tukey, D. F. Votaw, G. L. Walker, André Weil, Alexander Weinstein, Louis Weisner, R. A. Wetzel, George Whaples, A. P. Wheeler, John Williamson, Jack Wolfe, Fumio Yagi, Oscar Zariski, Antoni Zygmund.

The meeting opened at 10 A.M. in two sections, one for papers in Analysis, Professor Anna P. Wheeler presiding, and one for papers in Algebra, Geometry and Theory of Numbers, Professor Garrett Birkhoff presiding. At the general session immediately following these sections Professor Nelson Dunford gave an address entitled *Spectral theory*. Vice President C. C. MacDuffee presided at this session. In the afternoon at 2 P.M. there was a general session at which Professor R. H. Cameron gave an address entitled *Absolutely convergent trigonometric sums*. Vice President J. D. Tamarkin presided at this session.

The hospitality of Hunter College was very much appreciated by all who attended the meeting. By special arrangement, for the convenience of guests, an excellent lunch was served in the Hunter College Cafeteria.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers numbered 1 to 6 were read in the section for Analysis, papers 7 to 13 in the section for Algebra, Geometry and Theory of Numbers; papers 14 to 23 were read by title. Paper 11 was read by Professor Kasner. Professor Alexandroff was introduced by Professor Solomon Lefschetz, and Dr. Kober by Professor Einar Hille.

1. L. H. Loomis: *Spherical volume as a coordinate-free basis for Lebesgue measure theory.* (Abstract 49-5-143.)
2. Otto Szász: *On some trigonometric summability methods and Gibbs phenomenon.* (Abstract 49-5-148.)
3. Alexander Weinstein: *Differential equations with general boundary conditions.* (Abstract 49-5-154.)
4. Henry Blumberg: *On arbitrary point transformations.* (Abstract 49-1-25.)
5. C. N. Moore: *On the relationship between Nörlund means of a certain type.* (Abstract 49-1-37.)
6. R. J. Duffin: *Some representations for Fourier transforms.* (Abstract 49-5-132.)
7. J. E. Eaton: *A Galois theory for differential fields.* (Abstract 49-5-127.)
8. R. P. Dilworth: *Lattices with unique complements.* (Abstract 49-5-126.)
9. A. T. Brauer: *On the non-existence of odd perfect numbers of form* $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$. (Abstract 49-5-125.)
10. Jacques Hadamard: *On fractional iteration and connected questions.* (Abstract 49-5-162.)
11. Edward Kasner and John DeCicco: *Generalized transformation theory of isothermal families.* (Abstract 49-5-164.)
12. T. R. Hollcroft: *Plane curve systems with distinct nodes and cusps and of negative virtual dimension.* (Abstract 49-5-163.)
13. Jacques Dutka: *Transversality in higher space.* (Abstract 49-5-161.)
14. Paul Alexandroff: *On homological situation properties of complexes and closed sets.* (Abstract 49-7-198-*t*.)
15. H. R. Branson: *On the difference equation of a general quantum mechanical problem.* (Abstract 49-5-155-*t*.)
16. Nelson Dunford: *Spectral theory.* I. *Convergence to projections.* (Abstract 49-7-185-*t*.)
17. Edward Kasner and John DeCicco: *The congruence of element-series associated with a polygenic function.* (Abstract 49-5-165-*t*.)

18. Herman Kober: *On the approximation to integrable functions by integral functions.* (Abstract 49-7-187-*t*.)

19. E. R. Lorch: *The theory of analytic functions in normed abelian vector rings.* (Abstract 49-7-188-*t*.)

20. A. N. Lowan and H. E. Salzer: *Coefficients in the expansion of derivatives in terms of central differences.* (Abstract 49-7-196-*t*.)

21. Robert Schatten: *On certain properties of crossnorms.* (Abstract 49-5-146-*t*.)

22. A. E. Taylor: *Banach spaces of functions representable by Cauchy's integral formula.* (Abstract 49-5-149-*t*.)

23. L. I. Wade (National Research Fellow): *Transcendence properties of the Carlitz $\psi$-function.* (Abstract 49-5-130-*t*.)

T. R. HOLLCROFT,
*Associate Secretary*

# THE APRIL MEETING IN CHICAGO

The three hundred ninety-sixth meeting of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, on Friday and Saturday, April 23-24, 1943. The attendance was about one hundred fifty, including the following one hundred and nineteen members of the Society:

Leon Alaoglu, A. A. Albert, B. H. Arnold, Emil Artin, W. L. Ayres, Reinhold Baer, R. H. Bardell, J. C. Bell, C. A. Bennett, S. F. Bibb, W. A. Blankinship, G. A. Bliss, L. M. Blumenthal, D. G. Bourgin, E. L. Buell, H. E. Burns, Herbert Busemann, L. E. Bush, C. C. Camp, W. B. Caton, E. W. Chittenden, R. F. Clippinger, H. B. Curtis, J. Edgar Davis, M. M. Day, John DeCicco, M. W. Dehn, J. M. Dobbie, D. M. Dribin, Samuel Eilenberg, H. S. Everett, L. R. Ford, R. H. Fox, J. H. Giese, J. W. Givens, Michael Golomb, G. D. Gore, Cornelius Gouwens, L. M. Graves, L. W. Griffiths, V. G. Grove, P. R. Halmos, R. W. Hamming, M. L. Hartung, G. E. Hay, C. T. Hazard, M. H. Heins, E. D. Hellinger, Edward Helly, M. R. Hestenes, Carl Holtom, A. S. Householder, H. K. Hughes, H. D. Huskey, G. K. Kalisch, Wilfred Kaplan, William Karush, W. C. Krathwohl, E. P. Lane, R. E. Lane, R. E. Langer, Richard Lence, A. L. Lewis, A. O. Lindstrum, A. T. Lonseth, Eugene Lukacs, Janet MacDonald, G. W. Mackey, Morris Marden, H. J. Miser, C. W. Moran, Ivan Niven, P. B. Norman, J. I. Northam, C. D. Olds, Isaac Opatowski, M. E. Patno, G. B. Price, Tibor Radó, R. B. Rasmusen, Maxwell Reade, W. T. Reid, Haim Reingold, A. E. Ross, Arthur Saastad, R. G. Sanger, Peter Scherk, O. F. G. Schilling, K. C. Schraut, G. E. Schweigert, C. E. Sealander, M. E. Shanks, A. H. Smith, E. R. Smith, W. S. Snyder, Abraham Spitzbart, R. C. Stephens, A. H. Stone, D. M. Stone, E. B. Stouffer, R. L. Swain, H. P. Thielman, C. J. Thorne, M. K. Toft, S. M. Ulam, J. I. Vass, H. S. Wall, J. V. Wehausen, N. W. Wells, M. E. Wescott, M. D. Wetzel, G. W. Whitehead, L. R. Wilcox, K. P. Williams, L. A. Wolf, R. S. Wolfe, Y. K. Wong, J. W. T. Youngs, Daniel Zelinsky.

Sessions for the reading of contributed papers were held on Friday morning with Professor L. M. Blumenthal presiding, Friday afternoon with Professor Reinhold Baer presiding, and Saturday morning with Vice President L. M. Graves and Professor G. A. Bliss presiding. On Friday afternoon prior to the session for contributed papers, Professor Emil Artin of Indiana University gave an invited address on *The axiomatic treatment of number theory and class field theory*. Professor A. A. Albert presided at this lecture.

All sessions were held in the West lecture hall of the Museum of Science and Industry. At the Saturday morning session Professor G. B. Price read a resolution of appreciation to the Museum officials for providing the Society with a place of meeting.

The Council met on Friday, April 23, at 8 P.M. in the Windermere Hotel.

The acting secretary, Professor W. L. Ayres, announced the election of the following nine persons to membership in the Society:

Professor Charles Burton Barker, University of New Mexico;
Mr. William Joseph Barr, Naval Research Laboratory, Washington, D. C.;
Mr. Charles Alexander Cole, Jr., U. S. Marine Corps, Naval Research Laboratory, Washington, D. C.;
Mr. Albert Moore Freeman, Providence, R. I.;
Mr. Richard Delph Geckler, Standard Oil Company, Whiting, Ind.;
Dr. Harry Douglas Huskey, Ohio State University;
Mr. Charles Whaley Pflaum, University of Pennsylvania;
Mr. Tracy Whittelsey Simpson, Marchant Calculating Machine Company, Oakland, Calif.;
Mr. Charles Tyrrell West, Lieutenant (j.g.), U. S. Naval Reserve, Briggs Manufacturing Company, Detroit, Mich.

It was announced that the following person was elected as a nominee of the Mathematical Association of America:

Mr. James W. Beach, Iowa State College.

The following appointments by President M. H. Stone were reported: As a committee to consider the advisability of reducing the term of office of the President of the Society from two years to one, Professors G. C. Evans (chairman), G. D. Birkhoff, A. J. Kempner, Solomon Lefschetz, and R. L. Moore; as representative of the Society at the inaugural ceremonies of the Mexican Mathematical Society on April 9, 1943, Professor P. R. Rider.

It was reported that Professor W. T. Martin had accepted an invitation to give an address at the October, 1943, meeting in New York and Professor Antoni Zygmund had accepted an invitation to give an address at the 1943 Summer Meeting.

The Council decided that the meeting scheduled with the American Association for the Advancement of Science in December, 1943, in Cleveland, be canceled, and that arrangements would be made to hold the 1943 Annual Meeting at another time and place.

On request of the editors of the new Quarterly of Applied Mathematics, the Council authorized the President to appoint a liaison officer to consult with the editors of that journal on problems of policy.

Titles and cross references to the abstracts of papers read at the meeting follow below. Papers 1 to 8 were read Friday morning, papers 9 to 14 on Friday afternoon, papers 15 to 26 on Saturday morning, and papers 27 to 37, whose abstract numbers are followed by the letter *t*, were read by title. Paper 1 was read by Dr. Kaplan, 4 by Professor Eilenberg, 12 by Miss Wetzel, and 18 by Dr. Helsel.

1. Wilfred Kaplan and Max Dresden: *Topology of the molecular N-body problem.* (Abstract 49-5-158.)

2. J. W. T. Youngs: *On parametric representations of surfaces.* III. (Abstract 49-5-177.)

3. Tibor Radó: *On continuous path-surfaces of zero area.* (Abstract 49-5-174.)

4. Samuel Eilenberg and Saunders MacLane: *On a group construction by Hopf.* (Abstract 49-1-95.)

5. R. H. Fox: *Natural systems of homomorphisms.* Preliminary report. (Abstract 49-5-172.)

6. A. H. Stone: *Connectedness and coherence.* (Abstract 49-3-124.)

7. G. E. Schweigert: *Fixed elements and periodic types for homeomorphisms on semi-locally-connected continua.* (Abstract 49-5-175.)

8. B. H. Arnold: *On decompositions of $T_1$ spaces.* (Abstract 49-5-170.)

9. A. E. Ross: *Positive quaternary quadratic forms representing all integers with at most k exceptions.* (Abstract 49-5-129.)

10. A. A. Albert: *Algebras derived by non-associative matrix multiplication.* (Abstract 49-3-103.)

11. Leon Alaoglu: *Harmonic analysis of stochastic processes.* Preliminary report. (Abstract 49-5-169.)

12. H. S. Wall and Marion D. Wetzel: *Contributions to the analytic theory of J-fractions.* Preliminary report. (Abstract 49-5-153.)

13. R. E. Lane: *The values of continued fractions and their computation.* (Abstract 49-5-142.)

14. John DeCicco: *Extensions of certain dynamical theorems of Halphen and Kasner.* (Abstract 49-3-119.)

15. R. W. Hamming: *The asymptotic location of the roots of exponential sums having polynomial exponents.* (Abstract 49-5-136.)

16. S. M. Ulam: *Theory of the operation of product of sets.* II. Preliminary report. (Abstract 49-5-151.)

17. M. M. Day: *Uniform convexity.* III. (Abstract 49-3-111.)

18. R. G. Helsel and Tibor Radó: *The transformation of double integrals.* (Abstract 49-5-137.)

19. G. B. Price: *Cauchy-Stieltjes and Riemann-Stieltjes integrals.* (Abstract 49-1-38.)

20. W. S. Snyder: *A note on the interiority of real functions.* Preliminary report. (Abstract 49-5-147.)

21. P. R. Halmos: *Approximation theories for measure preserving transformations.* (Abstract 49-5-135.)

22. H. D. Huskey: *Contributions to the problem of Geöcze.* (Abstract 49-5-140.)

23. H. K. Hughes: *On a theorem of Newsom.* (Abstract 49-5-139.)

24. Isaac Opatowski: *An explicit formula for the refractive index in*

*electron optics*. (Abstract 49-5-159.)

25. M. R. Hestenes: *On the condition of Weierstrass in the calculus of variations*. (Abstract 49-7-186.)

26. D. G. Bourgin: *On a quasi-linear operation*. (Abstract 49-5-171.)

27. L. W. Cohen: *On linear equations in Hilbert space*. (Abstract 49-5-131-*t*.)

28. G. E. Albert and L. H. Miller: *Equiconvergence of series of orthonormal polynomials*. Preliminary report. (Abstract 49-3-108-*t*.)

29. George Piranian: *On the convergence of certain partial sums of a Taylor series with gaps*. (Abstract 49-3-112-*t*.)

30. G. E. Albert: *An extension of Korous' inequality for orthonormal polynomials*. Preliminary report. (Abstract 49-3-107-*t*.)

31. Rufus Oldenburger: *The characteristic of a sum of quadratic forms*. (Abstract 49-1-12-*t*.)

32. G. E. Schweigert: *Minimal A-sets, infinite orbits, and fixed elements*. (Abstract 49-5-176-*t*.)

33. W. J. Thron: *Twin convergence regions for continued fractions*. (Abstract 49-5-150-*t*.)

34. J. E. Wilkins: *A special class of surfaces in projective differential geometry*. (Abstract 49-5-166-*t*.)

35. J. F. Harding and Isaac Opatowski: *An approximate formula for the Legendre elliptic integral of the second kind*. (Abstract 49-5-157-*t*.)

36. George Piranian: *On the order of analytic functions in the sense of Hadamard*. (Abstract 49-7-189-*t*.)

37. Reinhold Baer: *A theory of crossed characters*. (Abstract 49-7-178-*t*.)

38. Reinhold Baer: *Radical extensions and crossed characters*. (Abstract 49-7-179-*t*.)

W. L. AYRES,
*Associate Secretary*

# THE APRIL MEETING AT STANFORD UNIVERSITY

The three hundred ninety-seventh meeting of the American Mathematical Society was held at Stanford University on Saturday, April 24, 1943. The attendance was about fifty-five, including the following twenty-nine members of the Society:

H. M. Bacon, K. E. Benson, B. A. Bernstein, H. F. Blichfeldt, T. C. Doyle, S. N. Hallam, M. A. Heaslet, J. G. Herriot, Alfred Horn, D. H. Lehmer, S. H. Levy, J. V. Lewis, A. D. Michal, F. R. Morris, W. H. Myers, George Pólya, R. M. Robinson, J. B. Robinson, Robert Russell, S. A. Schaaf, A. C. Schaeffer, D. C. Spencer, Gabor Szegö, A. E. Taylor, S. P. Timoshenko, J. V. Uspensky, A. R. Williams, František Wolf, Max Zorn.

The meeting opened in the morning with a general session for contributed papers at which Professor Gabor Szegö presided. By invitation of the Program Committee, Professor A. E. Taylor of the University of California at Los Angeles delivered an hour address on *Analysis in complex Banach spaces*. Professor B. A. Bernstein presided at this lecture.

The Symposium on Applied Mathematics was held in the afternoon with Professor A. D. Michal presiding. Professor S. P. Timoshenko of Stanford University spoke on *Theory of suspension bridges*, Dr. E. G. Keller of Lockheed Aircraft Corporation on *Some present nonlinear problems of the electrical and aeronautical industries*, and Dr. Hsu-shen Tsien of the California Institute of Technology on *The "limiting line" in mixed subsonic and supersonic flows of compressible fluids*.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 4 were read at the morning session; papers 5 and 10 were read in the afternoon; and those numbered 6 to 9 were read by title.

1. R. N. Robinson: *Analytic functions in circular rings*. (Abstract 49-3-113.)

2. B. A. Bernstein: *Postulate sets for Boolean rings*. (Abstract 49-5-167.)

3. George Pólya: *Inequalities for the area of the ellipsoid*. (Abstract 49-5-145.)

4. D. H. Lehmer: *On Ramanujan's numerical function* $\tau(n)$. (Abstract 49-5-128.)

5. Max Zorn: *Informal note on the second underivability theorem* (Abstract 49-5-168.)

6. R. C. James: *Orthogonality in normed linear topological spaces.* (Abstract 49-5-173-*t*.)

7. F. A. Valentine: *A Lipschitz condition preserving extension for a vector function.* (Abstract 49-5-152-*t*.)

8. Glynn Owens: *A boundary value problem for an ordinary nonlinear differential equation of the second order.* (Abstract 49-5-144-*t*.)

9. D. H. Hyers: *A note on the Michal differential in Banach spaces.* (Abstract 49-5-141-*t*.)

10. T. C. Doyle: *Tensor theory of invariants for the projective differential geometry of a curved surface.* (Abstract 49-7-194.)

A. D. MICHAL,
*Associate Secretary*

# BOOK REVIEWS

*Topics in topology*. By Solomon Lefschetz. (Annals of Mathematics Studies, no. 10.) Princeton University Press, 1942. 139 pp. $2 00.

This book is a companion volume to the author's recently published *Algebraic Topology*. In the latter volume the interest is focused, as the title implies, on the algebraic structure and properties of complexes and on their applications to the study of abstract spaces. Here, however, the interest lies in the topological structure of complexes and related spaces. Indeed, the major part of the book is devoted to a discussion of classes of spaces which are topological generalizations of complexes in much the same way that the abstract complex of *Algebraic Topology* is an algebraic generalization of a finite simplicial complex.

There have appeared in the literature definitions of two such classes which are successful from the point of view of giving rise to interesting and well developed bodies of theorems. The first of these, due to Borsuk, is the class of absolute neighborhood retracts, and the second, due to Lefschetz, is the class of $LC^*$ spaces. Both these classes have come to be of considerable topological importance, so the development given here of their essential properties, and of their mutual relations, will be welcomed by all topologists.

There are four chapters. The first is a discussion of various topologies which can be imposed on an infinite complex. A "natural" topology is obtained by taking as a basis for open sets the stars of vertices in all subdivisions. An interesting metric can also be defined and the two topologies are equivalent when the complex is locally finite, the case most used in later applications.

Chapter II is an extension of the discussion in *Algebraic Topology* of singular and continuous cycles. A few new notions along these lines are developed for later use.

In Chapter III a number of topics are dealt with, all dealing with existence theorems for various types of mappings of one space into another. First the Alexandroff theorem on the mapping of a space into the nerve of one of its coverings is generalized by giving necessary and sufficient conditions on the covering for the existence of such a mapping. This has interesting applications to Tychonoff and normal spaces, leading among other things to a characterization of the former.

Next are the proofs of two of the fundamental theorems from dimension theory: any $n$-dimensional separable metric space can be imbedded in a compact metric space of the same dimension and also in the Euclidean space of dimension $2n+1$.

The chapter concludes with a discussion of Borsuk's theory of retraction. An absolute retract (AR) is a separable metric space such that whenever it is a closed subset of a larger space then there is a continuous mapping (a "retraction") of the larger space onto the subset which leaves fixed each point of the subset. An absolute neighborhood retract (ANR) is defined in the same fashion but with the retraction required to exist only on a neighborhood of the subset. The most useful properties of such spaces stem from their characterizations in terms of extensions of mappings, e.g., a necessary and sufficient condition for a space A to be an AR is that whenever B is a closed subset of C, then every mapping of B into A can be extended to a mapping of C into A.

The fundamental theorem proved here, due originally to the author, is that to any separable metric space there may be added an infinite complex in such a fashion that the resulting union is an absolute retract.

The final chapter is devoted to the author's theory of homotopy local connectedness. Roughly speaking, a space is $p$-LC if small singular $p$-spheres in the space can be shrunk to points over small sets. The interesting case is where the space is $\mathrm{LC}^n$, i.e., $p$-LC for all $p \leqq n$. Such spaces can be characterized in terms of mappings of complexes into the space, and this leads to the slightly stronger notion of an $\mathrm{LC}^*$ space. The fundamental theorem is that a compact absolute neighborhood retract is an $\mathrm{LC}^*$ space and conversely.

Next the homology theory of such spaces is investigated. It turns out that all types of cycles (e.g., singular, Vietoris, Čech) yield the same homology groups. Furthermore these groups can be no more complicated than those of a finite complex; in particular they have finite bases. Using this result, the fixed point formula is shown to hold for these spaces. As an application the Schauder fixed point theorem for functional spaces and some of its generalizations are derived.

The chapter concludes with a brief outline of the results obtained by generalizing the local connectedness definitions by replacing homotopy by homology, i.e., singular spheres by cycles and shrinking to a point by bounding.

As with all the volumes of this series, this one is well printed and easy to read. It is marred, however, by an unfortunate number of misprints, although few of them will cause the reader any particular difficulty. There are also a few slips. The proof of the last part of the last theorem of Chapter III is incorrect but is easy to remedy. In theorem 23.1 of Chapter IV, the inequality should read "$q \leq p$." The proof here is not quite complete.

The terminology and notations used here are those of *Algebraic Topology*, so the index and bibliography are brief. There is, however, an additional bibliography of papers dealing with the subjects of retraction and homotopy local connectedness which is quite complete and useful.

EDWARD G. BEGLE

*The calculus of extension.* By Henry George Forder. (Including examples by Robert William Genese.) Cambridge University Press; New York, Macmillan, 1941. 15+490 pp. $6.75.

It will soon be the hundredth anniversary of the initial publication of Grassmann's monumental work *Die Lineale Ausdehnungslehre Ein Neuer Zweig der Mathematik*. Perhaps this fact will serve to arouse some belated interest in Grassmann's celebrated but otherwise neglected contributions to mathematics. If so, the excellent volume under review should indeed be a welcome addition to the comparatively meagre supply of general works treating this subject.

The history of the *Ausdehnungslehre* is extremely depressing. Grassmann had hoped that he would be able to secure the special opportunities for mathematical research that naturally accrue to a university post. In this he was bitterly disappointed. Such a position was not forthcoming. Cajori writes[1] "At the age of fifty-three this wonderful man, with heavy heart, gave up mathematics, and directed his energies to the study of Sanskrit, achieving in philology results which were better appreciated, and which vie in splendor with those in mathematics." Grassmann presented the *Ausdehnungslehre* in two books. The first, *Die Lineale Ausdehnungslehre*, appeared in 1844 (second edition, 1878) and the second, *Die Ausdehnungslehre, vollständig und in strenger Form bearbeitet*, in 1862. It has been said that only one person had read through the *Ausdehnungslehre* of 1844 eight years after its publication. Apparently, it was too new, general, and abstract to meet with popular approval. The *Ausdehnungslehre* of 1862 is easier to read and gives ample evidence of the wide range of the applications. It fared but little better than the former. In recent times Grassmann's work has been better appreciated. The historians of our subject[2] commend it, but it has not yet been adopted in America to the extent that it is frequently presented in courses and employed in exposition and research. This is in strong contrast to vector analysis which was "discovered" shortly after its extraction, by Gibbs and others from the *Ausdehnungslehre* and Hamilton's *Quaternions*. Tensor analysis, a second subject which had roots in the *Ausdehnungslehre*, was rather generally discovered twenty years after its formulation by Ricci and Levi-Civita.

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[1] See Florian Cajori, *A history of mathematics*, New York, 1938.

[2] See, E. T. Bell, *The development of mathematics*, New York, 1940; Florian Cajori, *A history of mathematics*, New York, 1938; J. L. Coolidge, *A history of geometrical methods*, Oxford, 1940.

Since the *Ausdehnungslehre* is not yet widely known, one or two remarks on its nature are in order. First, as to its formal aspects, this subject may be described as a study of the linear form $\sum x_i e_i$, in which the $x$'s are numbers and the $e$'s (the extensives) are mathematical entities following a prescribed algebra. In the geometrical applications, the extensives are geometrical entities, such as points, vectors, rotors (a rotor is a vector confined to a line through it), circles, etc. The definitions of the sums, differences, and products of the extensives form the means whereby geometrical theorems are reduced to algebraic identities. While the machinery contains numerous minutiae (definitions, rules of operation, etc.) the separate details are quite simple and the resulting theory is extremely powerful. In a word, the general subject matter of *The calculus of extension* is of considerable interest both historically and intrinsically.

With regard to the book itself, the important questions are the following: Does the author present new material and if so is the new material developed with clarity? Is old material rendered more easily attainable for the uninitiated? For what purposes is the book suited? Forder's *The calculus of extension* not only presents a wealth of specific applications of the subject to geometry, that are either new or not readily obtainable elsewhere; but in addition furnishes an admirable and fresh exposition of the *Ausdehnungslehre*. It is very carefully written and the individual proofs are uniformly short and easy to follow. The theorem density is exceptionally high and consequently despite the superior exposition it is not an easy book to work straight through—perhaps the key chapters suffer from lack of recapitulation. Books that contain a wealth of material are never easy to read through, and it is my conviction that *The calculus of extension* provides the best exposition of the fundamental processes of the *Ausdehnungslehre* and the most inclusive treatment of the geometrical applications available at present. It is a book that should be in the library of anyone who is interested in either algebra, the algebraic treatment of geometry, or vector and tensor analysis. Certain chapters could be used effectively as the basis for a college course.

Turning to the negative matter of the accuracy of the work, there is little that needs to be said. It is not feasible, of course, to produce a book without a blemish and I have noticed a few misprints or slips of the pen and one or two places that would benefit by revision. These flaws, however, are remarkably scarce and are not of the type that will cause the reader difficulty, assuming that he is ready for the subject. Consequently, they have but slight bearing on the purpose of this review which is, I presume, to enable the reader to decide whether or not he wishes to examine the book. I should have preferred a more modern treatment of differentiation, but this is not a serious matter since the book is obviously addressed to readers who have the maturity to supply the rigorous formulation. The statement on page 221 that a combination not satisfying $a(b+c)d=abd+acd$ would not be called a product seems a little strong in the light of the dot product of vector analysis; and number 13 on the preceding page needs revision. These are small matters, however, and as I have said the book is unusually accurate.

For the uninitiated who wish to get a general notion of the subject as quickly as possible a careful study of Chapter I, together with a sampling of Chapters II, V, VII, VIII, and IX will suffice. Chapter I (71 pages) begins with a polished treatment of the fundamentals of the algebra of extensives and then proceeds with the interpretation of the primary extensives as points in a plane. The product of a point and a real number $k$ is a point of weight $k$. The difference $b-a$ of two points of unit weight is the vector $\overrightarrow{ab}$. The sum of a point and a vector is the terminus of the vector when the vector's origin is placed at the point. The sum of two weighted points is a point at their center of gravity with weight equal to the total weights of the original points. The outer product of two points $a$ and $b$ is the rotor obtained by confining the vector $b-a$ to the line through $a$ and $b$. The outer product of three points $[abc]$ is twice the area of the triangle $(a, b, c)$. The difference of two rotors whose vectors are equal is called a bivector and in accordance with the equation $[(a-b)(p-q)]=[a(p-q)]-[b(p-q)]$ the outer product of two vectors is taken to be a bivector. The inner product of two vectors $u$ and $v$ is defined in terms of the supplement of $v$, i.e., the vector obtained by rotating $v$ through 90 degrees in the positive direction. Specifically, the inner product of $u$ and $v$ is the outer product of $u$ and the supplement of $v$. Since a bivector may be identified with its magnitude, namely, the area of an associated parallelogram, the inner product of two vectors is essentially the dot product of vector analysis. On the basis of these and certain other definitions the author establishes a wide variety of geometrical theorems including some in projective geometry.

Chapter II (pp. 71–110) is similar in character to Chapter I except that it is concerned with three-space. Chapter III (pp. 111–155), together with Chapters I, II, and VI, gives a reasonably complete account of projective geometry. Chapter V, Differentiation and Motion (pp. 177–196) is somewhat similar to the development by classical vector analysis. It treats among other things velocity, acceleration, curvature, central motion, and displacements in space. In Chapter VII, The General Theory (pp. 217–249), the central ideas are linear dependence and the various products of extensives. Determinants enter the picture by way of the fact that if $b_1, \cdots, b_r$ are each linearly related to $a_1, \cdots, a_r$, then $[b_1 \cdots b_r] = D[a_1 \cdots a_r]$, $D$ being the determinant of the coefficients in the linear relationship. Chapter VIII (pp. 250–264) gives the application to linear equations and determinants. With regard to linear equations the modus operandi is to multiply each equation of the set $a^r_\sigma x^\sigma = b^r$ by an extensive $e_r$ and add, thus $e_r a^r_\sigma x^\sigma = e_r b^r$. This reduces $N$ equations in $N$ unknowns to a single equation in extensives. The familiar theorems on the solvability of equations follow with marked simplicity. The treatment of determinants like the book as a whole is characterized by the wide scope of the material which is intelligibly presented in a relatively small space. Chapter IX (pp. 265–294) treats transformations, square matrices, and central quadrics—mostly well known material in slightly different dress. The remaining chapters are devoted to: the screw and linear complex, the general theory of inner products, circles (two chapters), the general theory of matrices, quadric spreads, and algebraic products.

HOMER V. CRAIG

*Introduction to the theory of relativity.* By Peter Gabriel Bergmann with a foreword by Albert Einstein. New York, Prentice-Hall, 1942. 287 pp. $4.50.

Many excellent books have been written on the Theory of Relativity. Although some of them appeared more than twenty years ago they are still read and studied, far from being regarded as antiquated. The books of Weyl, Pauli, Eddington are justly looked upon as classics in this subject. To say that Bergmann's book is in the same class as the books just mentioned means great, but deserved, praise.

Bergmann's book has its own character, and differs from the other books on Relativity Theory which have appeared up to now. First, it is more modern. The application of Relativity Theory to the Theory of Cosmic Rays, to Sommerfeld's relativitistic theory of the hydrogen atom, de Broglie's waves, Ives' experiments, Compton's effect, all these topics have found their proper places in Bergmann's book. But the advantage of having all these subjects included in a book on Relativity Theory does not, in itself, mean very much. Relativity Theory had changed but little in the last twenty-five years and all these additions are not very important from the point of view of Relativity Theory. Much more important for the evaluation of this book is the originality of the author's approach.

Someone may look for a book on Relativity Theory which states clearly and axiomatically the assumptions of this theory and develops deductively the conclusions from these assumptions. This is not what Bergmann's book tries to do. What it tries to do, and does excellently, is to show how we were compelled to adopt these assumptions, how the structure of Relativity grew from logical contradictions in the classical theory, how their removal leads naturally and simply to the Theory of Relativity. The author presents not the painful historical process, not how Relativity was discovered, but how it should have been discovered if we had known the simple and straight road of logic leading to its formulation. Even in Relativity Theory, created almost by the genius of one man, this difference between the historically and logically reconstructed process is remarkable; it is the difference between the broad highway and the pioneer's narrow pathway.

The author succeeds well in presenting the development of Relativity as a logical necessity. The book is always free from the dogmatic dull textbook tone and often achieves dramatic qualities. The style is clear and simple.

The book is divided into three parts. The first part (pp. 3–147) contains the Special Relativity Theory. Tensor algebra is included in this part. Tensor analysis is partially in this part, partially in the second part. Though from the structure of the book it is clear why the author did it, I have my doubts whether this was the best possible solution. Compared with the material of other books, one finds in this part modern examples from Quantum Theory, but misses the Minkowski representation.

The second part (pp. 151–242) contains the General Relativity Theory. The transition from the Special to General Relativity Theory is remarkably good. The whole chapter shows the natural development of the essential ideas in General Relativity Theory; difficult concepts are introduced in a very interesting and simple, though never superficial manner. When comparing it with other treatments, we note the omission of cosmological applications (treated so well in Tolman's book) and of time-space measurements in connection with the experimental verifications of General Relativity Theory. On the other hand much space is devoted to the equations of motion as deduced from the field equations.

The third part (pp. 245–279) is of much more special character and deals with the unification of the gravitational and electromagnetic field. Here we find an exposition of Weyl's and Kaluza's theories and of their generalizations on which the author collaborated with Einstein. This part will rather interest specialists than students.

The book is well designed. The title of the book is too modest. It is not an introduction; it is an excellent book on the principles of Relativity Theory.

L. INFELD

*Degree of approximation by polynomials in the complex domain.* By W. E. Sewell. (Annals of Mathematics Studies, no. 9.) Princeton University Press, 1942. 9+236 pp. $3.00.

The main subject of this book is the relation of the analytic and continuity properties of a function inside and on the boundary of a region, and the degree of approximation by polynomials. It is related to the analogous questions for functions of a real variable and approximation by trigonometric sums, as presented in treatises of C. de la Vallée Poussin, S. Bernstein, D. Jackson and G. Szegö. The problems of the present work originate with the fundamental theorem that a function analytic in a Jordan region and continuous in the closed region can be uniformly approximated by polynomials. This was proved by J. Walsh in 1926. His treatise *Interpolation and approximation by rational functions in the complex domain* (Amer. Math. Soc. Colloquium Publications, vol. 20, 1935) is an important forerunner to the present book.

This book is divided into two parts. Part 1 (Chapters II, III, and IV) is devoted to a study of the relation between the degree of convergence of certain sequences of polynomials $p_n(z)$ to a function $f(z)$ on a point set $E$ in the $z$-plane on the one hand and the continuity properties of $f(z)$ on the boundary $C$ of $E$ on the other hand (Prolem $\alpha$). Recent contributions to Problem $\alpha$ are due primarily to J. Curtiss, Sewell, and Walsh.

In Part II (Chapters V–VIII) a more delicate problem is considered. Let $E$ with boundary $C$ be a closed limited set, whose complement $K$ is connected and regular; thus a function $w=g(z)$ maps $K$ conformally on the exterior of the unit circle $|w| =1$ in the $w$-plane so that the points at infinity in the two planes correspond to each other. Denote by $C_\rho$ the image, under the conformal map, in the $z$-plane of the circle $|w|=\rho>1$. Let $f(z)$ be given analytic interior to some fixed $C_\rho$ and let its continuity properties on $C_\rho$ be prescribed. Problem $\beta$ is the study of the relation between the degree of convergence of various sequences $p_n(z)$ to $f(z)$ on $E$ on the one hand and the continuity properties of $f(z)$ on $C_\rho$ on the other hand. Special cases of this problem have been studied by Bernstein, G. Faber, and de la Vallée Poussin. The present work discusses in detail more recent material primarily due to Sewell and Walsh.

Two types of results are considered in either Problem $\alpha$ or $\beta$. A direct theorem is one in which the properties of $f(z)$ are in the hypothesis and the degree of convergence of $p_n(z)$ to $f(z)$ is the conclusion. An indirect (generally far more difficult) theorem is in the converse direction, that is the degree of convergence of $p_n(z)$ to $f(z)$ is in the hypothesis and the properties of $f(z)$ form the conclusion. Such properties are: (a) A Lipschitz condition of order $\alpha$ is satisfied by $f(z)$ on a set $E$; that is $|f(z_1)-f(z_2)| \leqq L|z_1-z_2|^\alpha$, where $\alpha$ is a fixed number, $0<\alpha\leqq 1$, $z_1$ and $z_2$ are arbitrary points of $E$, and $L$ is a constant independent of $z_1$ and $z_2$. (b) $f(z)$ belongs to the class $L(\kappa, \alpha)$ on a Jordan curve $C$ (boundary of $E$), if $f(z)$ is analytic in the interior points of $E$, is continuous in the closed set $E$, and the $\kappa$th derivative $f^{(\kappa)}(z)$ exists on $C$ in the one-dimensional sense and satisfies a Lipschitz condition of order $\alpha$ on $C$. (c) $f(z)$ belongs to the class log $(\kappa,1)$ on $C$ if $f(z)$ is analytic in the interior points of $E$, is continuous on the closed set $E$, and $f^{(\kappa)}(z)$ exists on $C$ in the one-dimensional sense and satisfies the condition

$$|f^{(\kappa)}(z_1) - f^{(\kappa)}(z_2)| \leqq L\,|z_1 - z_2|\,|\log|z_1 - z_2||,$$

where $z_1$, $z_2$ are arbitrary points of $C$, $|z_1-z_2|$ sufficiently small ($<1/2$ say); $L$ is a constant.

Occasionally an integrated Lipschitz condition is also considered. The degree of convergence or approximation is measured:

1. In the sense of Tchebycheff, that is by

$$\underset{z \text{ in } E}{\text{l. u. b.}}\,|f(z) - p_n(z)|.$$

2. By a line integral:

$$\int_C \Delta(z)\,|f(z) - p_n(z)|^q\,|dz|,$$

where $\Delta(z)$ is a non-negative weight function, $q$ is a fixed positive number.

As the author points out, many of the results are not final; at the end of each chapter possible extensions are indicated. The bibliography is restricted to a selected list. A few scattered misprints and errors can be easily corrected. For example on p. 13 the inequality $|\partial P/\partial\theta| \leqq A/(1-r)$ is false, but the proof of the theorem can be completed. On p. 126, where a beautiful theorem of Hardy and Littlewood is quoted, reference should be made to a paper of E. S. Quade, Duke Math. J., vol. 3 (1937), pp. 529–543.

On the whole the author deserves credit for his valuable contribution which will serve to stimulate further research on this important subject.

OTTO SZÁSZ

*Table of arc-tan x*. Federal Works Agency, Works Projects Administration for the City of New York, 1942. 25+173 pages. $2.00.

"This table of arc-tan $x$ is believed to be the most comprehensive so far published, in respect both to the number of decimal places given and to the smallness of tabular interval. It forms the first contribution to what it is hoped will be a complete set of tables of the inverse trigonometric and hyperbolic functions." The tables are computed to twelve decimals. The interval between successive arguments is 0.001 for $0 \leqq x \leqq 7$, it is 0.01 for $7 \leqq x \leqq 50$, 0.1 for $50 \leqq x \leqq 300$, 1 for $300 \leqq x \leqq 2{,}000$ and 10 for $2{,}000 \leqq x \leqq 10{,}000$. Linear interpolation provides for the whole range an accuracy of six decimal places. For interpolation to twelve places the second central differences are tabulated. The introduction gives the necessary formulae; correspondingly tables to six places for $p(1-p)$ and $1/6\ p(1-p^2)$ are given for the range $0 \leqq p \leqq 0.5$ and $0 \leqq p \leqq 0.999$. Tables for the conversion of radians into degrees and conversely are added. The introduction gives the necessary information for the use and scope of the tables, and ends in a report about the method of checking by a sixth difference test. The bibliography contains also a new list of errors in the tables of Hayashi.

O. NEUGEBAUER

## NOTES

The American Mathematical Society announces a new series of monographs, Mathematical Surveys. Each book will be about 150 pages in length and will make available the principal methods and results that have been obtained on a certain broad problem, without entering into all details of proof. The books of the new series will be distinguished from the Colloquium Publications in that the latter contain, to a large extent, new research material, while the monographs in the surveys will be mainly expository.

The first two books of the series are the following: *The problem of moments* by J. D. Tamarkin and J. A. Shohat and *Rings* by Nathan Jacobson. The price for Volume 1 is $2.50 and for Volume 2 $2.00. Members of the American Mathematical Society receive a discount of twenty-five per cent. Orders for either of these books should be addressed to the Society at 531 West 116th Street, New York, 27, N.Y.

The First National Congress of Mathematics was held at Saltillo, Mexico, November 1–7, 1942. Dr. A. Nápoles Gándara, who is director of the Institute of Mathematics at the University of Mexico, was secretary of the Congress. Professor P. R. Rider of Washington University and Professor M. S. Vallarta of the Massachusetts Institute of Technology participated in the Congress.

Dr. P. A. Alexandroff has been awarded a Stalin prize.

Professor L. E. J. Brouwer of the University of Amsterdam and Professor Godofredo García Díaz of the University of San Marcos have been elected to membership in the American Philosophical Society.

Professor E. A. Milne of the University of Oxford has been elected president of the Royal Astronomical Society.

Professor G. F. J. Temple of Kings College, London, has been elected to fellowship in the Royal Society.

Professor A. A. Albert of the University of Chicago has been elected to membership in the National Academy of Sciences.

Professor G. D. Birkhoff of Harvard University has been elected an honorary member of the Royal Irish Academy. He has also received an honorary doctorate of laws from the University of California at Los Angeles.

Professor M. H. Stone of Harvard University has been elected to membership in the American Philosophical Society.

Professor J. L. Synge of the University of Toronto has been elected a fellow of the Royal Society of London.

Dr. P. C. Rosenbloom has been appointed a National Research Fellow.

Assistant Professor R. G. Archibald of Queens College has been promoted to an associate professorship.

Assistant Professor H. R. Cooley of New York University has been promoted to an associate professorship.

Assistant Professor Nelson Dunford of Yale University has been promoted to an associate professorship.

Associate Professor J. S. Frame of Allegheny College has been appointed to a professorship at Michigan State College.

Associate Professor W. L. Hutchings of Rollins College, Winter Park, Florida, has been appointed to an associate professorship at Whitman College, Walla Walla, Washington.

Mrs. C. J. Keyser has been appointed dean of Nightingale-Bamford School, New York City.

Dr. B. A. Lengyel of the College of the City of New York has been appointed to an assistant professorship in the department of physics at the University of Rochester.

Assistant Professor W. T. Martin of Massachusetts Institute of Technology has been appointed to a professorship at Syracuse University.

Associate Professor A. F. Moursund of the University of Oregon has been promoted to a professorship.

Professor J. L. Synge of the University of Toronto has been appointed head of the department of mathematics at Ohio State University to replace Professor H. W. Kuhn, who has retired.

Dr. A. D. Wallace of the University of Pennsylvania has been promoted to an assistant professorship.

Associate Professor V. H. Wells of Williams College has been promoted to a professorship.

Dr. Max Wyman has been appointed lecturer in mathematics at the University of Alberta.

Dr. Leo Zippin of Queens College has been promoted to an assistant professorship.

The following appointments to instructorships are announced: University of Buffalo: Dr. Paul Civin; University of Michigan: Mr. P. S. Jones; Reed College; Dr. R. W. Shephard; University of Virginia: Mr. W. H. Gottschalk.

Dean H. E. Hawkes of Columbia University died May 4, 1943, at the age of seventy years. He had been a member of the Society since 1898.

Professor R. L. Menuet of Tulane University died May 9, 1943.

Dr. W. H. Metzler who retired as dean of the New York State College for Teachers in 1933 died April 18, 1943, at the age of seventy-nine years. He had been a member of the Society since 1891.

Professor Emeritus H. S. White of Vassar College died May 20, 1943. He had been a member of the Society since 1892.

Professor J. E. Williams of Virginia Polytechnic Institute died April 19, 1943, at the age of seventy-five years. He had been a member of the Society since 1905.

Professor Emeritus Clara E. Smith of Wellesley College died May 12, 1943. She had been a member of the Society since 1904.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

## ALGEBRA

### 178. Reinhold Baer: *A theory of crossed characters.*

If $G$ is a finite group, $C$ a homomorphism of $G$ into the group of automorphisms of the cyclic group $E$, then a $C$-character of $G$ is any single valued $G$ to $E$ function $f(x)$ satisfying the functional equation: $f(uv)=f(u)^{v}f(v)$ for $u$, $v$ in $G$ where the exponent $v$ indicates the automorphism upon which the element $v$ is mapped by $C$. The $C$-characters of $G$ form a finite abelian group; and it is the object of a theory of crossed characters to find conditions assuring the possibility of establishing a partial or complete duality between $G$ and its $C$-character group. (Received April 23, 1943.)

### 179. Reinhold Baer: *Radical extensions and crossed characters.*

An $m$-extension of the field $F$ is a finite, normal and separable extension of $F$ which is obtained by adjoining to $F$ $m$th roots of elements in $F$. A theory of these extensions may be obtained by applying the theory of crossed characters on the Galois group of these extensions. (Received April 23, 1943.)

### 180. C. J. Everett: *Closure operators and galois theory in lattices.*

Every $*$-closure ($A^{*}\supset A$; $A\subset B$ implies $A^{*}\subset B^{*}$) on a partially ordered set arises from a galois correspondence between $P$ and some partially ordered set $Q$. Every galois correspondence between complete lattices of subsets of two sets results from a binary relation on all elements of the two sets. Every closure on a complete lattice $P$ of subsets of a set is extensible to all subsets; if $(A\cup B)^{*}=A^{*}\cup B^{*}$ in $P$, the extension is also topological, for $P$ a boolean algebra. Every closure on all subsets of a set is defined by a binary relation. A generalization of Krull's topology for algebraic fields is obtained and used to characterize the regularly closed subspaces of the linear functional space of a Banach space. Necessary and sufficient conditions are given for the existence of a topology in a group on whose subgroup lattice a $*$-closure is defined, such that a subgroup is $*$-closed if and only if it is closed in the topology. This is combined with results of Baer on primary groups to characterize topologically the subgroups closed under the galois correspondence between $P$ and its automorphism group, $A$. Krull's method converts $A$ into a metric, totally disconnected, topological group. (Received May 29, 1943.)

### 181. Irving Kaplansky: *Solution of the "problème des ménages."*

The "problème des ménages" asks for the number of ways of seating $n$ husbands and $n$ wives at a circular table, men alternating with women, so that no husband sits next to his wife. By elementary methods, the author obtains the answer $4n\, n!\sum_{k=1}^{n}(-1)^k C_k^{2n-k}(n-k)!/(2n-k)$. (Received June 1, 1943.)

## Analysis

### 182. E. F. Beckenbach and R. H. Bing: *Conformal minimal varieties.*

A set of $m$ real functions of $n$ real variables, $m \geqq n \geqq 2$, defined in a domain $D$, has been called (for $m=n$ by N. Cioranesco, Bull. Sci. Math. vol 56 (1932) pp. 55–64) a set of conjugate harmonic functions provided the functions are harmonic and together satisfy the usual conditions for conformality. Such a set of $m$ functions gives a conformal map on Euclidean $n$-space of a minimal variety $V_n$ immersed in Euclidean $m$-space. But according to Haantjes, for $n \geqq 4$ the class of conformally flat spaces is quite narrow. It is now shown directly that for $n \geqq 3$ the only sets of conjugate harmonic functions necessarily are constants or linear functions, so that either the $V_n$ is a point or it can be obtained from $D$ by rigid motions, transformations of similitude, and reflections in hyperplanes. (Received May 7, 1943 )

### 183. R. P. Boas: *Almost periodic functions of exponential type.*

As an extension of a well known result on entire functions of exponential type which are periodic on the real axis, it is shown that an entire function of exponential type which is almost periodic on the real axis has its Fourier exponents bounded. Almost periodicity as general as Besicovitch (order 1) is admissible; in fact, the mere existence of the mean value of $|f(x)|$ implies the existence and vanishing of the mean value of $f(x)e^{i\lambda x}$ when $|\lambda|$ exceeds the type of the function $f(x)$. For the proof, one rotates the line of integration in $\int_0^T e^{i\lambda x}f(x)dx$ through an angle of $\pi/2$, and applies theorems which connect the growth of $|f(x+iy)|$ with that of $|f(x)|$. The estimate $f(x)=o(|x|)$ as $|x|\rightarrow\infty$ is a consequence of the existence of the mean value of $|f(x)|$. (Received May 11, 1943.)

### 184. Vincent Cowling and Walter Leighton: *On convergence regions for continued fractions.*

Let $k$ be any real number greater than 1 and $\epsilon$ any number such that $0<\epsilon<k$. If the complex numbers $a_{2n}$ and $a_{2n+1}=r_n e^{i\theta_n}$ satisfy the conditions (1) $|a_{2n}| \leqq k-\epsilon$, $a_{2n} \neq -1$, (2) $r_n \geqq 2[k-\cos\theta_n]$, $0 \leqq \theta_n \leqq 2\pi$, then the continued fraction $1+K[a_n/1]$ converges. It follows as a corollary that the continued fraction will converge if conditions (1) and $|a_{2n+1}| \geqq 2(k+1)$ hold. (Received April 28, 1943.)

### 185. Nelson Dunford: *Spectral theory.* I. *Convergence to projections.*

By a systematic use of an operational calculus suggested by the formula $f(T) = (2\pi i)^{-1}\int_C f(\lambda)(\lambda I-T)^{-1}d\lambda$ conditions are derived which are necessary and sufficient for the convergence of a given sequence $P_n(T)$ of polynomials in a linear operator $T$ (on a complex Banach space $X$) to a projection on a manifold of the form $\mathrm{M}[P] = E_x[x \in X, P(T)x=0]$. (Received April 24, 1943.)

### 186. M. R. Hestenes: *On the condition of Weierstrass in the calculus of variations.*

The present paper is devoted to the study of properties of the Weierstrass $E$-function $E(y, p, \lambda, q)$ associated with the problem of Bolza. It is shown that an extremal $C_0$ is nonsingular and satisfies the strengthened condition of Weierstrass if and only if there is a constant $b>0$ such that the inequality $E(y, p, \lambda, q) \geqq bE_L(p, q)$ holds for all differentiably admissible elements $(y, p)$, $(y, q)$ with $(y, p, \lambda)$ in a neighborhood of the values $(y, y', \lambda)$ on $C_0$. Here $E_L$ is the $E$-function for the length integral. Further equivalence relations are given. Moreover, it is shown that if these conditions are satisfied then the integrand $f(y, y')$ can be replaced by a function of form $f+\theta (y, y') \phi^\beta\phi^\beta$ such that the above inequality holds without the restriction that $\phi^\beta(y, p) = \phi^\beta(y, q) = 0$. (Received April 23, 1943.)

### 187. Herman Kober: *On the approximation to integrable functions by integral functions.*

Let $f(t) \in L_p(-\infty, \infty)$, $0<p\leqq\infty$, and suppose that $f(t)$ may be approximated (in mean if $0<p<\infty$, uniformly if $p=\infty$) by integral functions of fixed order not greater than $\rho$, where $\rho$ is fixed. If $\rho<1$, $f(t)$ is a constant (zero if $p=\infty$). Here 1 is the precise limit and it is always possible to find a sequence of integral functions of order one and normal type approximating $f(t)$ (if $p=\infty$, if and only if $f(t)$ is uniformly continuous in $(-\infty, \infty)$). If the types of the approximating functions have a finite inferior limit $\alpha$, then $f(t)$ is an integral function of order one and type not greater than $\alpha$, or a constant. The best approximation by integral functions of order one and given type $\alpha$ can be determined and is given by the Dirichlet singular integral $D_\alpha[f]$ when $p=2$. Approximation by functions of order $\rho$ implies approximation by functions of larger order. If $L_p(-\infty, \infty)$ be replaced by $L_p(0, \infty)$, the order one is to be replaced by the order 1/2 throughout. Extensions to other approximation problems are possible. (Received April 17, 1943.)

### 188. E. R. Lorch: *The theory of analytic functions in normed abelian vector rings.*

The development of an analytic function theory for these rings $\mathfrak{R}$ has been announced previously (Bull. Amer. Math. Soc., abstract 46-11-469). Certain applications are here considered. The function exp $(z)$ is simply periodic if and only if $\mathfrak{R}$ is irreducible; alternatively, if and only if the spectrum of each $a \in \mathfrak{R}$ is a continuum or a point. The periods of exp $(z)$ are $2\pi i\sum_1^n \pm j_k$ where the $j_k$ are any idempotents. The function log $z$ exists for all $z$ in the principal component of the topological group of non-singular elements in $\mathfrak{R}$. An $a \in \mathfrak{R}$ may be embedded in a continuous group, $a^s \cdot a^t = a^{s+t}$, if and only if log $a$ exists; then $a^s = \exp(s \log a)$ where the range of $s$ may be taken to be $\mathfrak{R}$. All points in the frontier of the spectrum of $a \in \mathfrak{R}$ are permanent singularities (cannot be removed by ring extensions). If $\mathfrak{R}$ is irreducible, a simple closed rectifiable curve $C$ in $\mathfrak{R}$ generates a closed set of singular elements, an open set of exterior elements, and an open set of interior elements. The open sets need not be connected. In case $C$ has a tangent at one point, the set of interior points is not empty. (Received April 19, 1943.)

### 189. George Piranian: *On the order of analytic functions in the sense of Hadamard.*

If in the region $|z|>R$ the function $f(z) = \sum a_n/z^n$ is holomorphic except for poles of aggregate multiplicity $P$, if $D_{nP}$ is Hadamard's determinant of order $P+1$ and rank $n$ for $f(z)$, and if the order of the function $F_P(z) = \sum D_{nP}/z^n$ on its circle of convergence is $\omega$, then the order of $f(z)$ on the circle $|z| = R$ is at least $\omega$. Moreover, if the order of $f(z)$ on $|z| = R$ is $\omega' > \omega$, there either exist at least two points of order $\omega'$ on this circle or the singularity of order $\omega'$ is non-Fuchsian. By means of an extension of Mandelbrojt's method for finding the singularities of an analytic function on its circle of convergence, the theorem gives a formula for the order of every pole lying outside of the convex hull of non-polar singularities of $f(z)$, and for the order of every Fuchsian singularity on the boundary $V$ of the convex hull, provided the singularity is not an interior point of a straight-line segment of $V$. (Received April 12, 1943.)

### 190. Harry Pollard: *A new criterion for completely monotonic functions.*

The Bernstein criterion for completely monotonic functions states that if (i) $f(0+)$ exists and (ii) $(-1)^k \Delta_\delta^k f(x) \geqq 0$ for $k \geqq 0$, $\delta > 0$, $x > 0$, then $f(x)$ is completely monotonic in $0 \leqq x < \infty$. It is established in this paper that (ii) can be weakened to $(-1)^k \Delta_{\delta_k}^1 f(x) \geqq 0$ for a suitable sequence $\{\delta_k\}$. (Received April 10, 1943.)

### 191. W. J. Thron: *A general theorem on convergence regions for continued fractions $b_0 + K(1/b_n)$.*

Let the regions $B_0$ and $B_1$ be defined by: $r \cdot e^{i\theta} \in B_0$ if $r > (1+\epsilon) \cdot f(\theta)$, $r \cdot e^{i\theta} \in B_1$ if $r > (1+\epsilon) g(\theta)$, where $\epsilon$ is an arbitrary small positive number and the functions $f(\theta)$ and $g(\theta)$ are positive in the interval $[0, 2\pi]$. If it is required that the complements of the regions $B_0$ and $B_1$ be both convex and if $f(\theta) \cdot g(\pi - \theta) \geqq 4$, then the continued fraction $b_0 + K(1/b_n)$ converges if $b_{2n} \in B_0$ and $b_{2n+1} \in B_1$, that is $B_0$ and $B_1$ are twin convergence regions for the continued fraction. The condition $f(\theta) g(\pi - \theta) \geqq 4$ is a necessary condition for two regions to be twin convergence regions. (Received April 19, 1943.)

### 192. W. J. Thron: *Convergence regions for the general continued fraction.*

It is shown that the continued fraction $K(a_n/b_n)$ converges if all $a_n = r \cdot e^{i\theta}$ lie in a bounded part of the parabola $r \leqq a^2/2(1 - \cos(\theta - 2\gamma))$, and if all $b_n$ lie in the half-plane $R(b_n e^{i\gamma}) \geqq a + \epsilon$. Here $a > 0$ and $\epsilon$ is an arbitrary small positive number. (Received April 24, 1943.)

### 193. Hassler Whitney: *On the extension of differentiable functions.*

Let $A$ be a bounded closed set in Euclidean space $E$. Suppose that for some number $\omega$ any two points of $A$ are joined by an arc in $A$ of length not more than $\omega$ times their distance apart. Then any function of class $C^m$ in $A$ which, with derivatives through the $m$th order, is sufficiently small in $A$, may be extended throughout $E$ so as to be small, with its derivatives. (Received May 11, 1943.)

## GEOMETRY

### 194. T. C. Doyle: *Tensor theory of invariants for the projective differential geometry of a curved surface.*

This paper completes the explicit determination of differential invariants of all orders for a curved two dimensional surface begun by E. J. Wilczynski, *Projective differential geometry of curved surfaces (fourth memoir)*, Trans. Amer. Math. Soc. vol. 10 (1909) pp. 176–200. The Lie theory of groups serves to determine the number of existing invariants of any specified order and these are then exhibited by contraction of fundamental surface tensors and their covariant derivatives appearing as coefficients of a system of partial differential equations defining the surface. The projective normal is attained by the formal elimination procedure of tensor analysis (Received April 10, 1943.)

### 195. H. T. Muhly: *Independent integral bases and a characterization of regular surfaces.*

Let $\mathfrak{o}^*$ denote the ring of homogeneous coordinates, $\xi_0^*, \xi_1^*, \cdots, \xi_n^*$ associated with a normal, nonsingular model $F$, of a field $\Sigma$ of algebraic functions of two variables, and assume that $\xi_0^*, \xi_1^*, \xi_2^*$ are selected so that they are algebraically independent and so that each element of $\mathfrak{o}^*$ depends integrally upon these three elements. Let the relative degree $[K(\xi_0^*, \xi_1^*, \cdots, \xi_n^*):K(\xi_0^*, \xi_1^*, \xi_2^*)]$ be denoted by $\nu$. It is shown in this paper that if there exists a set of $\nu$ elements $\lambda_1^*, \lambda_2^*, \cdots, \lambda_\nu^*$ in $\mathfrak{o}^*$ which form an independent modular base for $\mathfrak{o}^*$ over the ring $K[\xi_0^*, \xi_1^*, \xi_2^*]$, then the field $\Sigma$ is regular, that is, its arithmetic genus $p_a$ coincides with its geometric genus $p_g$. Furthermore, it is shown that if the field $\Sigma$ is regular then there exist projective models of $\Sigma$ which are normal and nonsingular, and which are such that the associated ring of homogeneous coordinates has a linearly independent modular base over a suitably chosen ring of independent variables In fact if $F$ is any normal, nonsingular model of a regular field $\Sigma$, and if $F_k$ is the derived normal model belonging to the character of homogeneity $k$, then $F_k$ has these properties for all sufficiently large values of $k$. (Received May 15, 1943.)

## NUMERICAL COMPUTATION

### 196. A. N. Lowan and H. E. Salzer: *Coefficients in the expansion of derivatives in terms of central differences.*

The coefficients in the expansion of the first 52 derivatives in terms of mean central and central differences (in most cases up to the 42nd difference) were computed by the Mathematical Tables Project. For the first 30 derivatives the exact values of the coefficients are given in the form of ordinary fractions for the first 30 differences and for some differences beyond the 30th. For all derivatives beyond the 30th exact fractional values are given for the coefficients of differences up to orders varying between the 41st and 52nd. Finally for most of the coefficients of differences of orders varying from the 31st to the 42nd, 18 significant figures are given. All fractions are believed to be in lowest terms. The tabulated coefficients are valuable in the evaluation of analytic functions of a complex variable when known along a straight line within the region. The coefficients were checked by means of two recursion formulas, which are not mentioned in the literature. (Received April 8, 1943.)

## STATISTICS AND PROBABILITY

### 197. Jacob Wolfowitz: *Asymptotic distributions of ascending and descending runs.*

Let $a_1, a_2, \cdots, a_N$ be any permutation of $N$ unequal numbers. Let there be assigned to each permutation the same probability. An element $a_i$ $(1<i<N)$ is called a turning point if $a_i$ is greater than or less than both $a_{i-1}$ and $a_{i+1}$. Let $a_j$ and $a_{j+k}$ be consecutive turning points; they are said to determine a "run" of length $k$. The author

obtains the asymptotic distributions of a large class of functions of these runs. An example of his results is the following. It is proved that the following are asymptotically normally distributed: (a) the total number of runs; (b) $R(p)$, the number of runs of length $p$; (c) $R(p)$ and $R(q)$ jointly. Similar results are obtained for runs defined by any of a large set of criteria, of which the one given above is of value in statistical applications. (Received May 1, 1943 )

## TOPOLOGY

### 198. Paul Alexandroff: *On homological situation properties of complexes and closed sets.*

The purpose of this paper is to find and to study topological invariants which connect the homological properties of a space $K$ with those of its closed subset $A$ and of the open complement $G=K\backslash A$, and thus contribute to characterize from the homological point of view the *situation of $A$ in $K$*. Thus the paper constitutes an extension of results already known when $K$ is simply connected (when its $\beta$ groups are 0 and also when $K$ is a manifold). (Received April 3, 1943.)

## NEW PUBLICATIONS

Bell, C., and Thomas, T. Y. Essentials of plane and spherical trigonometry. New York, Holt, 1943. 6+152 pp. $1.80.

Brenke, W. C. Plane and spherical trigonometry. New York, Dryden, 1943. 10+269 pp. $1.90.

Camm, F. J. Mathematical tables and formulae. London, Newnes, 1943. 144 pp. 3s. 6d.

Craig, H. V. Vector and tensor analysis. New York and London, McGraw-Hill, 1943. 14+434 pp. $3.50.

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# ON A NEW APPLICATION OF JACOBI POLYNOMIALS IN CONNECTION WITH THE MEAN VALUE THEOREM

E. G. GARNEA

Let us consider the classical theorem of mean value, which states that

$$(1) \qquad U(\xi) = \frac{1}{b-a}\int_a^b U(x)dx$$

or

$$(1') \qquad F'(\xi) = \frac{1}{b-a}[F(b) - F(a)]$$

can be satisfied by at least one value of $\xi$ inside[1] the interval $(a, b)$.

In the general case we can add no further precision concerning the position of the value $\xi$ inside the interval $(a, b)$. But if we consider only functions $U(x)$ belonging to a definite class of functions, we can, sometimes, give a more precise determination for this value $\xi$. We can, in particular, for some classes of functions, determine intervals $(a', b')$, concentric to $(a, b)$, with

$$(2) \qquad b' - a' = \theta(b - a), \qquad 0 \leqq \theta \leqq 1,$$

and such that (1) holds for at least one value $\xi$ inside $(a', b')$, for every function $U(x)$ belonging to the class considered and for every interval $(a, b)$ for which the classical mean value theorem holds. The smallest number $\theta$ which has the above mentioned property for a given class of functions is called its "contraction factor." It results from this definition that the value of the contraction factor depends only on the class of functions considered and is independent of all other factors, such as the interval $(a, b)$, and so on.

If we replace the equation (1) by (1′) and repeat the foregoing literally, we define in exactly the same way the contraction factors for classes of functions $F(x)$.

The existence of a contraction factor for certain classes of functions, particularly for polynomials of a real variable, has been proved by Paul Montel.[2] The value of $\theta$ as a function of the degree $n$ of the polynomials considered was found independently and almost at the

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[1] The expression "inside" in this paper means: within or at the ends.

[2] P. Montel, Bull. Soc. Math. France vol. 58 (1930) pp. 105–126. See also D. Pompeiu, Annales Scientifiques de l'Université de Jassy vol. 15 (1929) p. 335.

same time by Tchakaloff[3] and Biernacki.[4] Their results have been generalized since by many authors, among whom we quote Favard,[5] Anghelutza,[6] H. L. Krall[7] and Cioranescu,[8] whose paper is particularly important for us.

Let $U(y_1, y_2, \cdots, y_\nu)$ be an analytical function of $\nu$ independent variables, polyharmonical of the order $2m$ and of mean value zero within the hypersphere $\Sigma_\nu$ of its $\nu$-dimensional space. Therefore

$$\Delta^{(2m)}U = 0, \tag{3}$$

$$\int_{\Sigma_\nu} U d\tau = 0, \tag{4}$$

where $\Delta^{(2m)}$ is the operator of Laplace applied successively $2m$ times and the integral is extended over the inside of the hypersphere $\Sigma_\nu$ of volume element $d\tau$.

We consider now the system of $2m$ equations with the $2m$ unknowns $k_i$, $x_i$:

$$\sum_{i=1}^{m} k_i x_i^{2q} = \frac{\nu}{\nu + 2q}, \qquad q = 0, 1, 2, \cdots, 2m - 1, \tag{5}$$

and let $x_M$ be the largest of the solutions $x_i$ of this system. N. Cioranescu proved in his above mentioned paper that $x_M$ is the contraction factor of $U(y_1, y_2, \cdots, y_\nu)$ if:

(a) all $x_i$ are real and inside $(-1, +1)$;

(b) all $k_i$ are positive.

We remember that this statement means: If (3), (4), (a) and (b) are satisfied and $R$ is the radius of $\Sigma_\nu$, there is at least one point $P_0$ inside the hypersphere of radius $x_M R$ which makes $U(P_0) = 0$.

From Cioranescu's demonstration it follows also that if (a) and (b) are satisfied, but the order of $U$ is odd, so that

$$\Delta^{(2m+1)}U = 0, \tag{3'}$$

$$\int_{\Sigma_\nu} U d\tau = 0, \tag{4'}$$

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[3] L. Tchakaloff, C. R. Acad. Sci. Paris vol. 192 (1931) p. 32.

[4] M. Biernacki, Bulletin de Mathématiques et de Physique, pures et appliquées de l'École Polytechnique de Bucarest, II Année, no. 3, pp. 164–168.

[5] M Favard, C. R. Acad. Sci. Paris vol. 192 (1931) p. 716.

[6] T. Anghelutza, Mathematica, Cluj vol. 6 (1932) p. 140.

[7] H. L. Krall, *On the mean value theorem*, Amer. Math. Monthly vol. 42 (1935) pp. 604–606.

[8] N. Cioranescu, *Quelques propriétés* · · · , Mathematica, Cluj vol. 9 (1935) pp. 184–193.

the contraction factor is the highest of the values $x_i$, which are solutions of the system[9] of $2m+2$ equations with $2m+2$ unknowns $x_i$, $k_i$:

$$(5') \qquad \sum_{i=1}^{m+1} k_i x_i^{2q} = \frac{\nu}{\nu+2q}, \qquad x_{m+1} = 0, \qquad q = 0, 1, 2, \cdots, 2m.$$

Here below we shall prove that the conditions (a) and (b) are always satisfied, we shall find the values of the contraction factors connected with the zeros of a sequence of Jacobi (more generally of Tchebycheff) polynomials, and shall see that our results are a generalization of Tchakaloff's and Biernacki's theorems.

Let[10] us put $x_i^2 = u_i$ $(i = 1, 2, \cdots, m)$. We observe that the second member of (5) can be written:

$$(6) \qquad \frac{\nu}{\nu+2q} = \int_0^1 \nu u^{\nu+2q-1} du = \int_0^1 (\nu/2) u^{(\nu/2)-1} u^q du.$$

These are for $q = 0, 1, 2, \cdots, 2m-1$ the first $m$ moments of the function

$$\psi(x) = (1/2)\int_0^x \nu u^{(\nu/2)-1} du;$$

therefore, we can write (5) as:

$$(7) \qquad \sum_{i=1}^{m} k_i u_i^q = \int_0^1 u^q d\psi(u), \qquad q = 0, 1, 2, \cdots, 2m-1.$$

We observe that $\psi(x)$ is monotonically increasing in (0, 1); to such $\psi(x)$ correspond, as is well known,[11] a sequence $\{\phi_m(\nu; x)\}$ of orthonormal[12] polynomials, which, in turn, give rise to a mechanical quadrature formula:[11]

$$(8) \qquad \int_0^1 G_{2m-1}(u) d\psi(u) = \sum_{i=1}^{m} H_i G_{2m-1}(l_i)$$

where $G_{2m-1}(x)$ represents an arbitrary polynomial of degree at most $2m-1$, the abscissas $l_i$ are the zeros of $\phi_m(\nu; x)$, all real, distinct and inside (0, 1), and the coefficients $H_i$ are all positive. It follows, for $G_{2m-1}(u)=u^q$,

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[9] This system is given by Cioranescu in his above mentioned paper for $m \leq 2$.

[10] I am indebted for this interesting method to Professor J. A. Shohat, whose demonstration I follow very closely.

[11] J. A. Shohat, *Théorie générale des polynômes orthogonaux de Tchebycheff*, Mémorial des Sciences Mathématique, fascicule 66, pp. 8, 15.

[12] In this case they are Jacobi polynomials and coincide, for $\nu = 2$, with Legendre's polynomials.

$$\sum_{i=1}^{m} H_i l_i^q = \int_0^1 u^q d\psi(u), \qquad q = 0, 1, 2, \cdots, 2m-1. \tag{9}$$

Comparing (7) with (9) we see that a solution of (7) is given by

$$u_i = l_i, \qquad k_i = H_i, \tag{10}$$

with $0<l_i<1$, $H_i>0$, and, returning to (5):

$$x_i = \pm (l_i)^{1/2}, \qquad k_i = H_i, \qquad i = 1, 2, 3, \cdots, m. \tag{11}$$

Similar considerations on the $m$ equations 2 to $m+1$ of (5′) (putting for instance $k_ix_i^2=k_i'$) led also to $m$ values $x_i^2=l_i'$ $(i=1, 2, \cdots, m)$ to which we have only to add $x_{m+1}=0$, in order to have the complete set.

We may remark that if in $\Delta^{(n)}U=0$, $n=2m$, the moments of $\psi(x)$ are $\alpha_p=\nu/(\nu+2p)$ and the corresponding Jacobi polynomials form a complete sequence; if $n=2m-1$, the moments are $\alpha_p=\nu/(\nu+2(p+1))$ and the corresponding Jacobi polynomials form another complete sequence.

It follows that the systems (5) and (5′) have solutions, where all $x_i$ are real and in absolute value less than one, and all $k_i$ are positive; therefore, the conditions of Cioranescu are always satisfied.[13]

We could find the $x_i$ also solving (5) by the method of Sylvester,[14] which, however, requires lengthy considerations of determinants. Its result is that the solutions $x_i$ of (5), are the zeros of the polynomial:

$$E_{2m}(\nu; x) = \begin{vmatrix} \frac{x^2}{\nu}-\frac{1}{\nu+2} & \frac{x^2}{\nu+2}-\frac{1}{\nu+4} & \cdots & \frac{x^2}{\nu+2(m-1)}-\frac{1}{\nu+2m} \\ \frac{x^2}{\nu+2}-\frac{1}{\nu+4} & \frac{x^2}{\nu+4}-\frac{1}{\nu+6} & \cdots & \frac{x^2}{\nu+2m}-\frac{1}{\nu+2(m+1)} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{x^2}{\nu+2(m-1)}-\frac{1}{\nu+2m} & \frac{x^2}{\nu+2m}-\frac{1}{\nu+2(m+1)} & \cdots & \frac{x^2}{\nu+2(2m-2)}-\frac{1}{\nu+2(2m-1)} \end{vmatrix}. \tag{12}$$

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[13] The present method has many points in common with that of J. A. Shohat, *On a certain formula of mechanical quadratures with non-equidistant ordinates*, Trans Amer. Math. Soc. vol. 31 (1929) pp. 449–450.

[14] For Sylvester's method see for instance T. Muir, *Theory of determinants*, vol. II, pp. 332–335. It may be noted that the formula (in Sylvester's notation): $a_{n+1}-a_n\sum\lambda_1+a_{n-1}\sum\lambda_1\lambda_2-\cdots=0$, which we meet there, generalizes an analogous equation indicated by P. Montel in his quoted paper.

Similarly, the solutions of (5′) are the zeros of

$$E_{2m+1}(\nu; x) = x \begin{vmatrix} \frac{x^2}{\nu+2}-\frac{1}{\nu+4} & \frac{x^2}{\nu+4}-\frac{1}{\nu+6} & \cdots & \frac{x^2}{\nu+2m}-\frac{1}{\nu+2(m+1)} \\ \frac{x^2}{\nu+4}-\frac{1}{\nu+6} & \frac{x^2}{\nu+6}-\frac{1}{\nu+8} & \cdots & \frac{x^2}{\nu+2(m+1)}-\frac{1}{\nu+2(m+2)} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{x^2}{\nu+2m}-\frac{1}{\nu+2(m+1)} & \frac{x^2}{\nu+2(m+1)}-\frac{1}{\nu+2(m+2)} & \cdots & \frac{x^2}{\nu+2(2m-1)}-\frac{1}{\nu+4m} \end{vmatrix}. \tag{12′}$$

These formulae have already been found for $\nu=1$ by Biernacki.[15]

We can thus make the general statement:

*If the analytical function $U(P)$ of $\nu$ independent variables $y_1, y_2, \cdots, y_\nu$ satisfies the conditions $\Delta^{(n)}U=0$ and $\int_{\Sigma_\nu} U d\tau=0$, there is inside the hypersphere of radius $x_M R$ at least one point $P_0$ which makes $U(P_0)=0$, $R$ being the radius of $\Sigma_\nu$ and $x_M<1$ the square root of the highest zero of the Jacobi polynomial*[16] *$\phi_m(\nu; x)$, where $m=n/2$ or $m=(n-1)/2$, according to the parity of $n$; the same value $x_M$ is also the highest zero of the polynomial*[17] *$E_n(\nu; x)$.*

Taking into account the formulae of Tchebycheff's polynomials:[18]

$$\Phi_n(x) = \frac{1}{\Delta_n(\psi)} \begin{vmatrix} \alpha_0 & \alpha_1 \cdots \alpha_n \\ \alpha_1 & \alpha_2 \cdots \alpha_{n+1} \\ \cdot & \cdot \;\cdots\cdots \\ \alpha_{n-1} & \alpha_n \cdots \alpha_{2n-1} \\ 1 & x \;\cdots\; x^n \end{vmatrix} \tag{13}$$

$$= x^n - \delta_{n,n-1}x^{n-1} + \delta_{n,n-2}x^{n-2} - \cdots$$

with

$$\Delta_n(\psi) = \begin{vmatrix} \alpha_0 & \alpha_1 \cdots \alpha_{n-1} \\ \alpha_1 & \alpha_2 \cdots \alpha_n \\ \cdot & \cdot \;\cdots\cdots \\ \alpha_{n-1} & \alpha_n \cdots \alpha_{2n-2} \end{vmatrix} \tag{14}$$

and the formulae of Szegö:[19]

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[15] M. Biernacki, loc. cit. p. 166.

[16] We can write immediately $\phi_m(\nu; x)$, knowing the moments of $\psi(x)$.

[17] As we shall see, the polynomials $E_n(\nu, x)$ form a *single* sequence of Tchebycheff polynomials, without distinction as to whether $n$ is even or odd.

[18] J. A. Shohat, loc. cit. pp. 3–5.

[19] G. Szegö, *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören*, Math. Zeit. vol. 9 (1921) p. 218.

$$(15)\qquad \Delta_{2m}(\psi)\,\Phi_{2m}(x) = \begin{vmatrix} \alpha_0 & 0 & \cdots & \alpha_{2m} \\ 0 & \alpha_2 & \cdots & 0 \\ \alpha_2 & 0 & \cdots & \alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \alpha_{2m} & \cdots & 0 \\ 1 & x & \cdots & x^{2m} \end{vmatrix} = \begin{vmatrix} +\alpha_0 x & -\alpha_2 & \cdots & -\alpha_{2m} \\ -\alpha_2 & +\alpha_2 x & \cdots & +\alpha_{2m} x \\ -\alpha_2 x & +\alpha_4 & \cdots & +\alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ +\alpha_{2m-2} x & -\alpha_{2m} & \cdots & -\alpha_{4m-2} \\ -\alpha_{2m} & +\alpha_{2m} x & \cdots & +\alpha_{4m-2} x \end{vmatrix},$$

$$(15')\qquad \Delta_{2m+1}(\psi)\,\Phi_{2m+1}(x) = \begin{vmatrix} \alpha_0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & \alpha_{2m+2} \\ \alpha_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m} & 0 & \cdots & 0 \\ 1 & x & \cdots & x^{2m+1} \end{vmatrix} = \begin{vmatrix} +\alpha_0 x & -\alpha_2 & \cdots & +\alpha_{2m} x \\ -\alpha_2 & +\alpha_2 x & \cdots & -\alpha_{2m+2} \\ +\alpha_2 x & -\alpha_4 & \cdots & +\alpha_{2m+2} x \\ \cdot & \cdot & \cdots & \cdot \\ -\alpha_{2m} & +\alpha_{2m} x & \cdots & -\alpha_{4m} \\ +\alpha_{2m} x & -\alpha_{2m+2} & \cdots & +\alpha_{4m} x \end{vmatrix}$$

we find easily:

$$(16)\qquad \Delta_{2m}(\psi)\,\Phi_{2m}(x) = \begin{vmatrix} \alpha_0 & 0 & \cdots & \alpha_{2m} \\ 0 & \alpha_2 & \cdots & 0 \\ \alpha_2 & 0 & \cdots & \alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \alpha_{2m} & \cdots & 0 \\ 1 & x & \cdots & x^{2m} \end{vmatrix} = C_{2m+2} \begin{vmatrix} \alpha_0 x^2 - \alpha_2 & \alpha_2 x^2 - \alpha_4 & \cdots & \alpha_{2m-2} x^2 - \alpha_{2m} \\ \alpha_2 x^2 - \alpha_4 & \alpha_4 x^2 - \alpha_6 & \cdots & \alpha_{2m} x^2 - \alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m-2} x^2 - \alpha_{2m} & \alpha_{2m} x^2 - \alpha_{2m+2} & \cdots & \alpha_{4m-4} x^2 - \alpha_{4m-2} \end{vmatrix},$$

$$(16')\qquad \Delta_{2m+1}(\psi)\,\Phi_{2m+1}(x) = \begin{vmatrix} \alpha_0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & \alpha_{2m+2} \\ \alpha_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m} & 0 & \cdots & 0 \\ 1 & x & \cdots & x^{2m+1} \end{vmatrix} = C_{2m+2}\, x \begin{vmatrix} \alpha_2 x^2 - \alpha_4 & \alpha_4 x^2 - \alpha_6 & \cdots & \alpha_{2m} x^2 - \alpha_{2m+2} \\ \alpha_4 x^2 - \alpha_6 & \alpha_6 x^2 - \alpha_8 & \cdots & \alpha_{2m+2} x^2 - \alpha_{2m+4} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m} x^2 - \alpha_{2m+2} & \alpha_{2m+2} x^2 - \alpha_{2m+4} & \cdots & \alpha_{4m-2} x^2 - \alpha_{4m} \end{vmatrix}.$$

where

$$(17)\qquad C_{2m} = \begin{vmatrix} \alpha_0 & \alpha_2 & \cdots & \alpha_{2m-2} \\ \alpha_2 & \alpha_4 & \cdots & \alpha_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m-2} & \alpha_{2m} & \cdots & \alpha_{4m-4} \end{vmatrix},$$

$$C_{2m+1} = \begin{vmatrix} \alpha_2 & \alpha_4 & \cdots & \alpha_{2m} \\ \alpha_4 & \alpha_6 & \cdots & \alpha_{2m+2} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{2m} & \alpha_{2m+2} & \cdots & \alpha_{4m-2} \end{vmatrix}. \tag{17}$$

Equating in (16) the coefficients of $x^{2m}$, and in (16′) the coefficients of $x^{2m+1}$, we obtain,[10] taking into account (13):

$$\Delta_{2m}(\psi) = C_{2m}C_{2m+1}$$

and

$$\Delta_{2m+1}(\psi) = C_{2m+1}C_{2m+2}$$

or generally

$$\Delta_n(\psi) = C_nC_{n+1}. \tag{18}$$

Let us put now:

$$\alpha_{2m} = \frac{1}{\nu + 2m}, \qquad \alpha_{2m+1} = 0. \tag{19}$$

Then the $C_n$, easily calculable, being all positive, (18) shows that $\Delta_n(\psi) > 0$. Therefore, the conditions of Hamburger[21] being satisfied, there is a monotonically increasing function $\Psi(x)$[22], whose moments are (19). Comparing (12), (12′) with (16), (16′), we see that the polynomials $E_n(\nu; x)$ are just the sequence of polynomials of Tchebycheff corresponding to $\Psi(x)$. As they form a sequence of Sturm[23] and as $E_n(\nu; 1) > 0$, $E_{2m}(\nu; 0) = (-1)^m C_{2m+1}$, $\lim_{x\to 0} (1/x)E_{2m+1}(\nu; x) = (-1)^m D_{2m}$, with $D_{2m}$ an easily calculable positive determinant, we find also in this way that all the $x_i$, zeros of $E_n(\nu; x)$ are real, distinct and inside $(-1, +1)$.[24]

The function $U$, polyharmonical of order $n$, can be in particular a polynomial of degree $2n-1$, of $\nu$ independent variables. If a polynomial is of an even degree, it may be considered for our purpose as

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[20] I have learned that the same formulae have been derived by C. Rees in his thesis on *Elliptic orthogonal polynomials*.

[21] H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math. Ann. vol. 81 (1920) pp. 235–319; vol. 82 (1921) pp. 120–164 and pp. 168–187.

[22] Taking into account the further conditions of the problem, we easily find this function, which is: $d\Psi(x) = (1/2)|x|^{\nu-1}dx$ for $|x| \leq 1$, $d\Psi(x) = 0$ for $|x| > 1$.

[23] See also Brioschi, *Théorie des déterminants*, pp. 85–86.

[24] By solving completely the system (5) it can be proved also that $k_i > 0$, but this requires a very long calculation, whereas the ingenious method of J. A. Shohat yields immediately the result. Some rather complicated expressions of the $k_i$ in function of $\nu$ are given, for small values of $n$, in the quoted paper of N. Cioranescu in Mathematica, Cluj vol. 9 (1935) pp. 191–192.

of the next higher odd degree, with its first coefficient equal to zero.[25]

For $\nu=1$ we fall back on the known theorems of P. Montel, Tchakaloff[26] and Biernacki.[27] In fact, the formulae (12), (12′), with (16), (16′) and (19) show us [28] that for $\nu=1$, $E_n(1;x)$ are but the polynomials $P_n(x)$ of Legendre; moreover:

$$\Delta^{(n)}U(y) = \frac{d^{2n}U(y)}{dy^{2n}} = 0 \tag{3′′}$$

if the degree of $U$ is at most $2n-1$. Applying our results from above to this case we find that if $x_M$ is the highest zero of Legendre's polynomial $P_n(x)$ and

$$\int_a^b U(y)dy = 0 \tag{4′′}$$

where $U(y)$ is a polynomial of degree at most $2n-1$, then $U(y)$ has at least one zero inside the interval $(1/2)(b+a) \pm x_M(b-a)/2$. These are precisely the results of Tchakaloff and Biernacki and our main statement from p. 545 can be considered as a generalization of the precisions which the above mentioned authors have brought to the mean value theorem.

HAVANA, CUBA

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[25] This has been shown for $\nu=1$ by P. Montel in his quoted paper. A slight difference can be noted between his statement and ours, because P. Montel's refers not to the function $U(y)$, but to $F(y)=\int_{y_0}^{y} U(y)dy$

[26] Tchakaloff, loc. cit. p. 34.

[27] Biernacki, loc. cit. p. 167–168.

[28] We may remark that the formulae (13), (13′), (16) and (16′) attest the equivalence of the solutions found by the two authors in the problem of the contraction factor, in spite of their different form.

# ALTERNATIVE ALGEBRAS OVER AN ARBITRARY FIELD

R. D. SCHAFER[1]

The results of M. Zorn concerning alternative algebras[2] are incomplete over modular fields since, in his study of alternative division algebras, Zorn restricted the characteristic of the base field to be not two or three. In this paper we present first a unified treatment of alternative division algebras which, together with Zorn's results, permits us to state that any alternative, but not associative, algebra $A$ over an arbitrary field $F$ is central simple (that is, simple for all scalar extensions) if and only if $A$ is a Cayley-Dickson algebra[3] over $F$.

A. A. Albert in a recent paper, *Non-associative algebras* I: *Fundamental concepts and isotopy*,[4] introduced the concept of isotopy for the study of non-associative algebras. We present in the concluding section of this paper theorems concerning isotopes (with unity quantities) of alternative algebras. The reader is referred to Albert's paper, moreover, for definitions and explanations of notations which appear there and which, in the interests of brevity, have been omitted from this paper.

## 1. Alternative algebras.

A distributive algebra $A$ is called an *alternative algebra* if $ax^2=(ax)x$ and $x^2a=x(xa)$ for all elements $a$, $x$ in $A$. That is, in terms of the so-called right and left multiplications, $A$ is alternative if $R_{x^2}=(R_x)^2$ and $L_{x^2}=(L_x)^2$.

The following lemma, due to R. Moufang,[5] and the Theorem of Artin are well known.

LEMMA 1. *The relations* $L_xR_xR_y=R_{xy}L_x$, $R_xL_xL_y=L_{yx}R_x$, *and* $R_xL_{xy}=L_yL_xR_x$ *hold for all* $a$, $x$, $y$ *in an alternative algebra* $A$.

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[1] This paper is the essential portion of the author's doctoral dissertation, written at The University of Chicago under the direction of Professor A. A. Albert, whose kindness the author gratefully acknowledges.

[2] M. Zorn, *Theorie der Alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) pp. 123–147; *Alternativkorper und Quadratische Systeme*, ibid. vol. 9 (1933) pp. 395–402. See M. Zorn, *Alternative rings and related questions* I: *Existence of the radical*, Ann. of Math. (2) vol. 42 (1941) pp. 676–686, to verify that the Peirce decomposition in Zorn's first paper is valid for $F$ of characteristic two.

[3] Defined over an arbitrary field by A. A. Albert in his *Quadratic forms permitting composition*, Ann. of Math. (2) vol. 43 (1942) pp. 161–177.

[4] Ann. of Math. (2) vol. 43 (1942) pp. 685–708.

[5] R. Moufang, *Zur Struktur von Alternativkorpern*, Math. Ann. vol. 110 (1934) pp. 416–430.

THEOREM OF ARTIN. *The subalgebra generated by any two elements of an alternative algebra $A$ is associative.*

It follows from the Theorem of Artin that if $x$ is any element of an alternative algebra $A$ over $F$, and if $f(\lambda)$ is any polynomial in $\lambda$ with coefficients in $F$, then $f(R_x)=R_{f(x)}$ and $f(L_x)=L_{f(x)}$.

Moreover, if an alternative algebra $A$ contains a right nonsingular element $x$ and a left nonsingular element $y$, then $A$ has a unity quantity. For the identity transformation $I$ is then a polynomial in $R_x$ with coefficients in $F$, and the correspondence $x \rightarrow R_x$ is one-to-one. Thus the unity quantity of $A$ and the inverse $x^{-1}$ exist, and are polynomials in $x$, and $R_{x^{-1}}=(R_x)^{-1}$. Clearly these results hold for an alternative division algebra $A$.

In this paper we shall require the following lemma.

LEMMA 2. *In an alternative division algebra $A$, the norm of a product $xy$ is equal to the product of the norms of $x$ and $y$.*

For if $x=0$ the lemma is obvious. Otherwise $L_x$ is nonsingular, and $L_xR_xR_y=R_{xy}L_x$ by Lemma 1. Thus $|L_x|\cdot|R_x|\cdot|R_y|=|R_{xy}|\cdot|L_x|$, and $|R_x|\cdot|R_y|=|R_{xy}|$. The conclusion follows.

Since any simple algebra $A$ over $F$ is central simple over its transformation center, the determination of all simple alternative algebras consists of a determination of those which are central simple. Zorn's results imply that a central simple alternative algebra over an arbitrary field is either (1) a division algebra, (2) an associative algebra, or (3) a Cayley-Dickson algebra with divisors of zero.[6] We are led directly to this theorem.

THEOREM 1. *Let $A$ be an alternative, but not associative, central division algebra over $F$. Then $A$ is an algebra of degree two and order eight over $F$.*

For there exists a scalar extension $A_K$ of $A$ such that $A_K$ over $K$ is not a division algebra. Since $A_K$ is not associative, $A_K$ is a Cayley-Dickson algebra (with divisors of zero) over $K$. Hence $A$ is of degree two and order eight over $F$.

2. **Alternative division algebras of degree two.** We are able to make a study of alternative division algebras of degree two which is independent of Zorn's results, and (although a portion of the result is indicated in Theorem 1) we shall do so. For the proof of Theorem 2 we require the two lemmas which follow.

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[6] Zorn's so-called "vector matrix algebra" is (over any field, including those of characteristic two) merely a Cayley-Dickson algebra with divisors of zero.

LEMMA 3. *Let the principal function of any element $x$ of an alternative division algebra of degree two over $F$ be $x^2 - t(x)\cdot x + n(x)$, where $t(x)$ and $n(x)$ are in $F$. Then the linear transformation*

$$S:\quad x \leftrightarrow xS = t(x) - x \tag{1}$$

*is an involution of $A$ such that $x + xS = t(x)$, and $x(xS) = (xS)x = n(x)$.*

The conclusions are trivial except for showing that $S$ is an involution of $A$. Clearly $S^2 = I$. By Lemma 2 we have $(xy)\cdot(xy)S = n(xy) = n(x)\cdot n(y) = (x\cdot xS)\cdot n(y)$. It follows from the Theorem of Artin that $(xy)S = (xy)^{-1}\cdot(xy)\cdot(xy)S = y^{-1}x^{-1}x(xS)\cdot n(y) = n(y)\cdot y^{-1}(xS) = (yS\ y)y^{-1}(xS) = yS\cdot xS$. Hence $S$ is an involution of $A$.

LEMMA 4. *Let $B$ be an alternative algebra of order $2s$ over $F$ defined as the supplementary sum $B = f_1D + \cdots + f_sD$, where $D = (1, u_2)$, $u_2^2 = u_2 + \alpha$, $-4\alpha \neq 1$, $f_1 = 1$, $f_i^2 = \gamma_i \neq 0$, $\gamma_i$ in $F$, $d\ f_i = f_i\cdot dS$ for all $d$ in $D$, $S$ as in (1), $(i = 2, \cdots, s)$. Then, if $B$ is a proper subalgebra of an alternative division algebra $A$ of degree two over $F$, there exists an element $g$ in $A$, but not in $B$, such that*

$$g^2 = \gamma \neq 0,\ \gamma \text{ in } F, \text{ and } xg = g(xS) \tag{2}$$

*holds for all $x$ in $B$.*

It is evident that $f_jD\cdot f_jD \subseteq D$ and that the intersection $(f_jD\cdot f_kD) \cap D = 0$ for $j \neq k$, $j, k = 1, 2, \cdots, s$. In order to establish (2) it is sufficient to prove the existence of $g$ with the trace $t(xg) = 0$ for all $x$ of $B$. Now if $y$ is any element of $B$, then $y = d_1 + f_2d_2 + \cdots + f_sd_s$ with $d_i$ in $D$. But $t(f_jd_j) = 0$ for $j \neq 1$. Hence $t(y) = t(d_1)$.

Let $v$ be an element of $A$ not in $B$, and write $g = (\lambda_1 + \lambda_2u_2) + f_2(\lambda_3 + \lambda_4u_2) + \cdots + f_s(\lambda_{2s-1} + \lambda_{2s}u_2) + v$, where the $\lambda_i$ are undetermined coefficients in $F$. Denote $t(f_jv)$ by $\mu_{2j-1}$ and $t(f_ju_2\cdot v)$ by $\mu_{2j}$. Then the existence of $g$ satisfying (2) is equivalent to the existence of $\lambda_i$ $(i = 1, \cdots, 2s)$ in $F$ such that

$$\begin{aligned}
t(g) &= 2\lambda_1 + \lambda_2 + \mu_1 = 0,\\
t(u_2g) &= \lambda_1 + (1+2\alpha)\lambda_2 + \mu_2 = 0,\\
t(f_jg) &= 2\gamma_j\lambda_{2j-1} + \gamma_j\lambda_{2j} + \mu_{2j-1} = 0, \qquad j = 2, \cdots, s,\\
t(f_ju_2\cdot g) &= \gamma_j\lambda_{2j-1} - 2\alpha\gamma_j\lambda_{2j} + \mu_{2j} = 0, \qquad j = 2, \cdots, s.
\end{aligned}$$

But the determinant $\Delta$ of the coefficients of this system of linear equations is $\Delta = (1+4\alpha)\cdot\prod_2^s\gamma_j^2(-1-4\alpha) \neq 0$. Hence the desired solutions $\lambda_i$ exist.

An algebra $Q$ is called a *quaternion algebra* if $Q = (1, u_2, u_3, u_4)$, $u_4 = u_3u_2$, $u_2^2 = u_2 + \alpha$, $u_3^2 = \beta$, $u_2u_3 = u_3(1 - u_2)$, where $\alpha$ and $\beta \neq 0$ are in $F$,

$-4\alpha\neq 1$. Moreover, $Q$ is a division algebra if and only if there exist no $\lambda$ and $\mu$ in $F$ such that $\beta=\lambda^2+\lambda\mu-\alpha\mu^2$.

An algebra $C$ is called a *Cayley-Dickson algebra* if $C=Q+gQ$, with elements $z=x+gy$, where $x$, $y$ are quaternions, and multiplication is defined by

$$(3)\quad (x_1+gy_1)(x_2+gy_2)=(x_1x_2+\gamma y_2\cdot y_1S)+g(x_1S\cdot y_2+x_2y_1),$$

where $g^2=\gamma\neq 0$ in $F$ and $S$ is the involution (1) of $Q$.

THEOREM 2. *An alternative algebra $A$ over $F$ is a division algebra of degree two over $F$ if and only if $A$ is one of the following:*

(a) *a separable quadratic field, or an inseparable field of exponent two,*

(b) *a quaternion division algebra $Q$, or*

(c) *a Cayley-Dickson algebra $C=Q+gQ$, where $Q$ is a division algebra and there exist no $\lambda$, $\mu$, $\rho$, $\sigma$ in $F$ such that*

$$(4)\qquad \gamma=\lambda^2+\lambda\mu-\alpha\mu^2-\beta\rho^2-\beta\rho\sigma+\alpha\beta\sigma^2.$$

If $A$ is generated by less than three elements, then $A$ is associative and is either (a) or (b). Otherwise, if the characteristic of $F$ is not two, $A$ contains a quaternion subalgebra $Q$ as in (b), and an element $v$ which is not in $Q$. If the characteristic of $F$ is two, consider two cases: if $A$ is commutative, then $A$ is associative[7] and is either (a) or (b). If $A$ is not commutative, there exist two noncommutative elements $x$, $y$. These generate an associative, noncommutative subalgebra (a quaternion algebra $Q$) of $A$. Also there exists an element $v$ of $A$ which is not in $Q$.

The algebra $Q$ is a particular example, $s=2$, of the algebra $B$ defined in Lemma 4, $f_2=u_2$, $\gamma_2=\beta$. Thus there exists an element $g$ in $A$, but not in $Q=B$, satisfying (2). Then $A$ contains $A_0=B+gB$, the elements $z$ of $A_0$ being expressible uniquely as $z=x+gy$ for $x$, $y$ in $B$. We make effective use of equation (2), Lemma 1, and the Theorem of Artin in proving that $A_0$ is an algebra in which multiplication is defined by (3).

For $(gy_1)(gy_2)=(gy_1)(y_2S\cdot g)=[g(y_1\, y_2S)]g=[(y_2\cdot y_1S)g]g=(y_2\cdot y_1S)g^2=\gamma y_2\cdot y_1S$. Also from the fact that $y_1S=\gamma^{-1}y_1S\gamma=\gamma^{-1}y_1S\cdot g^2=\gamma^{-1}(y_1S\cdot g)g=\gamma^{-1}(gy_1)g$, it follows that $g(x_2y_1)=(y_1S\cdot x_2S)g=[\{\gamma^{-1}(gy_1)g\}x_2S]g=\gamma^{-1}(gy_1)[g(x_2S\cdot g)]=\gamma^{-1}(gy_1)[g(gx_2)]=\gamma^{-1}(gy_1)\cdot(\gamma x_2)=(gy_1)x_2$. But then $x_1=\gamma^{-1}(g\cdot x_1S)g$ and $g(y_2S\cdot x_1)=(gx_1)\cdot y_2S$ also. Hence $x_1(gy_2)=[\gamma^{-1}(g\cdot x_1S)g](gy_2)=\gamma^{-1}[g(x_1S\cdot g)](y_2S\cdot g)=\gamma^{-1}[g\{(x_1S\cdot g)y_2S\}]g=\gamma^{-1}[g\{(gx_1)y_2S\}]g=\gamma^{-1}[g\{g(y_2S\cdot x_1)\}]g=\gamma^{-1}(\gamma y_2S\cdot x_1)g=g(x_1S\cdot y_2)$. Hence equation (3) holds.

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[7] A commutative alternative algebra over a field of characteristic not three is associative. See M. Zorn, *Theorie der Alternativen Ringe*, op. cit.

Having shown that any alternative division algebra of degree two, which is not associative, must contain the algebra $A_0$, we need now only to prove that $A_0$ is actually an alternative algebra and is not a proper subalgebra of any alternative division algebra of degree two. We may readily verify these conclusions if we refer to the matrices corresponding to the linear transformations we have used. Thus, for $z=x+gy$ in $A_0$,

$$R_z = \begin{pmatrix} R_x & SR_y \\ \gamma SL_y & L_x \end{pmatrix}, \qquad L_z = \begin{pmatrix} L_x & R_y \\ \gamma R_{yS} & L_{Sx} \end{pmatrix} \tag{5}$$

where the matrices $R_x$, $L_x, \cdots, S$ are the 4-by-4 matrices of the linear transformations induced on $Q$ by the corresponding transformations on $A_0$. Computation, involving the associativity of $Q$, yields $R_{z^2}=(R_z)^2$ and $L_{z^2}=(L_z)^2$. Hence the Cayley-Dickson algebra $C=A_0$ is alternative. If $C$ were a proper subalgebra of $A$, then $A$ would contain a new $A_0=C+hC$ with multiplication defined by (5), $x$ and $y$ being in $C$. But computation with the matrices above reveals that if $A_0$ were alternative, $C$ would be associative, which it is not.

Condition (4) follows from Lemmas 2 and 3. Let $z=x+gy$ with $x, y$ in $Q$, and let $S$ be defined by (1). Then $n(z)=z\cdot zS=(x+gy)(xS+yS\cdot gS)$ $=(x+gy)(xS-gy)=x\cdot xS-\gamma y\cdot yS=n(x)-\gamma n(y)$. Now $C$ is a division algebra if and only if $n(z)\neq 0$ for every nonzero $z$ in $C$. But $n(x)-\gamma n(y)=0$ if and only if $\gamma=n(x)[n(y)]^{-1}=n(xy^{-1})=n(v)$, the norm of $v$ for some $v$ in $Q$. Let $v=\lambda+\mu u_2+\rho u_3+\sigma u_4$. Then $C$ is a division algebra if and only if there exist no $\lambda$, $\mu$, $\rho$, $\sigma$ in $F$ such that $\gamma=n(v)=v\cdot vS=\lambda^2+\lambda\mu-\alpha\mu^2-\beta\rho^2-\beta\rho\sigma+\alpha\beta\sigma^2$.

We may combine the results of Zorn and Theorems 1 and 2 in the following manner and say: an alternative, but not associative, algebra $A$ over an arbitrary field $F$ is central simple if and only if $A$ is a Cayley-Dickson algebra over $F$.

3. **Isotopes of alternative algebras.** Albert has proved that an algebra $A$ with a unity quantity is associative if and only if every isotope of $A$ with a unity quantity is associative and is equivalent to $A$. We consider here the corresponding problem for alternative algebras.

THEOREM 3. *Let $A$ be an alternative algebra, and $B$ be an isotope of $A$ with a unity quantity. Then $A$ has a unity quantity, and $B$ is alternative.*

For $B$ is equivalent to a principal isotope $A_0$ of $A$, in which products $(a, x)$ are defined by $(a, x) = aR_x^{(0)}$ with $R_x^{(0)} = PR_{xQ}$ for nonsingular transformations $P$, $Q$. (In terms of left multiplications, $(x, a) = aL_x^{(0)}$ with $L_x^{(0)} = QL_{xP}$.) Let $e$ be the unity quantity of $A_0$ and $h = eQ$. Then $I = R_e^{(0)} = PR_h$, and $R_h$ is nonsingular. Similarly if $k = eP$, then $L_k$ is nonsingular, and it follows from the proof in §1 that the algebra $A$ has a unity quantity.

Now $P = R_h^{-1} = R_{h^{-1}}$. Hence $R_x^{(0)} = R_{h^{-1}}R_{xQ} = L_h R_{h^{-1}(xQ)} L_{h^{-1}}$ by Lemma 1. That is, $R_x^{(0)} = HR_{(xT)H}H^{-1}$ where $H = L_h$ and $T = QL_h^{-1}$. Hence $A_0$ is equivalent to an isotope $A_1$ of $A$ in which products are denoted by $[a, x] = aR_x^{(1)}$ where $R_x^{(1)} = R_{xT}$. Also $[x, a] = aL_x^{(1)}$ where $L_x^{(1)} = TL_x$. Let $f$ be the unity quantity of $A_1$. Then $I = L_f^{(1)} = TL_f$ and $T = L_f^{-1} = L_{f^{-1}}$. Hence $R_x^{(1)} = R_{f^{-1}x}$. Therefore $R^{(1)}_{[x,x]} = R^{(1)}_{xR_x^{(1)}} = R_{xR_{f^{-1}x}} = R_{f^{-1}\{x(f^{-1}x)\}} = R_{(f^{-1}x)^2} = R_{f^{-1}x}R_{f^{-1}x} = R_x^{(1)}R_x^{(1)}$ since $A$ is alternative. Since $B$ is equivalent to $A_1$, it follows that half of the alternative law holds in $B$.

But similarly $B$ is equivalent to an isotope $A_2$ of $A$ in which products are denoted by $\{x, a\} = aL_x^{(2)}$ where $L_x^{(2)} = L_{xc^{-1}}$, the element $c$ being the unity quantity of $A_2$. Then $L^{(2)}_{\{x,x\}} = L_x^{(2)}L_x^{(2)}$, and the second half of the alternative law holds in $B$.

We complete the study of isotopy for simple alternative algebras by proving the following theorem.

THEOREM 4. *Let $A$ be a Cayley-Dickson algebra, and $B$ be any isotope of $A$ with a unity quantity. Then $B$ is equivalent to $A$.*

Any isotope with a unity quantity of a central simple algebra $A$ with a unity quantity is also central simple. Therefore, any isotope $B$ with a unity quantity of a Cayley-Dickson algebra $A$ is also a Cayley-Dickson algebra. We shall show that $B$ is equivalent to $A$.

For $B$ is equivalent to an algebra $A_1$ in which products are denoted by $[a, x] = aR_x^{(1)}$ where $R_x^{(1)} = R_{f^{-1}x}$, the element $f$ being the unity quantity of $A_1$. Now $f$, as an element in $A$, is contained in some quaternion subalgebra $Q$ of $A$. Let $x$ range over $Q$, and $R$ be the subspace of $A_1$ consisting of all elements $fx$. Then $x \leftrightarrow fx$ is an equivalence of $Q$ and $R$. For since $Q$ contains $f$ and is associative, $[fx, fy] = fxR_{f^{-1}fy} = f(xy)$ for all $x$, $y$ in $Q$.

Let $S$ be the involution of $Q$ defined by (1), and let $z = fx$. Then the transformation

$$U: \quad z \leftrightarrow zU = f(xS)$$

is the corresponding involution of $R$. Now $A = Q + gQ$ as in (3). Also $A_1 = R + wR$, where $w = fg$ (and where the multiplication defining $wR$

is of course the multiplication in $A_1$). By the proof of Theorem 2, in order to show the equivalence of $A$ and $A_1$ it is sufficient to show that $[w, w]=\gamma f$ and $[w, z]=[zU, w]$ for every $z$ of $R$. But $[w, w]=w(f^{-1}w)=(fg)(f^{-1}fg)=fg^2=\gamma f$, and $[w, z]=w(f^{-1}z)=(fg)(f^{-1}fx)=(fg)x=g(x\cdot fS)=(f\cdot xS)g=(f\cdot xS)(f^{-1}fg)=zU(f^{-1}w)=[zU, w]$. This proves the theorem.

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# ON FIBRE SPACES. I

RALPH H. FOX

In subsequent papers I propose to investigate various properties of fibre spaces.[1] The object of the fundamental Hurewicz-Steenrod definition[1] is to state a minimum[2] set of readily verifiable conditions under which the covering homotopy theorem[1] holds. An apparent defect of their definition is that it is not topologically invariant. In fact, for topological space $X$ and metrizable non-compact space $B$ the property "$X$ is a fibre space over $B$" depends on the metric of $B$. The object of this note is to give a topologically invariant definition of fibre space and to show that (when $B$ is metrizable) $X$ is a fibre space over $B$ in this sense if and only if $B$ has a metric in which $X$ is a fibre space over $B$ in the sense of Hurewicz-Steenrod. Since the definition of fibre space is controlled by the covering homotopy theorem, an essential part of my program is to give a topologically invariant definition of uniform homotopy.

Let $\pi$ be a continuous mapping of a topological space $X$ into another topological space $B$. Let $\Delta=\Delta(B)$ denote the diagonal set $\sum_{b\in B}(b, b)$ of the product space $B\times B$ and let $\bar\pi$ denote the mapping of $X\times B$ into $B\times B$ which is induced by the mapping $\pi$ according to the rule $\bar\pi(x, b)=(\pi(x), b)$. Thus the graph $G$ of $\pi$ is the set $\bar\pi^{-1}(\Delta)$, and $\bar\pi^{-1}(U)$ is a neighborhood of $G$ whenever $U$ is a neighborhood of $\Delta$.

Any neighborhood $U$ of $\Delta$ determines uniquely a covering of $B$ by neighborhoods $N_U(b)$ according to the rule $b'\in N_U(b)$ when $(b, b')\in U$.

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[1] W. Hurewicz and N. E. Steenrod, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) p. 61.

[2] How well they succeeded in this will be indicated in my next communication.

However not every covering of $B$ by neighborhoods need arise in this fashion—although the star neighborhoods of any open covering of $B$ may always be so generated.

A *slicing function* $\phi$ for $\pi$ is any continuous mapping defined over $\bar{\pi}^{-1}(U)$ for some neighborhood $U$ of $\Delta$, with values in $X$, which satisfies the conditions

$$\pi\phi(x, b) = b,$$
$$\phi(x, \pi(x)) = x,$$

whenever $\phi$ is defined. I shall call $\pi$ a *fibre mapping* relative to $U$ if it has a slicing function defined over $\bar{\pi}^{-1}(U)$. If $\pi$ is a fibre mapping I shall say that $X$ is a *fibre space* over the subset $\pi(X)$ of $B$. Since $U$ is a neighborhood of $\Delta$, $\pi(X)$ is open and closed in $B$.

This new definition is equivalent to the old one if the base space is compact metric (so that the Hurewicz-Steenrod definition is topologically invariant in this case). In fact, for metric space $B$, let $\sigma_\epsilon$ denote that neighborhood of $\Delta$ which determines the covering of $B$ by $\epsilon$-spheres. Clearly $X$ is a fibre space (relative to $\pi$) over the metric space $\pi(X)$ in the sense of Hurewicz-Steenrod if and only if $\pi$ has a slicing function defined over $\bar{\pi}^{-1}(\sigma_\epsilon)$ for some $\epsilon > 0$. Hence, *if $\pi$ is a fibre mapping and $\pi(X)$ is compact metrizable then $X$ is a fibre space over $\pi(X)$ in the sense of Hurewicz-Steenrod no matter how $\pi(X)$ is metrized.*

Now let $B$ denote an arbitrary metrizable space, let $U$ be a neighborhood of $\Delta$ and let $\pi$ be a fibre mapping whose slicing function is defined over $\bar{\pi}^{-1}(U)$. For simplicity, assume also that $\pi(X) = B$. To show that $X$ is a fibre space in the sense of Hurewicz-Steenrod when $B$ is properly metrized it is clearly sufficient to so metrize $B$ that $\sigma_\epsilon \subset U$ for some $\epsilon > 0$.

LEMMA.[3] *If $B$ is metrizable and $U$ is an open neighborhood of $\Delta(B)$ then $B$ can be so metrized that $\sigma_1 \subset U$.*

Choose any random metric $d$ for $B$. Since $B \times B$ is metric, hence normal, it is possible to define a continuous function $f \in [0, 1]^{B\times B}$ such that

$$f(b, b_0) = \begin{cases} 0 & \text{when} \quad (b, b_0) \in \Delta, \\ 1 & \text{when} \quad (b, b_0) \in B \times B - U. \end{cases}$$

Let $\phi$ denote the (continuous) mapping $b \rightarrow f_b$, where $f_b(b_0) = f(b, b_0)$.

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[3] This proof is modelled after a proof in André Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Actualités Scientifiques et Industrielles, no. 551, 1938, p. 15.

The graph $B' = \sum_{b \in B}(b, \phi(b))$ of $\phi$ is homeomorphic to $B$. The metric of $B'$ is induced by the metric of the product $B \times [0, 1]^B$ and is given by the formula

$$\delta(b_1', b_2') = \left\{ d^2(b_1, b_2) + d^2(\phi(b_1), \phi(b_2)) \right\}^{1/2},$$

where $b$ and $b'$ denote corresponding points of $B$ and $B'$. If now $(b_1', b_2') \in \sigma_1$ then $\delta(b_1', b_2') < 1$, hence $d(\phi(b_1), \phi(b_2)) < 1$, hence $\sup_{b \in B} |f(b_1, b) - f(b_2, b)| < 1$. It follows that $f(b_1, b_2) = |f(b_1, b_2) - f(b_2, b_2)| < 1$, so that $(b_1, b_2) \in U$ and $(b_1', b_2') \in U'$.

THEOREM. *If $\pi$ is a fibre mapping and $B$ is metrizable then the metric of $B$ can be so chosen that $X$ is a fibre space over $\pi(X)$ (relative to $\pi$) in the sense of Hurewicz and Steenrod.*

I conclude by defining uniform homotopy and stating the covering homotopy theorem for general fibre spaces. If $h$ is a homotopy in $B$ of a space $Y$ and $U$ is a neighborhood of $\Delta$ I shall say that $h$ is ***uniform*** with respect to $U$ if there is a $\delta > 0$ such that $|t - t'| < \delta$ implies that $\sum_{y \in Y}(h(y, t), h(y, t')) \subset U$. Let $E_\delta = \sum_{0 \leq t, t' \leq 1, |t - t'| < \delta} \sum_{y \in Y}(h(y, t), h(y, t'))$, so that $E_0 \subset \Delta$ and $E_1 \subset B \times B$. Clearly the neighborhoods $U$ with respect to which $h$ is uniform are those which contain an $E_\delta$ for some $\delta > 0$. Thus $h$ is always uniform with respect to $B \times B$; in the event that $Y$ is compact $h$ is uniform with respect to every neighborhood $U$. I shall call a homotopy $h^*$ in $X$ a *covering homotopy* (with respect to $\pi$) if

(1) $\pi h^* = h$,

(2) $h^*_{[0,1]}(y)$ degenerates to a point whenever $h_{[0,1]}(y)$ degenerates to a point.

I shall refer to the mappings $h_0$ and $h_0^*$ as the initial values of the homotopies $h$ and $h^*$, respectively. With these notations the covering homotopy theorem for fibre mappings reads thus.

THEOREM. *Given a fibre mapping $\pi \in B^X$ relative to $U$, a mapping $g \in X^Y$ and a homotopy $h$ in $B$, uniform with respect to $U$, with initial value $\pi g$, there exists a covering homotopy $h^*$ in $X$ with initial value $g$.*

The covering homotopy $h^*$ is constructed stepwise[1] and is easily seen to be uniform with respect to $U^* = \dot{\pi}^{-1}(U)$ where $\dot{\pi}(x, x') = (\pi(x), \pi(x'))$. Of course if $U$ is a $\sigma_\epsilon$ the neighborhood $U^*$ of $\Delta(X)$ need not be a $\sigma_\epsilon(X)$.

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# CHAINS IN PARTIALLY ORDERED SETS

OYSTEIN ORE

**1. Introduction.** Dedekind [1] in his remarkable paper on *Dualgruppen* was the first to analyze the axiomatic basis for the theorem of Jordan-Hölder in groups. Recently there have appeared a considerable number of investigations in this field; one should mention the contributions by Birkhoff [2], Klein-Barmen [3, 4], Uzkow [5], George [6], Kurosch [7], and the author [8, 9]. Most of these papers deal particularly with the conditions which it is necessary to impose upon a structure or lattice in order to assert that the chains between two elements have the same length.

In the present paper the general problem of comparing chains in partially ordered sets is considered. It seems remarkable and somewhat surprising that one can formulate a theorem about chains in arbitrary partially ordered sets which contains the theorem of Jordan-Hölder as a special case when applied to more restricted systems. This theorem yields for partially ordered sets, hence by specialization also for structures, a necessary and sufficient condition for the chains to have equal lengths.

**2. Simple cycles.** Let us indicate briefly the terminology which we shall use. The basic partially ordered set to be investigated shall be denoted by $P$. An element $x$ in $P$ *lies between* two elements $a \supset b$ when $a \supset x \supset b$ and *properly between* them when the two possibilities $a = x$ and $b = x$ are excluded. An element $a$ is *prime over* $b$ and $b$ is *prime under* $a$ when $a \supset b$ and there are no elements properly between $a$ and $b$. A *chain* is an *ordered subset* of $P$. A *finite chain* between $a$ and $b$ has the form

$$a = a_0 \supset a_1 \supset \cdots \supset a_{n-1} \supset a_n = b$$

where $n$ is the *length* of the chain. A chain is *complete* when it is not possible to intercalate further terms in it. Through transfinite induction one can prove that any chain is contained in some complete chain.

Now let $C_1$ and $C_2$ be two complete chains joining two elements $a \supset b$ in such a manner that they have no elements in common except the end points $a$ and $b$. Such two chains shall be said to form a *simple cycle* when the following condition is fulfilled:

*Simple cycle.* There shall exist no elements properly between $a$ and $b$

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containing or being contained in two different elements $c_1$ and $c_2$ belonging to $C_1$ and $C_2$, respectively.

The two chains $C_1$ and $C_2$ shall be called the *components* of the simple cycle they constitute.

To clarify the definition of a simple cycle let us observe that it is tacitly assumed that $c_1$ and $c_2$ are different from the end points $a$ and $b$ of the cycle. The definition clearly excludes that any $c_1$ in $C_1$ contains a $c_2$ in $C_2$ or conversely.

We shall denote by $C_b^a$ some complete chain from $a$ to $b$. If $C_b^a$ passes through the two elements $m \supset n$ we shall write $C_n^m$ for the part of the chain connecting $m$ with $n$. Now let $D_n^m$ be some other complete chain between $m$ and $n$ so that a new chain $D_b^a$ may be formed from $C_b^a$ by replacing $C_n^m$ by $D_n^m$. When in this case the two chains $C_n^m$ and $D_n^m$ together form the components of a simple cycle we shall say that $D_b^a$ has been obtained from $C_b^a$ by a *simple deformation*. Finally we shall say that two arbitrary chains $C_b^a$ and $D_b^a$ between $a$ and $b$ are *related* and one has been obtained by *deformation* from the other when one can proceed from one to the other by a series of successive simple deformations.

3. **Related chains.** By means of these concepts we can prove the following theorem which is basic for the comparison of chains in partially ordered sets:

THEOREM 1. *Let $P$ be a partially ordered set in which the chains between two elements have a bounded length. Then any two complete chains between two elements $a \supset b$ are related.*

PROOF. The theorem is evidently true when $a$ is prime over $b$. In this case the longest chain between $a$ and $b$ has the length 1. The theorem may therefore be proved by means of induction with respect to the length of the longest complete chain connecting the two elements $a \supset b$. We assume that the theorem holds for all pairs of elements $a' \supset b'$ for which the maximal chain has a length less than $n$ and prove it for any pair $a \supset b$ for which a maximal chain $C_b^a$ has the length $n$. We shall have to show that if $D_b^a$ is any other complete chain connecting $a$ and $b$ then $D_b^a$ can be obtained from $C_b^a$ by a series of simple deformations.

One can immediately exclude the possibility of $C_b^a$ and $D_b^a$ having elements in common other than $a$ and $b$ because in this case the desired result follows from the induction assumption. In the case where $C_b^a$ and $D_b^a$ form a simple cycle the result is trivially true. It may therefore be assumed for instance that there exists an element $m$ properly between $a$ and $b$ such that $m$ contains an element $c$ in $C_b^a$ and an element $d$ in $D_b^a$. Correspondingly there exist complete chains

$$(1)\qquad E_m^a,\quad F_c^m,\quad G_d^m.$$

Since the chain

$$(2)\qquad E_m^a + F_c^m$$

joins $a$ to $c$ and since the maximal length between $a$ and $c$ is less than $n$ it follows by the induction assumption that $C_c^a$ and the chain (2) are related. Thus also $C_b^a$ and the chain

$$(3)\qquad C' = E_m^a + F_c^m + C_b^c$$

are related. But the two chains

$$(4)\qquad F_c^m + C_b^c,\qquad G_d^m + D_b^d$$

are also related as a consequence of the induction so that $C'$ and the chain

$$(5)\qquad C'' = E_m^a + G_d^m + D_b^d$$

between $a$ and $b$ are related. Finally one sees that $C''$ must be related to

$$(6)\qquad D_b^a = D_d^a + D_b^d,$$

so that the theorem is proved.

As a referee of this paper Professor S. MacLane observed that Theorem 1 holds also when one assumes only that the chains between any two elements are finite. A proof of this fact will be published subsequently. This gives a more direct proof of Theorem 2.

4. **The main chain theorem.** To obtain the analogue of the Jordan-Hölder theorem that any two complete chains between $a$ and $b$ have the same length it is obviously necessary that the two components in a simple cycle have the same length. But conversely, if this is the case the length of a chain does not change by simple deformations, hence related chains have the same length. From Theorem 1 one concludes therefore that if the chains in the partially ordered set are bounded in length, the necessary and sufficient condition for the Jordan-Hölder theorem to hold is that the two components in a simple cycle have the same length. We shall not formulate this as a theorem since we shall derive the stronger result:

THEOREM 2. *Let $P$ be a partially ordered set in which a chain between* *two elements is always finite. Then the necessary and sufficient condition that any two complete chains connecting two elements $a \supset b$ have the same length is that the two components in a simple cycle have the same length. In this case the two complete chains are always related.*

Proof. The condition is obviously necessary. To prove the sufficiency we observe first that the theorem is true when $a$ is prime over $b$. Again we can use induction, but this time we shall prefer to use induction with respect to the length of the *shortest* complete chain connecting the two elements $a \supset b$. This procedure corresponds more closely to the ordinary proof of the theorem of Jordan-Hölder in groups and structures. We assume that the theorem holds for all pairs of elements $a' \supset b'$ which can be joined by a complete chain of length less than $n$ and prove it for any pair $a \supset b$ which can be joined by a complete chain $C_b^a$ of length $n$. We shall show that if $D_b^a$ is any other complete chain connecting $a$ and $b$ then $D_b^a$ can be obtained from $C_b^a$ by a series of simple deformations.

As before we can exclude the case when $C_b^a$ and $D_b^a$ have elements in common other than $a$ and $b$ and also the case where $a$ and $b$ form a simple cycle, since then the result is true by assumption. It may be assumed therefore as previously that there exists an element $m$ properly between $a$ and $b$ such that $m$ contains an element $c$ in $C_b^a$ and an element $d$ in $D_b^a$. Correspondingly there exist complete chains (1). Since the chain (2) joins $a$ to $c$ and since the chain $C_c^a$ has shorter length than $n$ it follows by the induction assumption that $C_c^a$ and the chain (2) are related. Thus also $C_b^a$ and the chain (3) are related and have the length $n$. But then the two elements $m \supset b$ are connected by a complete chain of shorter length than $n$ so that the two chains in (4) are related and have the same length according to the induction assumption. Consequently the two chains in (3) and (5) must be related and have the same length $n$. From this result one concludes finally that the two chains

$$E_m^a + G_d^m, \qquad D_d^a$$

must have a length shorter than $n$, so that they are related and we have shown that $D_b^a$ and (5) are related and have the length $n$. This concludes the proof since we have shown successively that the chains $C_b^a$, $C'$ in (3), $C''$ in (5) and $D_b^a$ are related and have the same length.

In Theorem 2 the finiteness condition for all chains in $P$ is not absolutely necessary. One can prove the following somewhat more general result:

THEOREM 3. *Let $P$ be a partially ordered set with the property that when at least one of the components in a simple cycle has a finite length then both of them have the same length. If two elements $a \supset b$ in $P$ can be connected by a complete finite chain then all complete chains between $a$ and $b$ are related and have the same length.*

The proof is a simple extension of the proof of Theorem 2. One uses induction with respect to the length $n$ of the shortest complete chain connecting $a \supset b$. With the same notations as before let $C_b^a$ be the finite complete chain of length $n$ between $a$ and $b$, and $D_b^a$ some complete chain to be compared with $C_b^a$. As before one can assume that the two chains have no elements in common and that they do not form a simple cycle. Consequently there exists an element $m$ and the corresponding chains (1). By the induction assumption the chain (2) must be finite and related to $C_b^a$, consequently the chains $C_b^a$ and $C'$ in (3) are related and have the same finite length. By the same argument one passes from $C'$ to $C''$ in (5) and again from $C''$ to $D_b^a$.

5. **The quadrilateral condition.** The theorems derived in the preceding express results on the comparison of chains in partially ordered sets which appear to be about as general as one can expect to obtain them. In most applications where such theorems have been derived they have been obtained under considerably stronger conditions. These conditions are in many cases included in the condition:

*Quadrilateral condition.* When $a$ and $b$ are two different elements in a partially ordered set, both contained in some element $u$ and both prime over the element $d$, then there shall exist at least one element $m$ also contained in $u$ and prime over both $a$ and $b$.

The geometric interpretation of this condition on the basis of the diagram of the partially ordered set is simply that when the two sides $(a, d)$ and $(b, d)$ occur in that part of the diagram which is below $u$, there shall also exist two other sides $(m, a)$ and $(m, b)$ in the same part of the diagram completing the quadrilateral $(a, d, b, m)$. Theorem 2 gives immediately this result:

THEOREM 4. *Let $P$ be a partially ordered set satisfying the quadrilateral condition. The chains between any two elements $a \supset b$ are supposed to be finite. Then any two complete chains between two elements $a \supset b$ have the same length and one can be obtained from the other by successively replacing two consecutive sides $(a_1, a_2)$ and $(a_2, a_3)$ by two other sides $(a_1, b_2)$ and $(b_2, a_3)$ in a quadrilateral.*

PROOF. Under the given conditions every simple cycle must be a quadrilateral.

It is possible to extend Theorem 4 to partially ordered sets in which the finiteness condition does not hold. It then takes the following form:

THEOREM 5. *Let P be a partially ordered set satisfying the quadrilateral condition. If in P there exist two complete finite chains connecting two elements* $a \supset b$ *then both chains are related and have the same length.*

PROOF. Except for one additional difficulty one can use the arguments of the ordinary proof of the Jordan-Hölder theorem. Let

$$\begin{gathered} a \supset a_1 \supset \cdots \supset a_{n-1} \supset a_n = b, \\ a \supset b_1 \supset \cdots \supset b_{r-1} \supset b_r = b \end{gathered} \tag{7}$$

be the two complete chains connecting $a$ and $b$. As before we use induction and assume that the theorem is true for all pairs connected by complete chains of length less than $n$. As a consequence one can assume that $a_{n-1} \neq b_{r-1}$. From the quadrilateral condition follows the existence of an element $m$ contained in $a$ and prime over $a_{n-1}$ and $b_{r-1}$. Corresponding to (1) we introduce the complete chains

$$E_m^a, \quad F_{a_{n-1}}^m, \quad G_{b_{r-1}}^m$$

where the last two have the length 1. If one can now assume that the chain $E_m^a$ may be taken to be of finite length, the arguments used in the proof of Theorem 1 can be applied to show that both chains in (7) are related and have the same length.

It remains therefore only to establish the existence of a finite complete chain between $a$ and $m = m_{n-2}$. If by chance $m_{n-2} = a_{n-2}$ this is trivial. But when $m_{n-2} \neq a_{n-2}$ there exists by the quadrilateral condition some element $m_{n-3}$ contained in $a$ and prime over $a_{n-2}$ and $m_{n-2}$. Again the finite chain exists when $m_{n-3} = a_{n-3}$. If this is not the case there is an element $m_{n-4}$ prime over $m_{n-3}$ and $a_{n-3}$. Clearly the repetition of this process must eventually give a finite chain between $a$ and $m$.

It should be noted that in Theorem 5 there is no statement about the infinite chains between $a$ and $b$ and the existence of infinite chains is by no means excluded.

## 6. Weak quadrilateral condition.

The quadrilateral condition may be weakened in the following manner:

*Weak quadrilateral condition.* When $a$ and $b$ are distinct elements prime over $d$ there shall exist at least one element $m$ prime over $a$ and $b$.

This condition is no longer sufficient to prove that any two finite complete chains between two elements $a\supset b$ in the partially ordered set have the same length. It is however sufficient to prove that certain special chains have the same length. This is expressed in the theorem:

THEOREM 6. *Let $P$ be a partially ordered set with a universal element $u$ containing all others and satisfying the weak quadrilateral condition. If there exist two finite complete chains between $u$ and some element $a$ then both chains have the same length and are related.*

The proof may be omitted since it involves only a slight modification of the proof of Theorem 5. For the case where $P$ is a partially ordered set in which all chains from $u$ to arbitrary elements $a$ are finite, the fact that all chains have the same length is a result which has recently been established by Newman [10].

7. **Application to structures.** The preceding theory may now be specialized to the case of structures or lattices. It is clear that in any structure $\Sigma$ both the quadrilateral condition and the weak quadrilateral condition take the form of the so-called

*Birkhoff condition.* When $a$ and $b$ are prime over their cross-cut $a\cap b$, the union $a\cup b$ shall be prime over $a$ and $b$.

When this condition is satisfied in a structure, Theorem 5 gives the main result that finite complete chains between two elements $a\supset b$ are related and have the same length. This is even a little more than the usual formulation for structures satisfying the Birkhoff condition since one ordinarily assumes that *all* chains between two elements are finite.

But the main Theorem 1 and its consequences in Theorems 2 and 3 go so much further that it is possible to use them to formulate a necessary and sufficient condition for the equality of the lengths of chains in a structure. Let us consider first the criterion for a simple cycle in a structure. We denote by $C_1^*$ and $D_1^*$ two complete chains connecting $a$ and $b$. Then one sees that these two chains are the components of a simple cycle if and only if for every pair of elements $c$ and $d$ different from $a$ and $b$ and belonging to $C_1^*$ and $D_1^*$, respectively, one has

$$c\cup d = a, \qquad c\cap d = b.$$

This form for the definition of a simple cycle makes it natural to introduce the concept of

*Cyclic structure.* A structure is cyclic when it consists of two chains

$C$ and $D$ connecting a universal element $a$ with a zero element $b$ such that

$$c \cup d = a, \qquad c \cap d = b$$

for all elements $c$ and $d$ in $C$ and $D$, respectively.

In other words a cyclic structure is a structure whose elements form a single simple cycle. Let us also say that a cyclic substructure $S$ of a structure $\Sigma$ is *complete* in $\Sigma$ when the two chains in $\Sigma$ defining $S$ are complete. Thus the simple cycles in a structure are the complete cyclic substructures.

We can now state the main theorem on the comparison of the lengths of chains in structures as a direct consequence of Theorem 2:

THEOREM 7. *Let $\Sigma$ be a structure in which the chains between two elements are finite. The necessary and sufficient condition that all complete chains in $\Sigma$ between two elements $a$ and $b$ have the same length is that in any complete cyclic substructure the two defining chains always have the same length.*

8. **Application to groups.** To conclude let us mention one further problem which is connected with this theory of chains. The theorem of Jordan-Hölder for *principal chains* (each subgroup in the chain normal in the whole group) in groups follows from the theory of Dedekind structures. The ordinary Jordan-Hölder theorem for *composition series* (each subgroup normal in the preceding) is however not so directly a consequence of the structure theory. This has been observed by various authors and extensions of the abstract theory have been proposed to remedy this deficiency (Uzkow, George, Kurosch). Let us remark that the preceding theory is sufficiently general to solve this difficulty. The partially ordered set of all subgroups occurring in a composition series of the group satisfies the (dual) quadrilateral condition and its complete chains are the composition series in the group, hence they are all related and of the same length. [In detail: If $A \supset B$ are two composition groups, that is, both occur in some composition series, there exists a composition series containing both. When $G \supset B_1 \supset \cdots \supset B_n \supset B \supset \cdots$ is the composition series containing $B$ one obtains the composition series from $A$ to $B$ explicitly as

$$A \supseteq B_1 \cap A \supseteq \cdots \supseteq B_n \cap A \supseteq B.$$

The quadrilateral condition expresses only that if $M_1$ and $M_2$ are maximal normal subgroups of $A$ then the intersection $M_1 \cap M_2$ is a maximal normal subgroup both of $M_1$ and $M_2$.]

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# A CONJECTURE OF ORE ON CHAINS IN PARTIALLY ORDERED SETS

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In a recent investigation, Ore[1] has given a form of the Jordan-Hölder theorem valid for an arbitrary partially ordered set $P$. This theorem involves essentially the deformation of one chain into another by successive steps, each step being like that used in the conventional Jordan-Hölder theorem. Ore observes that his first theorem would be slightly easier to apply if it were proved under a weaker hypothesis. The modified theorem runs as follows:[2]

THEOREM. *If $P$ is a partially ordered set in which every chain joining two elements is finite, then any complete chain between two elements $b<a$ can be deformed into any other complete chain between the same two elements.*

The proof rests on this lemma:

LEMMA. *Under the hypothesis of the theorem, if $C$ is a complete chain from $b$ to $a$ which cannot be deformed into the complete chain $D$ from $b$ to $a$, there exist in $P$ elements $b'<a'$ and complete chains $C'$ and $D'$ from $b'$ to $a'$ such that $C'$ cannot be deformed into $D'$ and such that $b \leqq b'$, $a' \leqq a$ where either $b<b'$ or $a'<a$.*

PROOF. *Case* 1. $C$ and $D$ have in common the element $e$, $b<e<a$. Then either $C_b^e$ cannot be deformed into $D_b^e$, or $C_e^a$ cannot be deformed into $D_e^a$. In these two cases, set $b'=b$, $a'=e$ or $b'=e$, $a'=a$, respectively.

*Case* 2. $C$ and $D$ have no elements in common. Since $C$ cannot be deformed into $D$, they cannot together constitute a simple cycle. There will then exist, say, elements $c$ in $C$ and $d$ in $D$ with $b<c<a$, $b<d<a$ and an element $m$ in $P$ with $c \leqq m<a$, $d \leqq m<a$. Because of the hypothesis that every chain in $P$ joining two elements is finite, there will exist in $P$ finite complete chains $E_m^a$, $F_c^m$, $G_d^m$. Then $b$ is joined to $a$ by four complete chains,

$$C_b^c + C_c^a \qquad C_b^c + F_c^m + E_m^a,$$
$$D_b^d + G_d^m + E_m^a, \qquad D_b^d + D_d^a.$$

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[1] Oystein Ore, *Chains in partially ordered sets*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 558–566.

[2] Terminology and notation follow the paper of Ore.

Since $C$ cannot be deformed into $D$, one of the following three deformations must be impossible:

$$C^a_c \to F^m_c + E^a_m, \qquad C^c_b + F^m_c \to D^d_b + G^m_d,$$
$$G^m_d + E^a_m \to D^a_d.$$

In the first case we set $a' = a$, $b' = c$; in the second case, $a' = m$, $b' = b$; in the third case $a' = a$, $b' = d$. In each case we have the conclusion of the lemma.

To prove the theorem, suppose that $P$ were to contain two complete chains $C$ and $D$ joining $b$ to $a$ in such wise that $C$ cannot be deformed into $D$. By induction on $n$, the lemma gives in $P$ elements $a = a_0 \geqq a_1 \geqq \cdots \geqq a_n$ and $b = b_0 \leqq b_1 \leqq b_n \leqq a_n$ such that for each $i$ either $a_{i-1} > a_i$ or $b_{i-1} < b_i$ $(i = 1, \cdots, n)$, and such that there are complete chains $C_n$, $D_n$ joining $b_n$ to $a_n$ with $C_n$ not deformable into $D_n$. This construction can be carried on indefinitely, using the axiom of choice to select at each stage a definite pair $a_{n+1}$, $b_{n+1}$. This produces two sequences of elements $a_i$, $b_i$ with

$$b_0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq \cdots \leqq a_2 \leqq a_1 \leqq a_0.$$

Furthermore, the inequality sign holds an infinite number of times here, so that we obtain an infinite chain joining $b = b_0$ to $a = a_0$, contrary to the hypothesis of the theorem.

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# ON IRREDUCIBLE CONTINUOUS CURVES

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This paper deals with the existence of continuous curves containing compact and closed point sets and with certain properties of continuous curves which are irreducible continua about point sets in spaces which are not necessarily metric. Previous results on these topics have been almost entirely for metric spaces. Thus Gehman[1] proved for the plane that, given a compact continuum, there exists a compact continuous curve, which is the sum of a countable number of arcs plus its limit points, containing it. Whyburn and Ayres[2] extended this to a space of a continuous curve in $n$ dimensions, and Zippin[3] indicated that their argument might be modified to give the following: If $T$ is a closed and compact subset of a complete metric continuous curve $S$, there exists a compact continuous curve which is a subset of $S$ and contains $T$. Zippin[4] proved that, given a complete metric continuous curve $S$ and a compact, closed, one-dimensional subset, $T$, of $S$, such that every component of $T$ is a continuous curve, and no more than a finite number of components of $T$ are of diameter greater than any positive number, there exists a compact continuous curve which is a subset of $S$ and an irreducible continuum about $T$. Miss Miller[5] proved that if, in a connected space satisfying Axioms $0-2$ of R. L. Moore's *Foundations of point set theory*,[6] $T$ be a compact and closed point set, there exists a compact continuum containing $T$.

The concept of a continuum irreducible about a subset of itself was first introduced by Wilson.[7] Most of the past work on continuous curves which are irreducible continua about point sets has been done

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[1] H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Ann. of Math. (2) vol. 27 (1926) p. 30.

[2] G. T. Whyburn and W. L. Ayres, *On continuous curves in n dimensions*, Bull. Amer. Math. Soc. vol 34 (1928) p. 350.

[3] L. Zippin, *On continuous curves irreducible about subsets*, Fund. Math. vol. 20 (1933) pp. 197–205.

[4] L. Zippin, loc. cit.

[5] H. C. Miller, *A theorem concerning closed and compact point sets which lie in connected domains*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 848.

[6] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications vol. 13, New York, 1932.

[7] W. A. Wilson, *On the oscillation of a continuum at a point*, Trans. Amer. Math. Soc. vol. 27 (1925) p. 433.

by Gehman, who obtained results[8] equivalent to the following theorem for the plane: If $M$ be a compact irreducible continuous curve about a closed point set $K$, then (1) $M$ is an irreducible continuum about $K$; (2) $K$ contains all the non-cut points of $M$; (3) if $H$ is a subcontinuum of $M$, $H$ is connected im kleinem at every point of $H(M-K)$. He[9] gave a later proof of part (2) which with very slight modification shows that if, in a space satisfying Axioms $0-1$, $M$ be a locally compact continuous curve which is an irreducible continuum about a closed subset $K$ of $M$, $K$ contains all the non-cut points of $M$. Zippin[10] indicated a proof of part (1) if $M$ be a complete metric space and $K$ be compact.

THEOREM 1. *In a connected space satisfying Axioms* $0-2$, *every closed and compact point set* $T$ *which has no continuum of condensation is a subset of a compact hereditary continuous curve.*

PROOF. Every component of $T$ is a continuous curve and hence is arcwise connected. By Axiom 2 regions may be considered as connected domains. There exists[11] an infinite sequence $W_1$, $W_2$, $W_3, \cdots$ such that (1) for each $n$ $W_n$ is a finite subcollection of $G_n$ covering $T$; (2) if $g$ be a member of $W_{n+1}$, $\bar{g}$ is a subset of some member of $W_n$; (3) if $H$ and $K$ be two mutually exclusive closed subsets of $T$, and $k$ be a positive integer, there exists a positive integer $m$ such that if $U$ be a coherent collection of $k$ regions, each belonging to a member of the sequence $W_m$, $W_{m+1}, \cdots$, and $U^*$ contain a point of $H$,[12] then $U^*$ contains no point of $K$; (4) every region of each $W_n$ contains a point of $T$, and if $P_1$, $P_2, \cdots$ be a sequence of points such that for each $n$ $P_n$ is a subset of $W_n$, then some subsequence of that sequence converges to a point of $T$. For each region $R_{i_1}$ in $W_1$ let $P_{i_1}$ be a point of $R_{i_1} \cdot T$. Let $\alpha_{i_1}$ denote an arc from $P_{1_1}$ to $P_{i_1}$ which is constructed to lie in the component of $T$ containing $P_{1_1}$ if that component contains $P_{i_1}$. Let $\alpha_1$ be the sum of all $\alpha_{i_1}$. For each pair of regions, $R_{i_1}$ of $W_1$ and $R_{i_2}$ of $W_2$, such that $R_{i_1}$ contains $R_{i_2}$, let $P_{i_1 i_2}$ be a point of $R_{i_2} \cdot T$. Let $\alpha_{i_1 i_2}$ denote an arc from $P_{i_1}$ to $P_{i_1 i_2}$ lying in $R_{i_1}$ or in the component of $T$ which contains $P_{i_1}$ according as that component does not or does contain $P_{i_1 i_2}$. Let $\alpha_2$ be the sum of $\alpha_1$ and all $\alpha_{i_1 i_2}$. For each triplet of regions, $R_{i_1}$ of $W_1$, $R_{i_2}$ of $W_2$, and $R_{i_3}$ of $W_3$, such that $R_{i_1}$ contains $R_{i_2}$ which contains $R_{i_3}$, let $P_{i_1i_2i_3}$ be a point of $R_{i_3}\cdot T$. Let $\alpha_{i_1i_2i_3}$ denote an arc from $P_{i_1i_2}$ to $P_{i_1i_2i_3}$ lying in $R_{i_2}$ or in the component of $T$ which contains $P_{i_1i_2}$ according as that component does not or does contain $P_{i_1i_2i_3}$. Let $\alpha_3$ be the sum of $\alpha_2$ and all $\alpha_{i_1i_2i_3}$. Continue the indicated constructions indefinitely. Let $\alpha$ be the sum of all $\alpha_n$.

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[11] R. L. Moore, loc. cit. pp. 58–59, Theorem 81.

[12] The point set $U^*$ is the sum of all point sets of the collection $U$.

The point set $\bar{\alpha}$ is a hereditary continuous curve containing $T$. First, since $\alpha$ is the sum of a monotonic sequence of connected sets, it is connected, so that $\bar{\alpha}$ is a continuum. Second, $\bar{\alpha}$ contains $T$ because, given a point $P$ of $T$ and a region $R$ containing $P$, there exists for some $n$ a region $R_{i_n}$ of $W_n$ which is a subset of $R$ and contains $P$; there then exists a monotonic sequence of regions $R_{i_1}$, $R_{i_2}, \cdots, R_{i_n}$, so that $R_{i_n}$ contains a point $P_{i_1\cdots i_n}$ of $\alpha$; whence $R$ contains a point of $\alpha$ and $P$ is a limit point of $\alpha$.

Suppose that $\alpha$ contains an infinite point set $K$ such that $\alpha+T$ contains no limit point of $K$. For each $n$, $\alpha_n+T$ is compact and therefore contains only a finite number of points of $K$. Since $W_n$ covers $\alpha-(\alpha_n+T)$, $W_n^*\cdot K$ is infinite. This contradicts part (4) of the condition on the $W_n$'s. It follows that $\bar{\alpha}$ is compact and that $T$ contains $\bar{\alpha}-\alpha$.

It remains to prove that every subcontinuum of $\bar{\alpha}$ is a continuous curve. Suppose that this is not true. Then there exists a subcontinuum $M$ of $\bar{\alpha}$, containing a point $P$ at which it is not connected im kleinem. Suppose, first, that $P$ is not a point of $T$. By part (4) of the condition on the $W_n$'s there exists an integer $k$ such that no region $W_{k-1}$ contains $P$; by part (2), if $R_{i_k}$ is a region of $W_k$, $\bar{R}_{i_k}$ does not contain $P$. As $W_k$ is a finite collection there exists a region $R$ containing $P$ that contains no point of $W_k^*$ and so contains no point of $T$. Hence $R\cdot\bar{\alpha}$ is a subset of $R\cdot\alpha_k$. As $R$ contains the point $P$ at which $M$ is not connected im kleinem, it contains a continuum of condensation of $M$ and so of $\alpha_k$, the sum of a finite number of arcs. This is impossible.

Suppose now that $P$ is a point of $T$. There exist a domain $D$ containing $P$, and a sequence of mutually exclusive continua $K$, $M_1$, $M_2, \cdots$ converging to $K$, such that (1) $K$ contains $P$; (2) every continuum of the sequence contains a point of $D$ and a point of $\bar{D}-D$; (3) each $M_i$ is a component of $M\,\bar{D}$. Since it contains no point where $M$ is connected im kleinem, $K\cdot D$ is a subset of $T$. If every point of $K\cdot D$ is a limit point of $T-K\cdot D$, then if $D'$ be a domain intersecting $K$, $\bar{D}'$ a subset of $D$, some component of $K\cdot\bar{D}'$ will be nondegenerate and a continuum of condensation of $T$. Hence some point $P'$ of $K\ D$ is not a limit point of $T-K\cdot D$. Let $D'$ be a domain containing $P'$, such that $\overline{D}'$ is a subset of $D$ and contains no point of $T-K\cdot D$. Let $D''$ be a domain containing $P'$ such that $\overline{D}''$ is a subset of $D'$. Let $D'''$ be a domain containing $P'$ such that $\overline{D}'''$ is a subset of $D''$. Let $K'$ be the component of $K\cdot\overline{D}'''$ containing $P'$. $K'$ is a nondegenerate subcontinuum of $T$ and a subset of the limiting set of the sequence $M_1\cdot\overline{D}''$, $M_2\cdot\overline{D}''$, $\cdots$. By parts (3) and (4) of the conditions on the $W_n$'s, there exists an integer $k$ such that no region of $W_n$, $n\geqq k$, intersects $\overline{D}''$ and $S-D'$. $\alpha_{i_1\cdots i_n}$ lies in $R_{i_{n-1}}+T$, so every arc $\alpha_{i_1\cdots i_n}$, $n>k$, which intersects $\overline{D}''$ and $S-D'$ lies in $T$. Hence every arc $\alpha_{i_1\cdots i_n}$, $n>k$, which intersects $\overline{D}''$ lies in $D'$ or in $T$; both end points of such an arc lie in $T$, by its construction, so that either the whole arc lies in $T$ or its end points lie in $T\cdot D'$ and so in $K$; in the latter case the end points lie in the same component of $T$ and so the arc lies in $T$. Therefore $\bar{\alpha}\cdot D''$ is a subset of $\alpha_k+T$.

It follows that $M_1\ \overline{D}''$, $M_2\ \overline{D}''$, $\cdots$ are all subsets of $\alpha_k+T$. Therefore $K'$ is a continuum of condensation of $\alpha_k+T$. $K'\ (\overline{T-K'})$ and $K'\ (\overline{\alpha_k-\alpha_k\cdot K'})$ are closed and totally disconnected. But their sum is the continuum $K'$. This involves a contradiction.

THEOREM 2. *In a connected space satisfying Axioms* $0-2$, *every closed and compact point set* $T$ *is a subset of a compact continuous curve.*

PROOF. By Axiom 2 regions may be considered as connected domains. Construct the $W_n$'s and the $P_{i_1\cdots i_n}$'s exactly as in the proof of Theorem 1. Define the $\alpha_{i_1\cdots i_n}$'s as follows: $\alpha_{i_1}$ is an arc from $P_{1_1}$ to $P_{i_1}$; $\alpha_{i_1\cdots i_n}$ is an arc from $P_{i_1\cdots i_{n-1}}$ to $P_{i_1\cdots i_n}$ in $R_{i_{n-1}}$. Define $\alpha_n$ and $\alpha$ as before on this basis. It will be proved that $\bar{\alpha}$ is a compact continuous curve containing $T$.

By the argument used in the proof of Theorem 1 it may be shown that $\bar{\alpha}$ is a compact continuum containing $T$. It remains to show that $\bar{\alpha}$ is connected im kleinem at every point. Let $P$ be any point of $\bar{\alpha}$, and $R$ a region containing $P$. There exists a positive integer $k$ such that if $R_1$ and $R_2$ are any two intersecting regions of $W_k$, one containing $P$, then $R$ contains $\overline{R}_1+\overline{R}_2$. For each set of indices $i_1, i_2, \cdots, i_k$ let $L_{i_1\cdots i_k}$ be the sum of all $\alpha_{i_1\cdots i_n}$'s, $n>k$, whose first $k$ indices are the members of that set. The set of $L_{i_1\cdots i_k}$'s is finite. If $\alpha_k$, the sum of a finite number of arcs, contains $P$, it is connected im kleinem there. Let $R'$ be a region containing $P$ such that (1) if $P'$ is a point of $\alpha_k\cdot R'$, there is a connected subset of $\alpha_k\cdot R$ containing $P$ and $P'$; (2) if $R'$ contains a point of $\overline{L_{i_1\cdots i_k}}$, $\overline{L_{i_1\cdots i_k}}$ contains $P$. Let $Q$ be any point of $R'\cdot\bar{\alpha}$. If $Q$ is a point of $\alpha_k$, there is a connected subset of $R\ \bar{\alpha}$ containing $P$ and $Q$. Suppose it does not belong to $\alpha_k$. Then it must belong to some $L_{i_1\cdots i_k}$. Then that $L_{i_1\cdots i_k}$ contains $P$ and is a subset of $\bar{R}_{i_k}$. Since $\bar{R}_{i_k}$ contains $P$ it is a subset of $R$. Hence $L_{i_1\cdots i_k}$ is a subset of $R\cdot\bar{\alpha}$ which is plainly connected and contains $P$ and $Q$. The theorem is therefore proved.

Theorem 2 does not remain true if "locally compact" replaces "compact." For let space be the sum of the intervals in the plane from (0, 0) to (1, 0) and from $(1/n, 0)$ to $(1/n, 1)$, $(n>0)$, and let $T$ be the set of points $(1/n, 1)$. Here space is a locally non-compact continuous curve which is an irreducible continuum about $T$. Furthermore, there is a non-cut point of space not belonging to $T$.

THEOREM 3. *If space satisfies Axioms* $0-1$, *and* $M$ *is an irreducible continuous curve about a compact and closed subset* $T$ *of* $M$, $M$ *is a compact irreducible continuum about* $T$.

PROOF. By Theorem 2, $M$ contains a compact continuous curve $U$ containing $T$. Since it is identical with $U$, $M$ is compact.

Suppose that there exists a proper subcontinuum $V$ of $M$ containing $T$. Let $P$ be a point of $M-V$, and let $C$ be the component of $M-P$ containing $V$. The set $C$ is a domain with respect to $M$ and hence considered as space satisfies Axioms $0-2$. Hence, by Theorem 2, $C$ contains a compact continuous curve $U'$ containing $T$. This is a contradiction.

THEOREM 4. *If space satisfies Axioms* $0-1$, *and* $M$ *is a locally compact continuous curve which is an irreducible continuum about a closed subset* $T$ *of* $M$, *every continuum of condensation of* $M$ *is a continuum of condensation of* $T$.

PROOF. The following lemma must first be demonstrated:

LEMMA. *Under the conditions of Theorem* 4, *every continuum of condensation of* $M$ *is a subset of* $T$.

PROOF OF LEMMA. Regard $M$ as space. Then Axiom 2 holds. Suppose that $M$ has a continuum of condensation $N$ which does not lie wholly in $T$. Since $M$ is locally compact, $N$ contains a nondegenerate compact continuum $N'$ which lies in $M-T$. $N'$ contains a subcontinuum $N''$ which is an irreducible continuum between two points $A$ and $B$. Every component of $M-N''$ intersects $T$ since its complement is a continuum. Since $M$ is locally compact, $N''$ is a subset of a domain $D$ such that $\bar{D}$ is a compact subset of $M-T$. The set $\bar{D}-D$ is the sum of its intersections with the components of $M-N''$, and each component intersects it since the component has a limit point in $N''$ and in $T$. Since no closed and compact point set is the sum of infinitely many mutually exclusive open subsets of itself, $M-N''$ has only a finite number of components. As every point of $N''$ is a cut point of $M$,[13] every non-cut point of $N''$ is the boundary of some component of $M-N''$, so the set of all non-cut points of $N''$ is finite. There exists a component $K$ of $M-N''$ such that $\overline{K}\ N''$ is infinite. There then exist three points of $\overline{K}\cdot N''$ which lie on the segment $AB$ of $N''$, such that no one of them is the entire boundary of any component of $M-N''$. Let $P$ be that one of them which lies between the other two on the segment $AB$ of $N''$.[14] Then $K+N''-P$ is plainly connected. Since $N''-P$ contains a limit point of every component of $M-N''$, $M-P$ is connected, which leads to a contradiction.

Suppose now that Theorem 4 is not true. Then, by the Lemma, $T$ contains a continuum $N$ which is a continuum of condensation of $M$ but not of $T$. Let $P$ be a point of $N$ not belonging to $\overline{T-N}$. Let $Q$ be a point of $N-P$. Let $C$ be any subcontinuum of $M$ which contains $\overline{T-N}+Q$. Since it contains $T$, the continuum $C+N$ is $M$. Hence $C$ contains $M-N$, and therefore $N$, so that $C$ is $M$ and $M$ is an irreducible continuum about $\overline{T-N}+Q$. As $N$ contains a point $P$ not in $\overline{T-N}+Q$, this contradicts the Lemma, and the theorem is proved.

THEOREM 5. *In a connected space satisfying Axioms* $0-2$, *every compact and closed point set* $T$ *with no continuum of condensation is a subset of a compact continuous curve which has no continuum of condensation.*

PROOF. By Theorem 1 there exists a compact hereditary continuous curve $K$ containing $T$. The set $K$ contains a continuum $M$ which is an irreducible continuum about $T$. Since $M$ is a continuous curve, by Theorem 4 $M$ has no continuum of condensation.

In conclusion I wish to express my deepest thanks to Professor R. L. Moore for invaluable guidance in the preparation of this paper.

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[13] H. M. Gehman, loc. cit.

[14] R. L. Moore, loc cit. pp. 35 ff.

# NEW SYSTEMS OF HYPERGEODESICS DEFINED ON A SURFACE

P. O. BELL

**Introduction.** Let a non-ruled surface $S$ be referred to its asymptotic net as parametric. As a point $P_y$ moves along a curve $C_\lambda$ of $S$, the tangents at $P_y$ to the $u$- and $v$-asymptotic curves of $S$ describe two ruled surfaces $R_\lambda^u$ and $R_\lambda^v$, respectively. Let $S_\rho$ and $S_\sigma$ denote arbitrary transversal surfaces of the congruences of $u$- and $v$-tangents of $S$, respectively. The purpose of the present paper is to introduce and study systems of curves of $S$ which will be called $\rho$- and *$\sigma$-tangeodesics*.

DEFINITION. *A curve $C_\lambda$ of $S$ whose associated ruled surface $R_\lambda^u$ intersects the surface $S_\rho$ in an asymptotic curve of $R_\lambda^u$ is a $\rho$-tangeodesic of $S$. Similarly, a curve $C_\lambda$ of $S$ whose associated ruled surface $R_\lambda^v$ intersects $S_\sigma$ in an asymptotic curve of $R_\lambda^v$ is a $\sigma$-tangeodesic of $S$.*

The $\rho$- and *$\sigma$-tangeodesics* of $S$ at $P_y$ are found to be associated in remarkable manners with the *edges of Green*, the *directrices of Wilczynski*, and the *projective normal of Fubini*. In fact, a new geometric characterization is obtained for each of these lines.

1. **Tangeodesics.** If the parametric net on a non-ruled surface $S$ is the asymptotic net, the homogeneous projective coordinates $y^{(i)}(u, v)$ $(i=1, 2, 3, 4)$ of a general point $P_y$ of $S$ are solutions of a system of differential equations which may be assumed to be reduced to Wilczynski's canonical form

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0. \tag{1.1}$$

The homogeneous coordinates of points $\rho$, $\sigma$ on arbitrarily selected transversal surfaces $S_\rho$ and $S_\sigma$ of the congruences of $u$- and $v$-tangents of $S$ are given by the vector forms

$$\rho = y_u - \beta y, \qquad \sigma = y_v - \alpha y, \tag{1.2}$$

wherein $\beta$, $\alpha$ are arbitrary analytic functions of $u$, $v$.

Let $l$ denote the line joining $\rho$, $\sigma$ and let $l'$ denote its reciprocal at $P_y$. The line $l'$ joins the points $P_y$ and $z$ where $z$ is given by

$$z = y_{uv} - \alpha y_u - \beta y_v \tag{1.3}$$

in which $\beta$ and $\alpha$ are the functions in (1.2). The line $l$, according to Green's classification, is a line of the first kind and generates a con-

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gruence $\Gamma$ of the first kind as $P_y$ moves over $S$. The line $l'$ is a line of the second kind and generates a congruence $\Gamma'$ of the second kind as $P_y$ moves over $S$.

Let $C_\lambda$ denote an integral curve of the curvilinear differential equation

$$dv - \lambda(u, v)du = 0. \tag{1.4}$$

Regarding $u$ as independent variable we write $v' = \lambda(u, v)$ and $v'' = \lambda_u + \lambda\lambda_v$, in which accents indicate differentiation with respect to $u$.

The homogeneous coordinates of a general point of the ruled surface $R_\lambda^*$ are represented by the vector form

$$\bar{y} = y_u + wy, \tag{1.5}$$

wherein $u$ and $w$ are independent variables and $v$ varies in accordance with the relation $v' = \lambda(u, v)$.

Let us put $u = u(t)$, $w = w(t)$, so that $\bar{y}$ describes a curve on $R_\lambda^*$ as $t$ varies. The necessary and sufficient condition that this curve be an asymptotic curve of $R_\lambda^*$ is that the determinant equation

$$(\bar{y},\ \bar{y}_u + v'\bar{y}_v,\ \bar{y}_w,\ d^2\bar{y}/dt^2) = 0 \tag{1.6}$$

be satisfied. If we transform equation (1.6) by making use of equations (1.5), (1.4) and (1.1) we obtain, in view of the inequality $(y, y_u, y_v, y_{uv}) \neq 0$, the equation

$$dw/du = \{2b^2 + (b_u - 2bw)v' + (w^2 + 2b_v + f)v'^2 - 2a'bv'^3 - bv''\}/v'^2. \tag{1.7}$$

As $P_y$ moves along $C_\lambda$ the point $\rho$ moves in the direction defined by (1.7) if and only if $w = -\beta$ satisfies (1.7). To obtain, therefore, the curvilinear differential equation for the $\rho$-tangeodesics we have merely to substitute $-\beta$ for $w$ in (1.7) and clear of fractions. The result is

$$bv'' - 2b^2 - (2b\beta + b_u)v' - (\beta^2 + 2b_v + f + \beta_u)v'^2 + (2a'b - \beta_v)v'^3 = 0. \tag{1.8}$$

The differential equation for the $\sigma$-tangeodesics may be obtained by making the substitution

$$\begin{pmatrix} v'' & v' & v & b & a' & \beta & f \\ -v''/v'^3 & 1/v' & u & a' & b & \alpha & g \end{pmatrix}$$

in (1.8). The result, on simplifying, is

$$a'v'' + \alpha_u - 2a'b + (\alpha^2 + 2a'_u + g + \alpha_v)v' + (2a'\alpha + a'_v)v'^2 + 2a'^2v'^3 = 0. \tag{1.9}$$

2. **Systems of hypergeodesics which have no cusp-axes.** The curves defined on a surface $S$ by a differential equation of the form

$$v'' = A + Bv' + Cv'^2 + Dv'^3 \tag{2.1}$$

in which the coefficients are functions of $u$, $v$ and accents indicate differentiation with respect to the independent variable $u$, are called *hypergeodesics*.[1] *The envelope of the osculating planes at a point $P_y$ of the hypergeodesics* (2.1) *is a cone which is ordinarily of the third class.* When this cone is of the third class it has three distinct cusp-planes which intersect in a line called *the cusp-axis of the cone*, or *the cusp-axis of the hypergeodesics at the point $P_y$*. The cusp-axis is the line $l'$ for which $\alpha$ and $\beta$ are given by

$$\alpha = C/2, \qquad \beta = -B/2. \tag{2.2}$$

We are interested here in those cases in which the class of the cone is less than three and the cone has no cusp-axis. The local equation of the osculating plane at $P_y$ of the curve $C_\lambda$ defined by (1.4) is

$$2\lambda(\lambda x_2 - x_3) + (\lambda' - 2b + 2a'\lambda^3)x_4 = 0, \tag{2.3}$$

when referred to the tetrahedron whose vertices have the general coordinates $y$, $y_u$, $y_v$, $y_{uv}$. Assuming $C_\lambda$ to be an integral curve of (2.1) we replace $\lambda'$ by the right member of (2.1) and put $\lambda$ in place of $v'$. The result is

$$2\lambda(\lambda x_2 - x_3) + (A - 2b + B\lambda + C\lambda^2 + [D + 2a']\lambda^3)x_4 = 0. \tag{2.4}$$

The *union curves of a congruence* $\Gamma'$ form a system of hypergeodesics, sometimes called an *axial system*, whose osculating planes at $P_y$ all pass through the line $l'$ of the congruence $\Gamma'$. Equation (2.1) represents such a system if $A = 2b$ and $D = -2a'$.

A system of hypergeodesics (2.1) for which

$$A = 2b, \qquad D \neq -2a' \tag{2.5}$$

will be called, for reasons which appear later, a *$u$-polar system*; and a system (2.1) for which

$$D = -2a', \qquad A \neq 2b \tag{2.6}$$

will be called a *$v$-polar system.*

If system (2.1) is a $u$-polar system, the equation for the envelope of its osculating planes at $P_y$ may be readily found from (2.4) to be

$$(2x_2 + Cx_4)^2 - 4(D + 2a')(Bx_4 - 2x_3)x_4 = 0. \tag{2.7}$$

Similarly, if (2.1) represents a $v$-polar system the equation for the envelope of its osculating planes at $P_y$ may be found to be

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[1] G. Fubini, *Fondamenti della geometria proiettivo-differenziale di una superficie*, Atti Accad. Sci. Torino vol. 53 (1918) p. 1034.

$$(2.8)\qquad (2x_3 - Bx_4)^2 - 4(A - 2b)(2x_2 + Cx_4)x_4 = 0.$$

Since the cones (2.7) and (2.8) are nondegenerate quadric cones, they have no cusp-axes at $P_y$. Hence we have that *neither a u-polar system nor a v-polar system of hypergeodesics has a cusp-axis at $P_y$.*

There are two generators of the cone (2.7) which are such that the tangent planes of the cone along these generators pass through the $u$-tangent to $S$ at $P_y$. One of these is the $v$-tangent to $S$ at $P_y$ and the other is the line $l'$ for which

$$(2.9)\qquad \alpha = C/2, \qquad \beta = -B/2,$$

wherein $B$ and $C$ are the functions appearing in (2.7). This line $l'$ will be called the *u-edge* of the *u-polar system.*

The *v-edge* of a *v-polar system* is characterized similarly.

Since equations (2.9) are of the same form as equations (2.2), we have immediately this theorem.

THEOREM 2.1. *If the coefficients $B$ and $C$ of the differential equation of a non-polar system of hypergeodesics are identical with the corresponding coefficients of the differential equation of a u-polar system of hypergeodesics, the cusp-axis of the non-polar system at $P_y$ coincides with the u-edge of the u-polar system at $P_y$.*

A similar theorem applies, of course, to a $v$-polar system of hypergeodesics.

The forms of the differential equations (1.8) and (1.9) show clearly that the $\rho$- and $\sigma$-tangeodesics form $u$- and $v$-polar systems of hypergeodesics. For the system (1.8) we have

$$(2.10)\qquad \begin{gathered} A = 2b, \quad B = 2\beta + b_u/b, \quad C = (\beta^2 + 2b_v + f + \beta_u)/b, \\ D = (\beta_v - 2a'b)/b. \end{gathered}$$

For the system (1.9) we have

$$(2.11)\qquad \begin{gathered} A = (2a'b - \alpha_u)/a', \quad B = -(\alpha^2 + 2a'_u + g + \alpha_v)/a', \\ C = -2\alpha - a'_v/a', \quad D = -2a'. \end{gathered}$$

The cone (2.7) is associated with the system (1.8) of $\rho$-tangeodesics if $A$, $B$, $C$, $D$ are given by (2.10). Similarly, if $A$, $B$, $C$, $D$ are defined by (2.11), the cone (2.8) is associated with the system (1.9) of $\sigma$-tangeodesics.

The *u-edge* of the *$\rho$-tangeodesics* (1.8) is the line $l_1'$ passing through the points $P_y$ and $z_1$, where $z_1$ is given by $z_1 = y_{uu} - \alpha_1 y_u - \beta_1 y_v$, in which

$$(2.12)\qquad \beta_1 = -\beta - b_u/2b, \quad \alpha_1 = (\beta^2 + 2b_v + f + \beta_u)/2b.$$

The $v$-edge of the $\sigma$-tangeodesics is the line $l_2'$ passing through $P_y$ and $z_2$ where $z_2$ is given by $z_2 = y_{uv} - \alpha_2 y_u - \beta_2 y_v$, in which

$$(2.13)\quad \beta_2 = (\alpha^2 + 2a_u' + g + \alpha_v)/2a', \quad \alpha_2 = -\alpha - a_v'/2a'.$$

3. **The edges of Green, the directrices of Wilczynski, and the projective normal.** Let us apply the results of §§1 and 2 to obtain new characterizations of the *edges of Green*, the *directrices of Wilczynski*, and the *projective normal of Fubini*. The plane which is tangent to the cone (2.7) of the $\rho$-tangeodesics along the *u-edge* intersects the plane which is tangent to the cone (2.8) of the $\sigma$-tangeodesics along the *v-edge* in a line $\bar{l}'$ of the second kind which will be called the *joint-edge of the systems of* $\rho$- *and* $\sigma$-*tangeodesics of* $S$ *at* $P_y$. This line passes through the points $P_y$ and $\bar{z}$ where the general coordinates of $\bar{z}$ are given by $\bar{z} = y_{uv} - \bar{\alpha} y_u - \bar{\beta} y_v$, in which

$$(3.1)\qquad \bar{\alpha} = -\alpha - a_v'/2a', \qquad \bar{\beta} = -\beta - b_u/2b.$$

Since the functions $\alpha$, $\beta$ associated with the edges of Green are given by

$$(3.2)\qquad \alpha = -a_v'/4a', \qquad \beta = -b_u/4b,$$

we have the following theorem.

THEOREM 3.1. *The second edge of Green at* $P_y$ *lies in the plane* $\pi$ *determined by the joint-edge of the systems of* $\rho$- *and* $\sigma$-*tangeodesics of* $S$ *at* $P_y$ *and the reciprocal* $l'$ *of the line* $l$ *joining* $\rho$, $\sigma$. *The joint-edge coincides with the line* $l'$ *if and only if* $l'$ *is the second edge of Green. Any two particular planes* $\pi_1$ *and* $\pi_2$ *of the plane* $\pi$ (*corresponding to selections* $\rho_1$, $\sigma_1$ *and* $\rho_2$, $\sigma_2$) *intersect in the second edge of Green.*

Let $\sigma_a$ denote the intersection of the tangent plane to $S_\rho$ at $\rho$ with the $v$-tangent to $S$ at $P_y$ and let $\rho_a$ denote the intersection of the tangent plane to $S_\sigma$ at $\sigma$ with the $u$-tangent to $S$ at $P_y$. It may be easily verified that the general coordinates of $\rho_a$ and $\sigma_a$ are given by $\rho_a = y_u - \beta_a y$, $\sigma_a = y_v - \alpha_a y$, wherein $\beta_a$, $\alpha_a$ are given by

$$(3.3)\qquad \beta_a = -(g + \alpha_v + \alpha^2)/2a', \quad \alpha_a = -(f + \beta_u + \beta^2)/2b.$$

The line $l_a$ joining $\rho_a$, $\sigma_a$ was introduced by the author in a previous paper[2] and called *the asymptotic associate of the line* $l$ joining $\rho$, $\sigma$.

The plane determined by the $v$-tangent of $S$ at $P_y$ and the $u$-edge of the $\rho$-tangeodesics at $P_y$ is the polar plane of the $u$-tangent of $S$ at $P_y$ with respect to the cone (2.7) of the $\rho$-tangeodesics. Similarly, the plane determined by the $u$-tangent of $S$ at $P_y$ and the $v$-edge of the $\sigma$-tangeodesics at $P_y$ is the polar plane of the $v$-tangent of $S$ at $P_y$ with respect to the cone (2.8) of the $\sigma$-tangeodesics. These two polar planes intersect in a line $l_3'$ which will be called the *polar-axis of the $\rho$- and $\sigma$-tangeodesics at $P_y$*. This line may be shown to pass through the points $P_y$ and $z_3$ where the general coordinates of $z_3$ are given by $z_3 = y_{uv} - \alpha_3 y_u - \beta_3 y_v$, in which

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[2] P. O. Bell, *A study of curved surfaces by means of certain associated ruled surfaces*, Trans. Amer. Math. Soc. vol. 46 (1939) p. 396.

$$(3.4) \qquad \alpha_3 = -\alpha_u + b_v/b, \qquad \beta_3 = -\beta_u + a_u'/a'.$$

Since the functions $\alpha$, $\beta$ for the directrix $l'$ of Wilczynski are given by

$$(3.5) \qquad \alpha = b_v/2b, \qquad \beta = a_u'/2a',$$

equations (3.4) are such that we have, immediately, this theorem.

THEOREM 3.2. *The second directrix of Wilczynski lies in the plane $p$ determined by the polar-axis of the $\rho$- and $\sigma$-tangeodesics at $P_y$ and the reciprocal $l_a'$, with respect to $S$ at $P_y$, of the asymptotic associate of $l$. Any two particular planes $p_1$ and $p_2$ of the plane $p$ (corresponding to lines $l_1$ and $l_2$) intersect in the second directrix of Wilczynski.*

Theorems 3.1 and 3.2 may be dualized by replacing the lines and planes of these theorems by their reciprocals with respect to $S$ at $P_y$. The dual of Theorem 3.1 is the following theorem.

THEOREM 3.3. *The first edge of Green contains the point $P$ of intersection of the reciprocal of the joint-edge of the systems of $\rho$- and $\sigma$-tangeodesics of $S$ and the line $l$ joining $\rho$, $\sigma$. These three lines coincide if and only if the line $l$ is the first edge of Green. Any two particular points $P_1$ and $P_2$ of the point $P$ (corresponding to selections $\rho_1$, $\sigma_1$ and $\rho_2$, $\sigma_2$) determine the first edge of Green.*

The statement of the dual of Theorem 3.2 will be left to the care of the reader.

Finally, since the *projective-normal* of $S$ at $P_y$ is the line for which the functions $\alpha$, $\beta$ are given by $\alpha = -(b_v/2b + a_v'/2a')$, $\beta = -(a_u'/2a' + b_u/2b)$, and the first directrix of $S$ at $P_y$ is the line $l$ for which $\alpha = b_v/2b$, $\beta = a_u'/2a'$, we have from equations (3.1) this theorem.

THEOREM 3.4. *If the line $l$ joining $\rho$, $\sigma$ is the first directrix of Wilczynski, the joint-edge of the systems of $\rho$ and $\sigma$-tangeodesics of $S$ at $P_y$ is the projective normal of Fubini.*

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# ZEROS AND POLES OF FUNCTIONS DEFINED BY TAYLOR SERIES

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**1. Introduction.** With the objective of providing a straightforward numerical method for the determination of poles and zeros of functions defined by Taylor series this note reexamines Hadamard's solution of this problem, which is found in his classical thesis.[1] The best known part of Hadamard's solution is the criterion which enables one to determine the meromorphic character of the expanded function and the total number of poles on the circle of convergence. But this solution also includes a method of determining these poles as functions of the Taylor coefficients, and Hadamard himself intimated that his results should prove useful in the numerical evaluation of poles and zeros. However, it seems that, as a device in numerical analysis, his method has attracted much less attention than it deserves. This may be due to the fact that Hadamard's criterion for the number of poles employs limits superior, which are impractical for numerical work.

In this paper no use is made of limits superior, and the number of poles on the circle of convergence is ascertained by the process of evaluating their affixes. Besides determining the polynomial whose zeros are all the poles of the expanded function on the circle of convergence with their proper multiplicities, the paper also determines the polynomial whose zeros are the different poles of highest order only. These results are based on an identity and an inequality for persymmetric determinants involving successive Taylor coefficients of rational and meromorphic functions, which seem to be new, and may also prove useful in other applications.

**2. A formula for persymmetric determinants.** We first are going to establish an identity for persymmetric determinants of the form

$$(1)\qquad d_n^{(m)} = \begin{vmatrix} c_{n+1} & c_{n+2} & \cdots & c_{n+m} \\ c_{n+2} & c_{n+3} & \cdots & c_{n+m+1} \\ \cdot & \cdot & \cdots & \cdot \\ c_{n+m} & c_{n+m+1} & \cdots & c_{n+2m-1} \end{vmatrix}, \qquad n = -1, 0, 1, \cdots,$$

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[1] J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de Taylor*, J. Math. Pures Appl. (4) vol. 8 (1892) pp. 101–186.

where the $2m-1$ numbers $c_{n+1}, c_{n+2}, \cdots, c_{n+2m-1}$ are successive coefficients in the Taylor series expansion of a rational function.

Suppose, at first, the rational function has only simple poles and is of the form $g_0(z)=\sum_{n=0}^{\infty}c_nz^n=-\sum_{k=1}^{p}\zeta_k/(z-z_k)$, $z_k\neq 0$. We wish to evaluate $d_n^{(p)}=d_n^{(p)}[g_0]$. Since $c_n=\sum_{k=1}^{p}\zeta_k z_k^{-(n+1)}$ $(n=0,\ 1,\cdots)$, we have

$$d_n^{(p)}=\sum\begin{vmatrix} \zeta_{k_1}z_{k_1}^{-(n+2)} & \zeta_{k_2}z_{k_2}^{-(n+3)} & \cdots & \zeta_{k_p}z_{k_p}^{-(n+p+1)} \\ \zeta_{k_1}z_{k_1}^{-(n+3)} & \zeta_{k_2}z_{k_2}^{-(n+4)} & \cdots & \zeta_{k_p}z_{k_p}^{-(n+p+2)} \\ \cdot & \cdot & \cdots & \cdot \\ \zeta_{k_1}z_{k_1}^{-(n+p+1)} & \zeta_{k_2}z_{k_2}^{-(n+p+2)} & \cdots & \zeta_{k_p}z_{k_p}^{-(n+2p)} \end{vmatrix},$$

where the sum is extended over all the combinations (with repetitions) $k_1, k_2, \cdots, k_p$ of the numbers $1, 2, \cdots, p$. Since the combinations with repeated elements lead to vanishing determinants we may also write

$$d_n^{(p)}=\frac{\zeta_1\zeta_2\cdots\zeta_p}{(z_1z_2\cdots z_p)^{n+2p}}\sum\begin{vmatrix} z_{k_1}^{2p-2} & z_{k_2}^{2p-3} & \cdots & z_{k_p}^{p-1} \\ z_{k_1}^{2p-3} & z_{k_2}^{2p-4} & \cdots & z_{k_p}^{p-2} \\ \cdot & \cdot & \cdots & \cdot \\ z_{k_1}^{p-1} & z_{k_2}^{p-2} & \cdots & z_{k_p}^{0} \end{vmatrix}.$$

The sum of determinants in the last formula is known to be equal to the square of the difference-product of the numbers $z_1, z_2, \cdots, z_p$; hence

$$d_n^{(p)}[g_0]=\Delta\zeta_1\zeta_2\cdots\zeta_p(z_1z_2\cdots z_p)^{-(n+2p)},$$
$$\Delta=\prod_{k<k'}(z_k-z_{k'})^2,\qquad k,\ k'=1,\ 2,\cdots,\ p. \tag{2}$$

Now suppose the $2p-1$ numbers $c_{n+1}, c_{n+2}, \cdots, c_{n+2p-1}$ are successive Taylor coefficients of the more general rational function

$$g(z)=\sum_{n=0}^{\infty}c_nz^n$$
$$=-\sum_{k=1}^{q}\left(\frac{\zeta_k}{(z-z_k)^{p_k}}+\frac{\zeta_{k,1}}{(z-z_k)^{p_k-1}}+\cdots+\frac{\zeta_{k,p_k-1}}{z-z_k}\right), \tag{3}$$
$$z_k\neq 0,$$

which has, at most, $p=p_1+p_2+\cdots+p_q$ poles (counted according to their multiplicities). We are going to evaluate $d_n^{(p)}=d_n^{(p)}[g]$ as the limit of determinants $d_n^{(p)}$ formed for rational functions with only simple poles.

With the numbers $z_{k,1}, z_{k,2}, \cdots, z_{k,p_k}$ $(k=1, 2, \cdots, q)$ which are assumed to be different from each other and from 0 but otherwise arbitrary, we define the polynomial

$$\pi_k(z) = \zeta_k + \zeta_{k,1}(z - z_k) + \cdots + \zeta_{k,p_k-1}(z - z_k)^{p_k-1}, \tag{4}$$

and consider, instead of the function $g(z)$, the function

$$g_1(z) = \sum_{n=0}^{\infty} c_n' z^n = - \sum_{k=1}^{q} \frac{\pi_k(z)}{(z - z_{k,1})(z - z_{k,2}) \cdots (z - z_{k,p_k})} . \tag{5}$$

$g_1(z)$ has only simple poles. Its partial fraction expansion is

$$g_1(z) = - \sum_{k,l} \frac{\zeta_{k,l}'}{z - z_{k,l}}, \qquad l = 1, 2, \cdots, p_k;\ k = 1, 2, \cdots, q,$$

where

$$\zeta_{k,l}' = \frac{\pi_k(z_{k,l})}{(z_{k,l} - z_{k,1}) \cdots (z_{k,l} - z_{k,l-1})(z_{k,l} - z_{k,l+1}) \cdots (z_{k,l} - z_{k,p_k})} .$$

Applying the first result (see formula (2)) to the function $g_1(z)$, we find $d_n^{(p)}[g_1] = \Delta' \prod_{k,l} \zeta_{k,l}' z_{k,l}^{-(n+2p)}$, where $\Delta'$ is the square of the difference-product of the numbers $z_{k,l}$. But $\prod_{k,l} \zeta_{k,l}' = \prod_{k,l} \pi_k(z_{k,l}) / \prod_k \Delta_k$, where $\Delta_k = (-1)^{p_k(p_k-1)/2} \prod_{l<l'} (z_{k,l} - z_{k,l'})^2$. Hence

$$\begin{aligned} d_n^{(p)}[g_1] &= \Delta'' \prod_{k,l} \pi_k(z_{k,l}) z_{k,l}^{-(n+2p)}, \\ \Delta'' &= (-1)^{\sum_k p_k(p_k-1)/2} \prod_{k<k';\, l,l'} (z_{k,l} - z_{k',l'})^2, \\ & \qquad l = 1, 2, \cdots, p_k;\ l' = 1, 2, \cdots, p_{k'}. \end{aligned} \tag{6}$$

We now assume that the numbers $z_{k,l}$ depend on a variable $\delta$, and that $\lim_{\delta \to 0} z_{k,l}(\delta) = z_k$ $(l=1, 2, \cdots, p_k;\ k=1, 2, \cdots, q)$. Instead of $g_1(z)$ and $c_n'$ we now have $g(z, \delta)$ and $c_n(\delta)$ respectively. Comparing (5) and (3), we see that $g(z, \delta)$ converges to $g(z)$, $c_n(\delta)$ converges to $c_n$, as $\delta \to 0$. Hence $d_n^{(p)}[g(\delta)]$ converges to $d_n^{(p)}[g]$, as $\delta \to 0$. Moreover, by (4), $\pi_k(z_{k,l})$ converges to $\zeta_k$, as $\delta \to 0$. Using all these limits, we obtain from (6)

$$\begin{aligned} d_n^{(p)}[g] &= \Delta \zeta_1^{p_1} \zeta_2^{p_2} \cdots \zeta_q^{p_q} (z_1^{p_1} z_2^{p_2} \cdots z_q^{p_q})^{-(n+2p)}, \\ \Delta &= (-1)^{\sum_k p_k(p_k-1)/2} \prod_{k<k'} (z_k - z_{k'})^{2p_k p_{k'}}, \quad k, k' = 1, 2, \cdots, q. \end{aligned} \tag{7}$$

**3. The equation for all the poles on the circle of convergence.** In the following, the numbers $a_n$ will represent the Taylor coefficients of any analytic function that has no singularities other than poles on its circle of convergence about 0. And besides such determinants as defined in (1), we shall consider determinants $d_{ni}^{(m)}$ ($i=1, 2, \cdots, m$), defined as

$$(8)\quad d_{ni}^{(m)}[f] = \begin{vmatrix} a_{n+1} & a_{n+2} & \cdots & a_{n+i-1} & a_n & a_{n+i+1} & \cdots & a_{n+m} \\ a_{n+2} & a_{n+3} & \cdots & a_{n+i} & a_{n+1} & a_{n+i+2} & \cdots & a_{n+m+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n+m} & a_{n+m+1} & \cdots & a_{n+i+m-2} & a_{n+m-1} & a_{n+i+m} & \cdots & a_{n+2m-1} \end{vmatrix},$$

$$d_{n-1}^{(m)}[f] = -d_{nm}^{(m)}[f], \qquad n = 0, 1, \cdots.$$

THEOREM 1. *Suppose* $f(z) = \sum_n a_n z^n = g(z) + f_1(z)$, *where*

$$g(z) = -\sum_{k=1}^{q}\left(\frac{\zeta_k}{(z-z_k)^{p_k}} + \frac{\zeta_{k,1}}{(z-z_k)^{p_k-1}} + \cdots + \frac{\zeta_{k,p_k-1}}{z-z_k}\right),$$

$$|z_k| = r > 0, \quad z_k \neq z_{k'}(k \neq k'), \quad \zeta_k \neq 0, \qquad k, k' = 1, 2, \cdots, q,$$

*and* $f_1(z)$ *is regular in* $|z| \leqq R = \tau r$, $\tau > 1$; *suppose*

$$\pi(z) = (z-z_1)^{p_1}(z-z_2)^{p_2} \cdots (z-z_q)^{p_q}$$
$$= z^p + \gamma_1 z^{p-1} + \cdots + \gamma_p, \qquad p = p_1 + p_2 + \cdots + p_q.$$

*Then, for* $i = 1, 2, \cdots, p$,

$$(9)\qquad \gamma_i + d_{ni}^{(p)}[f]/d_n^{(p)}[f] = O(\tau^{-n}), \qquad \text{as } n \to \infty.$$

PROOF. The function $\phi(z) = \sum_\nu \alpha_\nu z^\nu = \pi(z)f(z)$ has the coefficients

$$(10)\qquad \alpha_\nu = a_{\nu-p} + \gamma_1 a_{\nu-p+1} + \gamma_2 a_{\nu-p+2} + \cdots + \gamma_p a_\nu.$$

If, in (10), we put $\nu$ successively equal to $n+p$, $n+p+1, \cdots, n+2p-1$, we have $p$ equations for the unknowns $\gamma_1, \gamma_2, \cdots, \gamma_p$, whose solution is, provided $d_n^{(p)}[f] \neq 0$,

$$(11)\qquad \gamma_i + d_{ni}^{(p)}/d_n^{(p)} = e_{ni}^{(p)}/d_n^{(p)}, \qquad i = 1, 2, \cdots, p,$$

where $e_{ni}^{(p)}$ denotes the determinant obtained from $d_{ni}^{(p)}$ when $a_n$, $a_{n+1}, \cdots, a_{n+p-1}$ are replaced by $\alpha_{n+p}, \alpha_{n+p+1}, \cdots, \alpha_{n+2p-1}$ respectively.

The determinants $e_{ni}^{(p)}$ are easily estimated. Since $\phi(z)$ is regular in $|z| \leqq R$, and $f(z)$ is regular in $|z| < r$, there is some number $\sigma > 1$

such that $\alpha_n = O(\sigma^{-n}R^{-n})$, $a_n' = O(\sigma^{n/(p-1)}r^{-n})$ as $n \to \infty$; hence

$$e_{n1}^{(p)} = O(r^{-(p-1)n}R^{-n}) = O(\tau^{-n}r^{-pn}), \qquad \text{as } n \to \infty. \tag{12}$$

A lower bound for the determinants $d_n^{(p)}[f]$ is derived from identity (7). If $g(z) = \sum_n c_n z^n$, $f_1(z) = \sum_n b_n z^n$, then $a_n = b_n + c_n$, and $d_n^{(p)}[f]$ is the sum of $d_n^{(p)}[g]$ and several other determinants, each of which has at least one column of $b$'s and otherwise columns of $c$'s. Hence, as in (12),

$$d_n^{(p)}[f] = d_n^{(p)}[g] + O(\tau^{-n}r^{-pn}), \qquad \text{as } n \to \infty. \tag{13}$$

But since, by (7), for $n = 0, 1, 2, \cdots$, $|d_n^{(p)}[g]| = Mr^{-pn}$, where $M$ is a positive constant, we deduce from (13) that there is another constant $M_1$ such that, for sufficiently large $n$,

$$|d_n^{(p)}[f]| > M_1 r^{-pn}. \tag{14}$$

In particular, $d_n^{(p)}[f] \neq 0$ when $n$ is sufficiently large.

Inequalities (12) and (14), together with equation (11), prove the theorem.

Let it be noted that, since $\gamma_p = (-1)^p z_1^{p_1} z_2^{p_2} \cdots z_q^{p_q}$, we have $r^p = |\gamma_p|$, or by (9), $r^p = \lim_{n\to\infty} |d_{n-1}^{(p)}[f]/d_n^{(p)}[f]|$. This relation implies $r^p = \lim_{n\to\infty} |d_n^{(p)}[f]|^{-1/n}$.[2]

It is not necessary that all the poles $z_k$ should lie on the circle $|z| = r$. Indeed, we may state the following:

COROLLARY. *Estimate* (9) *still holds true if* $|z_k| = \tau_k r$, $\tau_k \geqq 1$ $(k = 1, 2, \cdots, q)$, *and* $R = \tau r \cdot \tau_1^{p_1}\tau_2^{p_2} \cdots \tau_q^{p_q}$, $\tau > 1$.

The proof of this corollary is an obvious variation of the proof given above.

If $f(z)$ is a rational function, say $f(z) = \psi(z)/\pi(z)$, where $\psi(z)$ is a polynomial whose degree we define to be $s \geqq p-2$, then $\phi(z) = \sum_\nu \alpha_\nu z^\nu = \pi(z)f(z) = \psi(z)$; hence $\alpha_\nu = 0$ for $\nu > s$, and consequently $e_{n1}^{(p)} = 0$ for $n > s-p$. On the other hand, $f(z)$ is the sum of a polynomial $f_1(z)$ of degree $s-p$ and a fraction $g(z)$ such as in (3). Hence, for $n > s-p$, $d_n^{(p)}[f] = d_n^{(p)}[g] \neq 0$, by (7). Thus, relation (9) holds, for $n > s-p$, with 0 substituted for the second term. In particular,

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[2] This "rather delicate limit relation" (see P. Dienes, *The Taylor series*, Oxford, 1931, p. 330) was first proved by Hadamard, op. cit. Other authors have tried to give simpler proofs for it, see, for example, A. Ostrowski, *Über einen Satz von Herrn Hadamard*, Jber. Deutscher Math. Verein. vol. 35 (1926) pp 179–182. It is an immediate consequence of identity (7). On the other hand, it is the only intricate point in the proof of Theorem 1.

$$\gamma_i = -\,d^{(p)}_{s-p+1,i}[f]/d^{(p)}_{s-p+1}[f], \qquad i = 1, 2, \cdots, p. \tag{15}$$

The numbers $\gamma_i$ having been found, the numbers $\alpha_\nu$ are given by (10). Thus, the coefficients of the polynomials $\pi(z)$ and $\psi(z)$ are expressed, by (15) and (10), as rational functions of the Taylor coefficients $a_0, a_1, \cdots, a_{p+s}$ of $f(z) = \psi(z)/\pi(z)$.

4. **An inequality for persymmetric determinants.** In the following as in the foregoing we shall assume the function $f(z) = \sum_n a_n z^n$ to have no singularities other than poles on the circle of convergence $|z| = r$ of the Taylor series. We shall be interested no longer in all the poles, but only in the poles of highest order on $|z| = r$. We shall say that $f(z)$ is of order $m$ on $|z| = r$ if $f(z)$ has there at least one pole of order $m$, and no other singularities but poles of order less than or equal to $m$.

We are going to establish an estimate for the determinants $d_n^{(q)}[f]$ $(n = 0, 1, \cdots)$ where $q$ is the number of poles of highest order of $f(z)$ on $|z| = r$. Suppose that $f(z) = \sum_n a_n z^n = g(z) + f_1(z)$, where $g(z) = -\sum_{k=1}^{q} \zeta_k/(z - z_k)^m$, $|z_k| = r > 0$, and $f_1(z)$ is regular in $|z| < r$ and of order less than $m$ on $|z| = r$. The coefficient of $z^n$ in the Taylor series expansion of $-\zeta_k(z - z_k)^{-m}$ is

$$(-1)^{m-1} \frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!\, z_k^{m+n}} \zeta_k$$
$$= (-1)^{m-1} \frac{n^{m-1}}{(m-1)!\, z_k^{m+n}} \zeta_k + O\left(\frac{n^{m-2}}{r^n}\right), \qquad \text{as } n \to \infty.$$

The coefficient of $z^n$ in the Taylor series expansion of $f_1(z)$ is, for the same reason, $O(n^{m-2} r^{-n})$. Hence we have the estimate

$$\begin{aligned} a_n &= \sum_{k=1}^{q} \frac{\xi_k}{z_k^{n+1}} + O\left(\frac{n^{m-2}}{r^n}\right), \qquad \text{as } n \to \infty, \\ \xi_k &= (-1)^{m-1} \frac{n^{m-1}}{(m-1)!\, z_k^{m-1}} \zeta_k. \end{aligned} \tag{16}$$

It implies the less exact estimate

$$a_n = O(n^{m-1}/r^n), \qquad \text{as } n \to \infty. \tag{17}$$

Corresponding to the decomposition (16), the determinant $d_n^{(q)}[f]$ splits into the sum of a persymmetric determinant whose terms are $c_\nu = \sum_{k=1}^{q} \xi_k z_k^{-(\nu+1)}$ $(\nu = n+1, n+2, \cdots, n+2q-1)$ and several other determinants, each of which has at least one column made of terms that are $O(n^{m-2} r^{-n})$, and otherwise columns made of terms that are

$O(n^{m-1}r^{-n})$. The first determinant, as was shown in §2, is equal to $\Delta\cdot\xi_1\xi_2\cdots\xi_q\,(z_1z_2\cdots z_q)^{-(n+2q)}$, where $\Delta$ is the square of the difference-product of the numbers $z_1, z_2, \cdots, z_q$. The sum of the other determinants is $O(n^{q(m-1)-1}r^{-qn})$. Hence, using (16), we have the estimate

$$(18)\qquad \begin{aligned} d_n^{(q)}[f] &= C_f\,\frac{n^{q(m-1)}}{(z_1z_2\cdots z_q)^n} + O\left(\frac{n^{q(m-1)-1}}{r^{qn}}\right), \quad \text{as } n\to\infty,\\ C_f &= (-1)^{q(m-1)}\Delta\cdot\frac{\zeta_1\zeta_2\cdots\zeta_q}{[(m-1)!]^q(z_1z_2\cdots z_q)^{2q+m-1}}. \end{aligned}$$

5. **The equation for the poles of highest order.** In §3 we have established the algebraic equation whose roots are *all* the poles of $f(z)$ (counted according to their multiplicities) on $|z|=r$. We shall now establish the algebraic equation whose roots are the different poles of highest order of $f(z)$ on $|z|=r$.

THEOREM 2. *Suppose that* $f(z)=\sum_n a_nz^n=g(z)+f_1(z)$, *where*

$$g(z) = -\left(\zeta_1/(z-z_1)^m + \zeta_2/(z-z_2)^m + \cdots + \zeta_q/(z-z_q)^m\right),$$

$$|z_k| = r > 0,\quad z_k\neq z_{k'}\,(k\neq k'),\quad \zeta_k\neq 0,\quad k,\,k' = 1, 2, \cdots, q,$$

*and* $f_1(z)$ *is regular in* $|z|<r$ *and of order less than* $m$ *on* $|z|=r$; *suppose* $\pi(z)=(z-z_1)(z-z_2)\cdots(z-z_q)=z^q+\gamma_1z^{q-1}+\cdots+\gamma_q$. *Then, for* $i=1, 2, \cdots, q$,

$$(19\text{a})\qquad \gamma_i + d_{n,i}^{(q)}[f]/d_n^{(q)}[f] = O(1/n), \qquad \text{as } n\to\infty,$$

*and*

$$(19\text{b})^3\qquad m = 1 + \lim_{n\to\infty}\frac{\log|d_n^{(q)}[f]| + nq\log r}{q\log n}.$$

PROOF. The function $\phi(z)=\sum_\nu\alpha_\nu z^\nu=\pi(z)f(z)$, whose coefficients are

$$(20)\qquad \alpha_\nu = a_{\nu-q} + \gamma_1a_{\nu-q+1} + \cdots + \gamma_qa_\nu,$$

is regular in $|z|<r$, and of order less than or equal to $m-1$ on $|z|=r$. Hence, as in (17), the coefficients $\alpha_\nu$ are $O(\nu^{m-2}r^{-\nu})$, as $\nu\to\infty$. If, in

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[3] Hadamard, see op. cit., defined the order of any series $\sum a_nz^n$ and proved that, on its circle of convergence, the order is given by the formula $m=1+\limsup\,(\log|a_n|+n\log r)/\log n$. Formula (19b) represents, in accordance with the objective of this paper, the order as a regular limit, but it holds only for series which are meromorphic on their circle of convergence.

(20), we put $\nu$ successively equal to $n+q, n+q+1, \cdots, n+2q-1$, we have $q$ equations for the unknowns $\gamma_1, \gamma_2, \cdots, \gamma_q$, whose solution is, provided $d_n^{(q)}[f] \neq 0$,

$$\gamma_i + d_{ni}^{(q)}/d_n^{(q)} = e_{ni}^{(q)}/d_n^{(q)}, \qquad i = 1, 2, \cdots, q, \tag{21}$$

where $e_{ni}^{(q)}$ denotes the determinant that is obtained from $d_{ni}^{(q)}$ when $a_n, a_{n+1}, \cdots, a_{n+q-1}$ are replaced by $\alpha_{n+q}, \alpha_{n+q+1}, \cdots, \alpha_{n+2q-1}$ respectively. Using estimate (17) for the $a$'s and a corresponding estimate for the $\alpha$'s, we find readily

$$e_{ni}^{(q)} = O(n^{q(m-1)-1} r^{-qn}), \qquad \text{as } n \to \infty. \tag{22}$$

Taking into account that, by assumption, the poles $z_1, z_2, \cdots, z_q$ are different from each other, we deduce from relation (18) that there are constants $M_1, M_2$ such that, for sufficiently large $n$,

$$d_n^{(q)}[f] = \mu_n n^{q(m-1)} r^{-qn}, \qquad M_1 < |\mu_n| < M_2. \tag{23}$$

In particular, $d_n^{(q)}[f] \neq 0$ when $n$ is sufficiently large.

Inequalities (22) and (23), together with equation (21), prove the theorem.

6. **Converses of Theorems 1 and 2.** In problems where the poles of $f(z)$ are to be determined from the given Taylor coefficients, it is, in general, not known, a priori, what and how many singularities $f(z)$ has on the circle of convergence $|z| = r$ of its Taylor series expansion. Even if $f(z)$ is known to be meromorphic, Theorems 1 and 2 do not always suffice to establish the polynomial $\pi(z)$ whose zeros are all the poles, or the poles of highest order, of $f(z)$ on $|z| = r$, as the numbers $p$ or $q$ may not be known. In such a case the procedure to follow is to investigate the behavior of the quotients $d_{ni}^{m}[f]/d_n^{(m)}[f]$, as $n \to \infty$, successively for $m = 1, 2, \cdots$. From the convergence properties of these sequences the nature and number of the singularities of $f(z)$ on $|z| = r$ can be deduced, as will be shown in the following converses of Theorems 1 and 2. It will always be assumed that $f(z) = \sum_n a_n z^n$ has a positive radius of convergence.

THEOREM 3. *Suppose that, for a fixed $p$, there are $p$ numbers $\gamma_1, \gamma_2, \cdots, \gamma_p$ and a number $\tau > 1$ such that, for $i = 1, 2, \cdots, p$,*

$$\gamma_i + d_{ni}^{(p)}[f]/d_n^{(p)}[f] = O(\tau^{-n}), \qquad \textit{as } n \to \infty; \tag{24}$$

*suppose that this is not the case for any $p' < p$. Then $f(z)$ is regular in* *the circle* $|z| \leqq r = |\gamma_p|^{1/p}$ *but for* $p$ *poles (counted according to their multiplicities) on* $|z| = r$, *which are the zeros of the polynomials* $\pi(z) = z^p + \sum_{i=1}^{p} \gamma_i z^{p-i}$.

PROOF. According to the definition of the determinants $d_{ni}^{(p)}$, $d_n^{(p)}$, and since the quotients $d_{ni}^{(p)}/d_n^{(p)}$ exist, by assumption, for sufficiently large $n$, we have

$$(25) \qquad a_n - (1/d_n^{(p)})(a_{n+1}d_{n1}^{(p)} + a_{n+2}d_{n2}^{(p)} + \cdots + a_{n+p}d_{np}^{(p)}) = 0.$$

Let $r$ be the radius of convergence of $\sum_n a_n z^n$; then $a_n = O(r'^{-n})$, as $n \to \infty$, for any $r' < r$. If we choose $r'$ so that $R = rr' > r$, we obtain, by substituting (24) in (25),

$$\alpha_{n+p} = a_n + \gamma_1 a_{n+1} + \gamma_2 a_{n+2} + \cdots + \gamma_p a_{n+p} = O(R^{-n}), \quad \text{as } n \to \infty.$$

Hence the function $\phi(z) = \pi(z)f(z) = \sum_\nu \alpha_\nu z^\nu$ is regular in $|z| \leqq R > r$. This implies that $f(z)$ has no other singularities than, at most, $p$ poles on the circle $|z| = r$, which are among the zeros of $\pi(z)$.

If $f(z)$ had less than $p$ poles on $|z| = r$, some relation like (24) would, by Theorem 1, hold for $p' < p$, in contradiction with the hypothesis. Hence $f(z)$ has exactly $p$ poles on $|z| = r$, which are the zeros of $\pi(z)$. Since $(-1)^p\gamma_p$ is the product of the $p$ zeros, $r = |\gamma_p|^{1/p}$.

THEOREM 4. *Suppose* $f(z)$ *has no singularities but poles on* $|z| = r$; *suppose that, for a fixed* $q$, *there are* $q$ *numbers* $\gamma_1, \gamma_2, \cdots, \gamma_q$ *such that, for* $i = 1, 2, \cdots, q$,

$$(26) \qquad \gamma_i + \lim_{n\to\infty} d_{ni}^{(q)}[f]/d_n^{(q)}[f] = 0;$$

*suppose that this is not the case for any* $q' < q$. *Then* $f(z)$ *is regular in the circle* $|z| < r = |\gamma_q|^{1/q}$, *and of order*

$$m = 1 + \lim_{n\to\infty} (\log |d_n^{(q)}| + nq \log r)/q \log n$$

*on* $|z| = r$. $f(z)$ *has* $q$ *different poles of order* $m$ *on* $|z| = r$, *which are the zeros of the polynomial* $\pi(z) = z^q + \sum_{i=1}^{q} \gamma_i z^{q-i}$.

PROOF. Let $r$ be the radius of convergence of $\sum_n a_n z^n$, and let $m$ be the order of $f(z)$ on $|z| = r$. If $f(z)$ had less than $q$ poles on $|z| = r$, some relation like (26) would, by Theorem 2, hold true for $q' < q$, in contradiction with the hypothesis. Hence $f(z)$ has $q' \geqq q$ poles of order $m$ on $|z| = r$.

As in the proof of Theorem 3, we have, for sufficiently large $n$,

$$(27) \qquad a_n - (1/d_n^{(q)})(a_{n+1}d_{n1}^{(q)} + a_{n+2}d_{n2}^{(q)} + \cdots + a_{n+q}d_{nq}^{(q)}) = 0.$$

In §4 (see (17)) it was shown that $a_n = O(n^{m-1}r^{-n})$, as $n \to \infty$, whereas by assumption, $d_{ni}^{(q)}/d_n^{(q)} = -\gamma_i + o(1)$, as $n \to \infty$. Hence (27) implies

$$\alpha_{n+q} = a_n + \gamma_1 a_{n+1} + \gamma_2 a_{n+2} + \cdots + \gamma_q a_{n+q} = o(n^{m-1}r^{-n}), \tag{28}$$

as $n \to \infty$.

Now suppose $q' > q$. Since $\pi(z)$ is of degree $q$, $\phi(z) = \sum_n \alpha_n z^n = \pi(z)f(z)$ is, like $f(z)$, of order $m$, and has $q'' \geqq q' - q$ different poles $z_1, z_2, \cdots, z_{q''}$ of order $m$ on $|z| = r$. Hence, as was shown in §4 (see (16)) $\alpha_n$ has the form

$$\alpha_n = n^{m-1} \sum_{k=1}^{q''} \frac{\theta_k}{z_k^n} + O\left(\frac{n^{m-2}}{r^n}\right) \tag{29}$$

as $n \to \infty$, and $\theta_k \neq 0$ $(k = 1, 2, \cdots, q'')$.

It is easily shown[4] that there is a positive number $M$ such that, for $n = 0, 1, \cdots$, at least one of the $q''$ numbers $|r^{n+\nu} \sum_{k=1}^{q''} \theta_k / z_k^{n+\nu}|$, $\nu = 0, 1, \cdots, q'' - 1$ is greater than $M$. This fact, together with (29), implies that, to any given $n_0$, there are numbers $n > n_0$ such that $|\alpha_n| > (1/2)Mn^{m-1}r^{-n}$, whereas, because of (28), $n_0$ may be found such that, for *all* $n > n_0$, $|\alpha_n| < (1/2)Mn^{m-1}r^{-n}$. Thus the assumption $q' > q$ is proved to be wrong.

Since we have proved that the number of poles of $f(z)$ of order $m$ on $|z| = r$ is $q$, the other statements of the theorem follow immediately from Theorem 2.

COROLLARY. *If* $f(z) = \sum_n a_n z^n$ *has no singularities but poles on the circle of convergence* $|z| = r$ *of* $\sum_n a_n z^n$, *and if* $z_1 = \lim_{n\to\infty} a_n/a_{n+1}$ *exists, then* $f(z)$ *is of order* $m = 1 + \lim_{n\to\infty} (\log |a_n| + n \log r)/\log n$ *on* $|z| = r$, $z_1$ *being the only pole of order* $m$ *on* $|z| = r$.[5]

This is obviously Theorem 4 for $q = 1$.

The following theorem is the logical product of Theorems 3 and 4.

THEOREM 5. *Suppose that, for a fixed* $p$, *there are* $p$ *numbers* $\gamma_1, \gamma_2, \cdots, \gamma_p$ *and a number* $\tau > 1$ *such that, for* $i = 1, 2, \cdots, p$, $\gamma_i + d_{ni}^{(p)}[f]/d_n^{(p)}[f] = O(\tau^{-n})$, *as* $n \to \infty$; *suppose that there is no* $p' < p$ *such that the limits* $\lim_{n\to\infty} d_{ni}^{(p')}[f]/d_n^{(p')}[f]$ $(i = 1, 2, \cdots, p')$ *exist. Then* $f(z)$ *is regular in the circle* $|z| \leqq r = |\gamma_p|^{1/p}$ *but for* $p$ *poles of the first order on* $|z| = r$, *which are the zeros of the polynomial* $\pi(z) = z^p + \sum_{i=1}^{p} \gamma_i z^{p-i}$.

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[4] See A. Ostrowski, *Über Singularitäten gewisser mit Lucken behafteten Potenzreihen*, Jber. Deutschen Math. Verein. vol. 35 (1926) p. 269.

[5] Cf. footnote 3.

PROOF. As in the proof of Theorem 3 it is seen that $f(z)$ has no other singularities than, at most, $p$ poles (counted according to their multiplicities) on the circle of convergence $|z| = r$ of $f(z) = \sum_n a_n z^n$. If, on $|z| = r$, $f(z)$ had less than $p$ poles, or if some of the poles were of higher than the first order, then $f(z)$ would have $q < p$ poles of highest order on $|z| = r$. This would imply, by Theorem 2, the existence of the limits $\lim_{n\to\infty} d_{n+1}^{(i)}/d_n^{(i)}$ $(i = 1, 2, \cdots, q)$, in contradiction with the hypothesis of the theorem. $f(z)$ has, therefore, exactly $p$ poles of the first order on $|z| = r$. The other statements of the theorem follow immediately.

7. **Application to the evaluation of roots.** The zeros of smallest absolute value of the function $F(z) = \sum_n A_n z^n$ are the poles of the function $f(z) = \sum_n a_n z^n = 1/F(z)$ on its circle of convergence. Hence the theory just developed contains a method of evaluating zeros of smallest absolute value of functions given by Taylor series. The procedure to follow is that outlined at the beginning of §6. It requires the evaluation of the determinants $d_n^{(m)}[f]$ $(n = 0, 1, \cdots)$ for several values of $m$. These determinants are defined, in (8), in terms of the coefficients $a_n$ of $f(z)$. They also can be expressed[6] easily in terms of the coefficients $A_n$ of the given function $F(z)$, with the determinant $d_n^{(m)}[f]$ of order $m$ in the elements $a_n, a_{n+1}, \cdots, a_{n+2m-1}$ replaced by a determinant of order $m+n+1$ in the elements $A_0, A_1, \cdots, A_{n+2m-1}$.

Since the order of these latter determinants increases with $n$, they are not suited for practical purposes. If numerical evaluation of the zeros of $F(z)$ is required, it is advisable to compute first the coefficients $a_n$ of the reciprocal function $f(z)$, and then to work with the determinants $d_n^{(m)}[f]$ as given in (8). If $f(z) = \sum_n a_n z^n$ is the negative reciprocal of $F(z) = \sum_n A_n z^n$, and the coefficient $A_0$ is made to be $-1$, then the coefficients $a_n$ are given by the simple recursion formulae:

$$\begin{aligned}
a_0 &= 1,\\
a_1 &= A_1,\\
a_2 &= A_2 + a_1 A_1,\\
a_3 &= A_3 + a_1 A_2 + a_2 A_1,\\
&\cdots\cdots\cdots\cdots.
\end{aligned}$$

To illustrate the numerical efficiency of the method, the two conjugate complex roots $z_1, z_2$ of smallest absolute value of $F(z) = z - e^z = 0$ were computed. The results are exhibited in the following table. The second column of the table contains the Taylor coefficients $a_n$ of $f(z)=1/(e^z-z)$ multiplied by $n!$ in order to avoid fractions; the third column contains the quotients $d_{n1}^{(2)}[f]/d_n^{(2)}[f]$, which represent successive approximations to $-\gamma_1=z_1+z_2$; the fourth column contains the quotients $d_{n-1}^{(2)}[f]/d_n^{(2)}[f]$, which represent successive approximations to $\gamma_2=z_1z_2$. The final values in the table give

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[6] See J. Hadamard, loc. cit., p. 136.

$$z_1+z_2=0.6362630104, \qquad z_1z_2=1.889406976$$

with a relative error of $1\cdot 10^{-10}$. As the table shows, the fifth approximation, which is obtained with very little effort, already gives the values of $z_1+z_2$ and $z_1z_2$ with a relative error of $1\cdot 10^{-6}$.

| $n$ | $a_n$ | $d_{n1}^{(2)}[f]/d_n^{(2)}[f]$ | $d_{n-1}^{(2)}[f]/d_n^{(2)}[f]$ |
|---|---:|---|---|
| 1 | 0 | 0.67 | 2 |
| 2 | −1 | 0.632 | 1.89 |
| 3 | −1 | 0.6368 | 1.891 |
| 4 | 5 | 0.63616 | 1.8897 |
| 5 | 19 | 0.6362624 | 1.88942 |
| 6 | −41 | 0.6362628 | 1.88941 |
| 7 | −519 | 0.63626291 | 1.8894073 |
| 8 | −183 | 0.63626306 | 1.8894072 |
| 9 | 19 223 | 0.63626299 | 1.88940700 |
| 10 | 73 451 | 0.636263011 | 1.889406978 |
| 11 | −847 067 | 0.6362630099 | 1.889406977 |
| 12 | −8 554 547 | 0.6362630105 | 1.889406976 |
| 13 | 32 488 611 | 0.6362630104 | 1.889406976 |
| 14 | 977 198 559 | 0 6362630104 | 1.889406976 |
| 15 | 1 325 135 969 | | |
| 16 | −116 987 762 287 | | |

PURDUE UNIVERSITY

# ON EXTENSION OF WRONSKIAN MATRICES

DAVID M. KRABILL

1. **Introduction.** By an interval $J$ we shall understand a finite interval of the type $a \leqq x \leqq b$. If $u_1(x), u_2(x), \cdots, u_n(x)$ are real functions possessing finite derivatives of the first $t$ orders in an interval $J$ and $0 \leqq s \leqq t$, we call the functional matrix

$$M_s(u_1, \cdots, u_n) = \left\| \begin{matrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ u_1^{(s)} & u_2^{(s)} & \cdots & u_n^{(s)} \end{matrix} \right\|$$

their Wronskian matrix of order $s$. The Wronskian $W(u_1, \cdots, u_n)$ is the determinant of the matrix $M_{n-1}(u_1, \cdots, u_n)$.

The principal result we obtain is that if $n \leqq s \leqq t$ and the Wronskian matrix of order $s$ for $n$ arbitrary functions $u_1(x), u_2(x), \cdots, u_n(x)$ of class[1] $C^{(t)}$ in an interval $J$ has constant rank $n$, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that the extended matrix $M_s(u_1, \cdots, u_{n+1})$ has constant rank $n+1$ in $J$. We employ a theorem of Curtiss[2] which may be stated in the form:

THEOREM C. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ in an interval $J$ and their Wronskian matrix of order $t$ has rank $n$ throughout $J$, then the Wronskian $W(u_1, \cdots, u_n)$ has at most isolated zeros.*

From the extension property of Wronskian matrices we obtain a sufficient condition, in terms of the rank of a certain functional matrix, that an arbitrary set of functions having suitable class properties be solutions of an ordinary homogeneous linear differential equation.

2. **Lemmas.** We first prove two lemmas.

LEMMA 1. *If $\delta, c_1, c_2, \cdots, c_n$ are given constants with $\delta > 0$, there exists a function $f(x)$ of class $C^{(n)}$ in the interval $-1 \leqq x \leqq 1$ which satisfies the conditions:* (1) $|f(x)| \leqq \delta$, $-1 \leqq x \leqq 1$; (2) $f^{(i)}(-1) = f^{(i)}(1) = 0$, $i = 0, 1, \cdots, n$; (3) $f(0) = 0$, $f^{(i)}(0) = c_i$, $i = 1, 2, \cdots, n$.

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[1] Each function has continuous derivatives of the first $t$ orders at every point of $J$.

[2] D. R. Curtiss, *The vanishing of the Wronskian and the problem of linear dependence*, Math. Ann. vol. 65 (1908) Theorem 4.

PROOF. Let $p(x)$ be the polynomial

$$c_1x + (c_2/2!)x^2 + \cdots + (c_n/n!)x^n.$$

Then $p(0)=0$, $p^{(i)}(0)=c_i$ and for some $\beta$ with $0<\beta<1$, $|p(x)| \leqq \delta$, $-\beta \leqq x \leqq \beta$. Take $\alpha$ such that $0<\alpha<\beta$. Let

$$\theta = (-1)^n(x+\alpha)^{n+1}/(x+\beta)^2, \qquad \phi = -(x-\alpha)^{n+1}/(x-\beta)^2,$$

and let

$$g_1(x) = \begin{cases} 0, & x \leqq -\beta, \\ e^\theta, & -\beta < x < -\alpha, \\ 1, & -\alpha \leqq x \leqq 0, \end{cases} \qquad g_2(x) = \begin{cases} 1, & 0 \leqq x \leqq \alpha, \\ e^\phi, & \alpha < x < \beta, \\ 0, & \beta \leqq x. \end{cases}$$

The $g$'s are of class $C^{(n)}$ in their respective intervals of definition and $g_1^{(i)}(0)=g_2^{(i)}(0)$ for $i=0, 1, \cdots, n$. Then the function

$$f(x) = \begin{cases} p(x)g_1(x), & -1 \leqq x \leqq 0, \\ p(x)g_2(x), & 0 \leqq x \leqq 1, \end{cases}$$

is of class $C^{(n)}$ in the interval $-1 \leqq x \leqq 1$ and satisfies the prescribed conditions.

LEMMA 2. *Given an interval $J$ and constants $a_i$ and $b_i$, $i=0, 1, \cdots, n$ with $a_0b_0>0$, there exists a function $f(x)$ which satisfies the conditions*: (1) $f(x)$ *is nonzero and of class* $C^{(n)}$ *in* $J$; (2) $f^{(i)}(a)=a_i$, $f^{(i)}(b)=b_i$, $i=0, 1, \cdots, n$.

PROOF. Let $\delta=1/3 \min(|a_0|, |b_0|, b-a)$. Take $f_k(x)$, $k=1, 2$, of class $C^{(n)}$ in the interval $-1 \leqq x \leqq 1$ such that: $|f_k(x)| \leqq \delta$, $-1 \leqq x \leqq 1$; $f_k^{(i)}(-1)=f_k^{(i)}(1)=f_k(0)=0$, $i=0, 1, \cdots, n$; $f_1^{(i)}(0)=a_i$, $f_2^{(i)}(0)=b_i$, $i=1, 2, \cdots, n$. Let

$$u(x) = f_1((x-a)/\delta) + a_0, \qquad a \leqq x \leqq a + \delta,$$
$$v(x) = f_2((x-b)/\delta) + b_0, \qquad b - \delta \leqq x \leqq b.$$

Then $u(a)=u(a+\delta)=a_0$; $|u(x)-u(a)| \leqq \delta$, $a \leqq x \leqq a+\delta$; $u^{(i)}(a)=a_i$, $u^{(i)}(a+\delta)=0$, $i=1, 2, \cdots, n$; and $v(b)=v(b-\delta)=b_0$; $|v(x)-v(b)| \leqq \delta$, $b-\delta \leqq x \leqq b$; $v^{(i)}(b)=b_i$, $v^{(i)}(b-\delta)=0$, $i=1, 2, \cdots, n$. Moreover, both $u(x)$ and $v(x)$ are nonzero in their respective mutually exclusive intervals of definition by our choice of $\delta$. Let $w(x)=(a_0-b_0)e^\theta+b_0$, $a+\delta \leqq x \leqq b-\delta$, where $\theta=-(x-a-\delta)^{n+1}/(x-b+\delta)^2$. Then $w(x)$ is nonzero throughout its interval of definition and

$$w(a+\delta) = a_0,\ w(b-\delta) = b_0,\ w^{(i)}(a+\delta) = w^{(i)}(b-\delta) = 0, \qquad i = 1, 2, \cdots, n.$$

Therefore the function

$$f(x) = \begin{cases} u(x), & a \leqq x \leqq a + \delta, \\ w(x), & a + \delta \leqq x \leqq b - \delta, \\ v(x), & b - \delta \leqq x \leqq b, \end{cases}$$

satisfies the conditions of the lemma.

3. **Extension of Wronskian matrices.** Let $u(x)$ be a function of class $C^{(t)}$, $t \geqq 1$, in an interval $J$ with $M_1(u)$ of rank 1 at every point of $J$. If $v(x)$ of class $C^{(t)}$ exists such that $M_1(u, v)$ has rank 2 throughout $J$, the Wronskian $W(u, v)$ is nonzero in $J$. Thus in this case there exists a function $K(x)$ of class $C^{(t-1)}$ and nonzero in $J$ such that $v(x)$ is a solution of the differential equation

$$u(x)y' - u'(x)y = K(x). \tag{1}$$

This leads us to ask: does (1) have a solution of class $C^{(t)}$ in $J$ for arbitrary $K(x)$ of class $C^{(t-1)}$ which is nonzero in $J$? We give a counter-example.

*Example.* Let $J$ be the interval $0 \leqq x \leqq 2$ and take $u(x) = 1 - x$, $K(x) = 1 + |1 - x|^{1/2}$. Then the differential equation (1) becomes

$$(1 - x)y' + y = 1 + |1 - x|^{1/2}, \tag{2}$$

for which the general solution in the interval $0 \leqq x < 1$ is

$$y(x) = u(x) \int_0^x (K(s)/u^2(s))ds + cu(x)$$

where $c$ is constant. Thus

$$y'(x) = (1/u(x))[K(x) + u'(x)y(x)] \equiv 3 - c - (1 - x)^{-1/2}$$

and $\lim_{x \to 1-0} y'(x)$ does not exist, so that the differential equation (2) has no solution of class $C'$ defined in the interval $0 \leqq x \leqq 2$.

We now consider the special case of extending a Wronskian matrix of $n$ columns and $n+1$ rows which has rank $n$ throughout an interval $J$.

THEOREM 1. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ $(t \geqq n)$ in an interval $J$ and their Wronskian matrix of order $n$ has rank $n$ at every point of $J$, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that $M_n(u_1, \cdots, u_{n+1})$ has rank $n+1$ throughout $J$.*

PROOF. Since $M_n(u_1, \cdots, u_n)$ has rank $n$ throughout $J$, it follows from Theorem C that $W(u_1, \cdots, u_n)$ has at most a finite number of zeros in $J$. Denote by

$$f_r(x), \qquad r = 0, 1, \cdots, n, \tag{3}$$

the cofactor of $y^{(r)}$ in the determinant $W(u_1, \cdots, u_n, y)$ in which $y$ is unknown. If we prove the existence of a function $K(x)$ which is nonzero and of class $C^{(t-n)}$ in $J$ for which the differential equation

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = K(x) \tag{4}$$

has a solution defined in $J$, we shall have proved our theorem since the coefficients in (4) are of class $C^{(t-n)}$. If $f_n(x)$ does not vanish in $J$, standard existence theorems apply for arbitrary $K(x)$.

Let $f_n(x)$ vanish in $J$. Since $f_n(x) \equiv W(u_1, \cdots, u_n)$, its zeros are finite in number. Assume that the zeros are at the points $c_1, c_2, \cdots, c_m$ with $c_1 < c_2 < \cdots < c_m$. If the $c$'s are not all interior points of $J$, let

$$U_i(x) \equiv \begin{cases} \sum_{j=0}^{t} \frac{1}{j!} u_i^{(j)}(a)(x-a)^j, & x < a, \\ u_i(x), & a \leqq x \leqq b, \\ \sum_{j=0}^{t} \frac{1}{j!} u_i^{(j)}(b)(x-b)^j, & x > b, \end{cases} \qquad i = 1, 2, \cdots, n.$$

Then $W(U_1, \cdots, U_n)$ is identical with $W(u_1, \cdots, u_n)$ in $J$, and is a polynomial for $x$ not in $J$. Therefore $W(U_1, \cdots, U_n)$ has isolated zeros and for some $a^* < a$ and $b^* > b$, the $c$'s are its only zeros in the interval $a^* \leqq x \leqq b^*$. Consequently we assume without loss of generality that $c_1 > a$ and $c_m < b$.

Since $M_n(u_1, \cdots, u_n)$ has rank $n$ throughout $J$, for each $i$, $i = 1, 2, \cdots, m$, at least one function in (3) does not vanish at $c_i$. Denote by $f_{k_i}(x)$ a function from (3) such that $f_{k_i}(c_i) \neq 0$, $i = 1, 2, \cdots, m$. Define

$$q_i(x) \equiv \pm (1/k_i!)(x - c_i)^{k_i}, \qquad i = 1, 2, \cdots, m,$$

where the ambiguous sign is so chosen that each Wronskian $W(u_1, \cdots, u_n, q_i)$, which has the value $\pm f_{k_i}(c_i)$ at $c_i$, has the same sign at $c_i$. Then, for some $\epsilon > 0$ with $a < c_1 - \epsilon < c_1 + \epsilon < c_2 - \epsilon < \cdots < c_m + \epsilon < b$,

$$Q_i(x) \equiv W(u_1, \cdots, u_n, q_i) \neq 0, c_i - \epsilon \leqq x \leqq c_i + \epsilon, i = 1, 2, \cdots, m,$$

and $q_i(x)$ is a solution of the differential equation

$$W(u_1, \cdots, u_n, y) = Q_i(x), \qquad i = 1, 2, \cdots, m.$$

Denote the interval

$$(5) \qquad c_{k-1}+\epsilon \leqq x \leqq c_k-\epsilon, \quad k=1,2,\cdots,m+1,$$

where $c_0+\epsilon=a$ and $c_{m+1}-\epsilon=b$, by $J_k$. Let $P_k(x)$ be any function (Lemma 2) which is nonzero and of class $C^{(t-n)}$ in $J_k$ and satisfies the conditions

$$(6) \qquad \begin{aligned} P_k^{(j)}(c_{k-1}+\epsilon+0) &= Q_{k-1}^{(j)}(c_{k-1}+\epsilon-0), \\ P_k^{(j)}(c_k-\epsilon-0) &= Q_k^{(j)}(c_k-\epsilon+0), \end{aligned}$$

for $j=0, 1, \cdots, t-n$, where

$$\begin{aligned} Q_0^{(j)}(c_0+\epsilon-0) &= Q_1^{(j)}(c_1-\epsilon+0), \\ Q_{m+1}^{(j)}(c_{m+1}-\epsilon+0) &= Q_m^{(j)}(c_m+\epsilon-0). \end{aligned}$$

Then each function

$$K_k(x) \equiv \begin{cases} P_k(x), & c_{k-1}+\epsilon \leqq x \leqq c_k-\epsilon, \\ Q_k(x), & c_k-\epsilon \leqq x \leqq c_k+\epsilon, \end{cases} \qquad k=1,2,\cdots,m+1,$$

where $c_{m+1}+\epsilon=b$, is nonzero and of class $C^{(t-n)}$ in its interval of definition by the second set of conditions in (6). Thus the function $K(x)$ defined in $J$ by

$$(7) \qquad K(x) \equiv K_k(x), \quad c_{k-1}+\epsilon \leqq x \leqq c_k+\epsilon, \quad k=1,2,\cdots,m+1,$$

is nonzero and of class $C^{(t-n)}$ in $J$ by the first set of conditions in (6).

Since $f_n(x)\neq 0$ in $J_1$, the differential equation $W(u_1, \cdots, u_n, y) = P_1(x)$ has a solution $p_1(x)$ which satisfies the initial conditions

$$(8) \qquad p_1^{(j)}(c_1-\epsilon-0) = q_1^{(j)}(c_1-\epsilon+0), \qquad j=0, 1, \cdots, n-1.$$

It follows from the second set of conditions in (6) that (8) also holds for $j=n, n+1, \cdots, t$. The function

$$y_1(x) \equiv \begin{cases} p_1(x), & a \leqq x \leqq c_1-\epsilon, \\ q_1(x), & c_1-\epsilon \leqq x \leqq c_1+\epsilon, \end{cases}$$

is therefore a solution of the differential equation $W(u_1, \cdots, u_n, y) = K_1(x)$.

In $J_2$ the differential equation

$$(9) \qquad W(u_1, \cdots, u_n, y) = P_2(x)$$

has a solution $p_2(x)$ with $p_2^{(j)}(c_1+\epsilon+0) = y_1^{(j)}(c_1+\epsilon-0)$, $j=0, 1, \cdots, t$, and also a solution $r(x)$ with $r^{(j)}(c_2-\epsilon-0)=q_2^{(j)}(c_2-\epsilon+0)$, $j=0, 1, \cdots, t$. Now the difference of any two solutions of (9) is a solution of the corresponding homogeneous equation for which the $u$'s are linearly independent solutions. Thus, for some set of constants $a_1, a_2, \cdots, a_n$,

$$p_2(x) = r(x) + a_1u_1(x) + \cdots + a_nu_n(x), \quad c_1 + \epsilon \leqq x \leqq c_2 - \epsilon.$$

Then the function

$$y_2(x) = \begin{cases} p_2(x), & c_1 + \epsilon \leqq x \leqq c_2 - \epsilon, \\ q_2(x) + a_1u_1(x) + \cdots + a_nu_n(x), & c_2 - \epsilon \leqq x \leqq c_2 + \epsilon, \end{cases}$$

is of class $C^{(t)}$ in its interval of definition with

$$y_2^{(j)}(c_1 + \epsilon + 0) = y_1^{(j)}(c_1 + \epsilon - 0), \qquad j = 0, 1, \cdots, t,$$

and is a solution of the differential equation $W(u_1, \cdots, u_n, y) = K_2(x)$.

We continue in the above manner, obtaining $y_k(x)$, $c_{k-1}+\epsilon \leqq x \leqq c_k + \epsilon$ $(k=2, 3, \cdots, m+1)$ of class $C^{(t)}$ in its interval of definition with

$$y_k^{(j)}(c_{k-1} + \epsilon + 0) = y_{k-1}^{(j)}(c_{k-1} + \epsilon - 0), \qquad j = 0, 1, \cdots, t,$$

which is a solution of the differential equation $W(u_1, \cdots, u_n, y) = K_k(x)$. Then the function $u_{n+1}(x)$ defined in $J$ by

$$u_{n+1}(x) = y_k(x), \quad c_{k-1} + \epsilon \leqq x \leqq c_k + \epsilon, \quad k = 1, 2, \cdots, m + 1,$$

is of class $C^{(t)}$ in $J$ and is a solution of (4) with $K(x)$ defined by (7).

In the following theorem we consider the general case of extending a Wronskian matrix.

THEOREM 2. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ $(t \geqq n)$ in an interval $J$ and their Wronskian matrix of order $s$ $(n \leqq s \leqq t)$ has rank $n$ at every point of $J$, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that $M_s(u_1, \cdots, u_{n+1})$ has rank $n+1$ throughout $J$.*

PROOF. The case $s=n$ is treated in Theorem 1. Assume now that $M_n(u_1, \cdots, u_n)$ does not have rank $n$ throughout $J$. Since $M_s(u_1, \cdots, u_n)$ has rank $n$ at every point, it follows from Theorem C that $W(u_1, \cdots, u_n)$ has isolated zeros. Define $f_r(x)$, $r=0, 1, \cdots, n$, as in (3). Since the $f$'s are the $n$-rowed minor determinants of $M_n(u_1, \cdots, u_n)$ they have at least one zero in common, but only a finite number for $f_n(x) = W(u_1, \cdots, u_n)$. Assume that the common zeros of the $f$'s are at the points $c_1, c_2, \cdots, c_m$ with $c_1 < c_2 < \cdots < c_m$. There is no loss of generality to assume that $c_1 > a$ and $c_m < b$.

For each $i$, $i=1, 2, \cdots, m$, some $n$-rowed determinant of $M_s(u_1, \cdots, u_n)$, which we shall denote by $\Delta_i$, does not vanish at $c_i$. If $u(x)$ is a function of class $C^{(t)}$ in $J$ and row $r$ of $M_s(u_1, \cdots, u_n)$ is not a row of $\Delta_i$, denote by $\Delta_{i,r}$ the determinant of $M_s(u_1, \cdots, u_n, u)$ which consists of row $r$ and the rows represented in $\Delta_i$.

Let row $r_i$ be the first row of $M_s(u_1, \cdots, u_n)$ which is not a row of $\Delta_i$. Define

$$q_i(x) = (1/(r_i - 1)!)(x - c_i)^{r_i - 1}, \qquad i = 1, 2, \cdots, m.$$

Then at $c_i$ the determinant $\Delta_{i,r_i}$ of $M_s(u_1, \cdots, u_n, q_i)$ has the value $(-1)^{n+1+r_i}\Delta_i(c_i)$, hence for some $\delta>0$ with $a<c_1-\delta<c_1+\delta<c_2-\delta < \cdots < c_m+\delta<b$,

$$\Delta_{i,r_i} \neq 0,\ c_i - \delta \leq x \leq c_i + \delta, \qquad i = 1, 2, \cdots, m. \tag{10}$$

Since $M_s(u_1, \cdots, u_n, q_i)$ has rank $n+1$ throughout the interval $c_i-\delta \leq x \leq c_i+\delta$ by (10), $W(u_1, \cdots, u_n, q_i)$ has isolated zeros in this interval. Thus, for some $\epsilon>0$ with $\epsilon \leq \delta$,

$$Q_i(x) \equiv W(u_1, \cdots, u_n, q_i) \neq 0 \text{ at } x = c_i \pm \epsilon,\ i = 1, 2, \cdots, m. \tag{11}$$

We shall assume that $Q_{i+1}(c_{i+1}-\epsilon)$, $i=1, 2, \cdots, m-1$, is positive or negative according as $Q_i(c_i+\epsilon)$ is positive or negative, since $q_{i+1}(x)$ could be replaced by $-q_{i+1}(x)$ and the expressions corresponding to (10) and (11) would hold.

Define the interval $J_k$ as in (5). Since $f_0(x), f_1(x), \cdots, f_n(x)$ have no common zero in $J_k$, $M_n(u_1, \cdots, u_n)$ has rank $n$ at every point of $J_k$. Thus, by the method employed in the proof of Theorem 1, we can define a function $P_k(x)$, $k=1, 2, \cdots, m+1$, which is nonzero and of class $C^{(t-n)}$ in $J_k$, which satisfies the conditions

$$\begin{aligned} P_k^{(j)}(c_{k-1} + \epsilon + 0) &= Q_{k-1}^{(j)}(c_{k-1} + \epsilon - 0), \\ P_k^{(j)}(c_k - \epsilon - 0) &= Q_k^{(j)}(c_k - \epsilon + 0) \end{aligned} \tag{12}$$

for $j=0, 1, \cdots, t-n$, where

$$\begin{aligned} Q_0^{(j)}(c_0 + \epsilon - 0) &= Q_1^{(j)}(c_1 - \epsilon + 0), \\ Q_{m+1}^{(j)}(c_{m+1} - \epsilon + 0) &= Q_m^{(j)}(c_m + \epsilon - 0), \end{aligned}$$

and for which the differential equation $W(u_1, \cdots, u_n, y)=P_k(x)$, $k=1, 2, \cdots, m+1$, has a solution of class $C^{(t)}$ defined in $J_k$.

In $J_1$ the differential equation

$$W(u_1, \cdots, u_n, y) = P_1(x) \tag{13}$$

has a solution $p_1(x)$ with $p_1^{(j)}(c_1-\epsilon-0)=q_1^{(j)}(c_1-\epsilon+0)$, $j=0, 1, \cdots, t$, in view of the second set of conditions in (12). Then the function

$$y_1(x)=\begin{cases} p_1(x), & a\leqq x\leqq c_1-\epsilon, \\ q_1(x), & c_1-\epsilon\leqq x\leqq c_1+\epsilon, \end{cases}$$

is of class $C^{(t)}$, and $M_t(u_1, \cdots, u_n, y_1)$ has rank $n+1$ at every point of the interval $a\leqq x\leqq c_1+\epsilon$ since $p_1(x)$ is a solution of (13) and the determinant $\Delta_{1,r_1}$ does not vanish in the interval $c_1-\epsilon\leqq x\leqq c_1+\epsilon$ by (10).

In $J_2$ the differential equation

$$W(u_1, \cdots, u_n, y) = P_2(x) \tag{14}$$

has a solution $p_2(x)$ with $p_2^{(j)}(c_1+\epsilon+0)=y_1^{(j)}(c_1+\epsilon-0)$, $j=0, 1, \cdots, t$, and also a solution $r(x)$ with $r^{(j)}(c_2-\epsilon-0)=q_2^{(j)}(c_2-\epsilon+0)$, $j=0, 1, \cdots, t$. Since $u_1(x), u_2(x), \cdots, u_n(x)$ are linearly independent solutions of the homogeneous equation corresponding to (14), for some set of constants $a_1, a_2, \cdots, a_n$,

$$p_2(x) = r(x) + a_1u_1(x) + \cdots + a_nu_n(x), \quad c_1+\epsilon\leqq x\leqq c_2-\epsilon.$$

Then the function

$$y_2(x)=\begin{cases} p_2(x), & c_1+\epsilon\leqq x\leqq c_2-\epsilon, \\ q_2(x)+a_1u_1(x)+\cdots+a_nu_n(x), & c_2-\epsilon\leqq x\leqq c_2+\epsilon, \end{cases}$$

is of class $C^{(t)}$ in its interval of definition with

$$y_2^{(j)}(c_1+\epsilon+0) = y_1^{(j)}(c_1+\epsilon-0), \quad j=0, 1, \cdots, t,$$

and $M_t(u_1, \cdots, u_n, y_2)$ has rank $n+1$ throughout the interval $c_1+\epsilon\leqq x\leqq c_2+\epsilon$, since $p_2(x)$ is a solution of (14) and the determinant $\Delta_{2,r_2}$ does not vanish in the interval $c_2-\epsilon\leqq x\leqq c_2+\epsilon$ by (10).

We continue in the above manner, obtaining $y_k(x)$, $c_{k-1}+\epsilon\leqq x\leqq c_k+\epsilon$ $(k=2, 3, \cdots, m+1$, where $c_{m+1}+\epsilon=b)$ of class $C^{(t)}$ in its interval of definition with

$$y_k^{(j)}(c_{k-1}+\epsilon+0) = y_{k-1}^{(j)}(c_{k-1}+\epsilon-0), \quad j=0, 1, \cdots, t,$$

and with each matrix $M_t(u_1, \cdots, u_n, y_k)$ of rank $n+1$ throughout the interval $c_{k-1}+\epsilon\leqq x\leqq c_k+\epsilon$. Then the function $u_{n+1}(x)$ defined in $J$ by

$$u_{n+1}(x) = y_k(x), \quad c_{k-1}+\epsilon\leqq x\leqq c_k+\epsilon, \quad k=1, 2, \cdots, m+1,$$

is of class $C^{(t)}$ in $J$ and the Wronskian matrix $M_t(u_1, \cdots, u_{n+1})$ has rank $n+1$ throughout $J$.

**4. Application to the theory of differential equations.** The extension property of Wronskian matrices leads us to the following sufficient condition that a set of $n$ given functions be solutions of an ordinary homogeneous linear differential equation.

THEOREM 3. *If* $u_1(x)$, $u_2(x)$, $\cdots$, $u_n(x)$ *are functions of class* $C^{(t)}$ $(t \geqq n)$ *in an interval* $J$ *and their Wronskian matrix of order* $s$ $(s<t)$ *has rank* $n$ *at every point of* $J$, *then the u's are linearly independent solutions of a homogeneous linear differential equation of order* $s+1$ *of the type*

$$y^{(s+1)} + f_s(x)y^{(s)} + \cdots + f_1(x)y' + f_0(x)y = 0 \tag{15}$$

*in which the* $f_i(x)$ *are functions of class* $C^{(t-s-1)}$ *in* $J$.

PROOF. Let $m=s-n+1$. The case $m=0$ is well known and $m<0$ is impossible. Assume now that $m>0$. Then $n \leqq s < t$ and, by $m$ successive applications of Theorem 2, there exist $m$ functions, $u_{n+1}(x)$, $u_{n+2}(x)$, $\cdots$, $u_{n+m}(x)$, of class $C^{(t)}$ in $J$ such that the Wronskian matrices

$$M_s(u_1, \cdots, u_{n+1}),\ M_s(u_1, \cdots, u_{n+2}),\ \cdots,\ M_s(u_1, \cdots, u_{n+m}) \tag{16}$$

have the respective ranks $n+1$, $n+2$, $\cdots$, $n+m$ at every point of $J$. Since $n+m=s+1$, the last matrix in (16) is the Wronskian matrix of order $s$ for $s+1$ functions of class $C^{(t)}$ $(t>s)$ with $W(u_1, \cdots, u_{s+1}) \neq 0$ throughout $J$. Therefore $u_1(x)$, $u_2(x)$, $\cdots$, $u_{s+1}(x)$ constitute a fundamental system of solutions of a homogeneous equation of type (15) with the $f_i(x)$ of class $C^{(t-s-1)}$ in $J$. Hence the $n$ given functions have the asserted property.

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# ON HARMONIC AND ANALYTIC FUNCTIONS

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If we study the behavior of a harmonic function on the boundary of the unit circle along an arc $\alpha<\theta<\beta$, it is sometimes of advantage, if the function behaves in the simplest possible way outside this arc. This problem of *isolating* the singular arc can easily be solved for a harmonic function which is bounded in the unit circle. For such a function can be expressed by means of a Poisson integral

$$u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\vartheta)+r^2}\, u(1, \vartheta)d\vartheta,$$

and then

$$v(r, \theta) = \frac{1}{2\pi}\int_{\alpha}^{\beta} \frac{1-r^2}{1-2r\cos(\theta-\vartheta)+r^2}\, u(1, \vartheta)d\vartheta$$

is known *to behave in the same way* as $u(r, \theta)$ in the neighborhood of the arc $(\alpha, \beta)$—in fact the difference of the two functions tends uniformly to zero inside the arc—and $v(r, \theta)$ can be extended so as to make it harmonic and equal to zero on the rest of the circumference.

It is equally easy to solve the problem for a harmonic function $u(r, \theta)$ which is $O(1/(1-r)^n)$ near the circumference and is, therefore, the $(n+2)$nd derivative of a harmonic function, bounded in $r\leqq 1$.

The purpose of this paper is to show that the problem can be solved for any function harmonic in $r<1$. The result can be generalized to any domain which can be represented conformally on the unit circle.

THEOREM. *If $u(r, \theta)$ is a function, harmonic in the unit circle, then, given the arc $(r=1,\ \alpha<\theta<\beta)$, there is a function $v(r, \theta)$ harmonic in $r<1$, such that $u(r, \theta)-v(r, \theta)$ can be extended across the arc $(\alpha, \beta)$ so as to make it harmonic and zero on the arc; and $v(r, \theta)$ is harmonic and zero along the rest of the circumference.*

PROOF. (i) If

$$u(r, \theta) = \sum_{1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta),$$

we can find a nonincreasing sequence $\epsilon_n \to 0$, such that

$$|a_n|,\ |b_n| \leqq ((1/2)e^{\epsilon_n n} - 2)/n^2. \tag{1}$$

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We have indeed

$$r^n \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} d\theta.$$

If we put $r = e^{-\epsilon}$, we find that whatever $\epsilon > 0$ we select, $|a^n|$, $|b^n| \leqq U e^{\epsilon n}$ for all $n$. Hence

$$\limsup \frac{\log |a_n|}{n} \leqq 0, \qquad \limsup \frac{\log |b_n|}{n} \leqq 0$$

and, therefore,

$$\epsilon_n = \max_{k=n,n-1,\cdots} \left[0,\ k^{-1}\log(2|a_k|\cdot k^2 + 4),\ k^{-1}\log(2|b_k|\cdot k^2 + 4)\right]$$

will satisfy our requirements.

(ii) There exists an entire function $\phi(z)$ such that $\phi(z) = O(e^{\epsilon|z|})$ for any $\epsilon > 0$ and sufficiently large $|z|$, $\overline{\phi(z)} = \phi(\bar{z})$, $\phi(z) = \phi(-z)$, $\phi(0) = 0$ and

$$(2) \qquad (|a_n| + |b_n|)/\phi(n) \leqq 2/n^2,$$

for all $n$ so large that $2\epsilon_n \leqq 1$. We define $m_n$ as the least integer such that $\sum_{k>m_n} n^{2k}/(2k)! \leqq 1$. Then

$$\phi(z) = \sum_1^{m_1} \frac{(\epsilon_1 z)^{2k}}{(2k)!} + \sum_{m_1+1}^{m_2} \frac{(\epsilon_2 z)^{2k}}{(2k)!} + \cdots = \sum_{k=1}^{\infty} \frac{(\delta_k z)^{2k}}{(2k)!}.$$

This function satisfies our requirements. Evidently, it is sufficient to show that (2) is satisfied.

Since

$$\sum_{k>m_n}^{\infty} \frac{(\delta_k n)^{2k}}{(2k)!} \leqq \epsilon_{n+1}^{2m_n} \sum_{k>m_n}^{\infty} \frac{n^{2k}}{(2k!)} \leqq \epsilon_{n+1}^{2m_n},$$

we have

$$\phi(n) \geqq \sum_1^{m_n} \frac{(\delta_k n)^{2k}}{(2k)!} - \epsilon_n^{2m_n} \geqq \sum_1^{\infty} \frac{(\epsilon_n n)^{2k}}{(2k)!} - 2\epsilon_n^{2m_n}$$
$$= (1/2)(e^{\epsilon_n n} + e^{-\epsilon_n n}) - 1 - 2\epsilon_n^{2m_n},$$

and, for $2\epsilon_n \leqq 1$, this yields, by (1),

$$|a_n| n^2, \ |b| n^2 \leqq \phi(n).$$

(iii) From (2) it follows that

$$U(r, \theta) = \sum_1^{\infty} r^n \frac{a_n \cos n\theta + b_n \sin n\theta}{\phi(n)}$$

is a harmonic function, continuous in the closed unit circle. Hence we can express it by means of a Poisson integral

$$\begin{aligned} U(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} U(1,\vartheta)\frac{(1-r^2)d\vartheta}{1-2r\cos(\vartheta-\theta)+r^2} \\ &= \frac{1}{2\pi}\left(\int_\alpha^\beta + \int_{(0,2\pi)-(\alpha,\beta)}\right)U(1,\vartheta)\,\frac{(1-r^2)d\vartheta}{1-2r\cos(\vartheta-\theta)+r^2} \\ &= V(r,\theta)+W(r,\theta). \end{aligned}$$

The harmonic function $V(r, \theta)$ expressed by the first integral can be extended across the arc complementary to $(\alpha, \beta)$, so that it becomes harmonic and zero on the arc. For $W(r,\theta)$ the same is true with respect to the arc $(\alpha, \beta)$ itself. We see that the function $V(r,\theta)$ bears to $U(r, \theta)$ the same relation as the function $v(r, \theta)$, required by the theorem, should bear to $u(r, \theta)$.

(iv) Wigert's theorem asserts that if $\phi(z)$ is an entire function satisfying the first condition in (ii), then

$$f(z) = \sum_0^\infty \phi(n)z^n$$

is a function having its only singular point at $z=1$. We get easily

$$f(e^w) = \sum_{k=1}^\infty \frac{(\delta_k)^{2k}}{(2k)!}\,\frac{d^{2k}}{dw^{2k}}\left(\frac{1}{1-e^w}\right).$$

In a proof of the above theorem, given by Bieberbach,[1] it is shown that this last series converges uniformly in a neighborhood of any point different from $w=0$. Therefore, we have

$$\begin{aligned} f(e^{-w}) &= \sum_1^\infty \frac{\delta_k^{2k}}{(2k)!}\,\frac{d^{2k}}{dw^{2k}}\left(\frac{1}{1-e^{-w}}\right) = -\sum_1^\infty \frac{\delta_k^{2k}}{(2k)!}\,\frac{d^{2k}}{dw^{2k}}\left(\frac{1}{1-e^{w}}\right) \\ &= -f(e^w). \end{aligned}$$

For $w=i\theta$, we get $(1/2)(f(e^{i\theta})+f(e^{-i\theta}))=R[f(e^{i\theta})]=0$. Hence $K(r, \theta)$ $=\sum_1^\infty r^n\phi(n)\cos n\theta$ is a function, harmonic in the whole plane except at $r=1$, $\theta=0$, and is zero for $r=1$, $\theta\neq 0$.

(v) Using Parseval's identity, we find without difficulty that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K(r,\theta-x)U(1,x)dx = \sum r^n(a_n\cos n\theta + b_n\sin n\theta) = u(r,\theta).$$

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[1] *Lehrbuch der Funktionentheorie*, vol. 2, pp. 288–292.

Further

$$v(r, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} K(r, \theta - x)V(1, x)dx = \frac{1}{\pi}\int_{\alpha}^{\beta} K(r, \theta - x)V(1, x)dx$$

is clearly a function, harmonic in $r<1$ and it is easy to deduce from the properties of $K(r, \theta)$ and $V(1, \theta)$ that $v(r, \theta)$ satisfies all the conditions of the theorem.

COROLLARY. *If $f(z)$ is analytic in $|z|<1$, then, given the arc ($|z|=1$, $\alpha<\arg z<\beta$), there is a function $g(z)$ analytic in $|z|<1$ and on the arc of $|z|=1$ complementary to $(\alpha, \beta)$, such that $f(z)-g(z)$ can be extended analytically across $(\alpha, \beta)$.*

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# ON THE COMPLEX ZEROS OF THE BESSEL FUNCTIONS

E. HILLE AND G. SZEGÖ

1. **Introduction.** Various proofs have been given for the following classical theorem of A. Hurwitz:

THEOREM 1. *The entire function*

$$z^{\beta/2}J_{-\beta}(2z^{1/2}) = \sum_{m=0}^{\infty} \frac{(-z)^m}{m!}\,\frac{1}{\Gamma(m+1-\beta)} \tag{1.1}$$

*has precisely $[\beta]$ nonpositive zeros. Here $J_{-\beta}$ is the Bessel function of order $-\beta$ and $\beta \geqq 0$.*

In case $\beta$ is an integer these nonpositive zeros are all at the origin; in case $\beta$ is not an integer and $[\beta]$ is odd there is precisely one negative zero and we have $\frac{1}{2}([\beta]-1)$ pairs of conjugate complex zeros; in case $\beta$ is not an integer and $[\beta]$ is even there are $\frac{1}{2}[\beta]$ pairs of conjugate complex zeros.

Most of the proofs for this theorem (see the papers [2, 4, 6, 7, 9] of the Bibliography at the end of the text) make use of polynomial approximations of the Bessel function. The present proof follows the same line by obtaining the Bessel function as the limit of *Laguerre polynomials*. The study of the complex zeros of these polynomials is

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particularly simple and for this reason the present arrangement may have certain advantages.[1] Our proof yields also certain bounds for the nonpositive zeros of the function (1.1) in terms of $\beta$.

In §2 we give a compilation of the definitions and results on Laguerre polynomials which are essential for our proof. We follow the notation of G. Szegö, *Orthogonal polynomials* (Amer. Math. Soc. Colloquium Publications, vol. 23, New York, 1939; quoted as Sz.).

§3 contains certain bounds for the nonpositive zeros of Laguerre polynomials from which Theorem 1 follows immediately.

§4 gives bounds for the nonpositive zeros of the function (1.1).

In §3 we use as an important instrument an elementary formula involving a solution of a second order linear differential equation which was employed for the study of the zeros of various functions by A. Hurwitz and E. Hille [5, 3].

2. **Laguerre polynomials.** We define the Laguerre polynomials by [Sz. (5.1.6)]

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} C_{n+\alpha,n-m} \frac{(-x)^m}{m!}. \tag{2.1}$$

This is equivalent to the formula [Sz. (5.1.5)]

$$e^{-x}x^{\alpha}L_n^{(\alpha)}(x) = (n!)^{-1}(d/dx)^n(e^{-x}x^{n+\alpha}). \tag{2.2}$$

It is easy to verify the differential equation [Sz. (5.1.2)]

$$\begin{cases} w'' + G(x)w = 0, & w = w(x) = e^{-x/2}x^{(\alpha+1)/2}L_n^{(\alpha)}(x), \\ G(x) = Ax^{-1} - Bx^{-2} - 1/4, & A = n + (\alpha+1)/2, \\ & B = (\alpha^2 - 1)/4. \end{cases} \tag{2.3}$$

Let $x = u + iv \neq 0$; we note that

$$\Im\{G(x)\} = v\{-A(u^2+v^2)^{-1} + 2Bu(u^2+v^2)^{-2}\}. \tag{2.4}$$

Finally we conclude from (2.1) by comparing the corresponding powers of $x$ [Sz. (5.1.14)]

$$(d/dx)L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x). \tag{2.5}$$

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[1] The proof given by Pólya [7] deduces Theorem 1 from certain theorems of a general character. It may be of interest to observe that his theorem on the so-called Jensen polynomials [7, p. 165] furnishes the result of Hurwitz; indeed, the Jensen polynomials associated with the function (1.1) are (except for a factor independent of $x$) the Laguerre polynomials $L_n^{(-\beta)}(x)$.

LEMMA 1. *Let $n \geqq 1$, $\alpha$ an arbitrary real number, $\alpha \neq -1, -2, \cdots, -n$. Then the zeros of $L_n^{(\alpha)}(x)$ are not equal to* 0 *and distinct. Furthermore the number of the positive zeros is $n$ if $\alpha > -1$; it is $n + [\alpha] + 1$ if $-n < \alpha < -1$; it is* 0 *if $\alpha < -n$. The number of the negative zeros is* 0 *or* 1. [Sz., Theorem 6.73.]

Let $\alpha = -\beta$ be negative, $\beta$ not an integer. If $n > \beta$ the polynomial $L_n^{(-\beta)}(x)$ has $[\beta]$ nonpositive zeros which are all complex, except a single negative one in case $[\beta]$ is odd.

This theorem is due to E. Stridsberg [8] and W. Hahn [1]. It follows from older results on Jacobi polynomials by a limiting process [Sz. loc. cit.]. A continuity proof based on the variation of $\beta$ is also possible.

For the sake of completeness we give here the following very simple proof of Lemma 1.

From (2.1) and (2.3) we conclude that all the zeros are not equal to 0 and distinct. In case $\alpha > -1$ Rolle's theorem applied to formula (2.2) furnishes exactly $n$ positive zeros. Further let $-k-1 < \alpha < -k$ $(1 \leqq k \leqq n)$. The same theorem furnishes then at least $n-k$ positive zeros, and by Descartes' rule of signs we obtain exactly this number of positive zeros. Inspection of (2.1) shows that for $k \geqq n$ no positive zeros exist. Now let $x$ be negative and $\alpha < -1$. Then[3] $G(x) < 0$ so that $w$ and $w''$ are of the same sign. This excludes the possibility of two negative zeros. Hence the number of the negative zeros is 0 or 1.

LEMMA 2. *Let $\alpha$ be an arbitrary real number. Then*

$$\lim_{n \to \infty} n^{-\alpha} L_n^{(\alpha)}(z/n) = z^{-\alpha/2} J_\alpha(2z^{1/2}) \tag{2.6}$$

*uniformly if $z$ is bounded.* [Sz. Theorem 8.1.3.]

This follows immediately from (2.1).

In view of the limit formula (2.6) Hurwitz's theorem results, provided we can show that the nonpositive zeros of $L_n^{(\alpha)}(x)$ are in absolute value not greater than $Cn^{-1}$, where $C = C(\alpha)$ is independent of $n$. (Indeed, the zeros of the functions on the left-hand side of (2.6) tend to those of the function on the right-hand side, as $n \to \infty$.) The only possibility which has to be excluded is the *confluence* of two conjugate complex zeros of $L_n^{(\alpha)}(x)$ into a real zero of $z^{-\alpha/2} J_\alpha(2z^{1/2})$, which would be necessarily multiple and not equal to 0. But this is out of the question since this function as a solution of a second order linear differential equation with the only finite singularity $z=0$ can not have a multiple zero different from 0.

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[3] $G(x) \leqq (\alpha+1)/2x - (\alpha^2-1)/4x^2 - 1/4 \leqq (1/2)/(\alpha-1) < 0$.

3. **Bounds for the nonpositive zeros of $L_n^{(\alpha)}(x)$.** In what follows we use the notation of §2.

LEMMA 3. *Let $\alpha=-\beta$ be negative, $\beta$ not an integer and $[\beta]$ odd. Then for the only negative zero $x_0$ of $L_n^{(-\beta)}(x)$ we have*

$$x_0 > -\beta^2/4n \tag{3.1}$$

*provided $n$ is sufficiently large.*

According to (2.5) $L_n^{(-\beta)}(x)$ is monotonic for negative $x$; furthermore $L_n^{(-\beta)}(0)<0$. Thus $x_0>-cn^{-1}$, $c>0$, provided $L_n^{(-\beta)}(-cn^{-1})>0$. Since, $n>[\beta]$,

$$\begin{aligned} L_n^{(-\beta)}(-cn^{-1}) &= \sum_{m=0}^{[\beta]} C_{n-\beta,n-m}\frac{(cn^{-1})^m}{m!} + \sum_{m=[\beta]+1}^{n} C_{n-\beta,n-m}\frac{(cn^{-1})^m}{m!} \\ &> \sum_{m=0}^{[\beta]} C_{n-\beta,n-m}\frac{(cn^{-1})^m}{m!} \end{aligned} \tag{3.2}$$

and

$$\lim_{n\to\infty} n^{\beta}\sum_{m=0}^{[\beta]} C_{n-\beta,n-m}\frac{(cn^{-1})^m}{m!} = \sum_{m=0}^{[\beta]}\frac{c^m}{m!}\,\frac{1}{\Gamma(m+1-\beta)}, \tag{3.3}$$

we have $L_n^{(-\beta)}(-cn^{-1})>0$ for sufficiently large $n$ provided the expression on the right-hand side of (3.3) is positive. This is the case if $c$ is chosen sufficiently large.

It is easy to specify the constant $c$. The above expression is certainly positive provided

$$c > m(\beta - m), \qquad m = 1, 2, 3, \cdots, [\beta]. \tag{3.4}$$

We can take, for instance,

$$c = \beta^2/4. \tag{3.5}$$

The formula of A. Hurwitz and E. Hille mentioned in the introduction applied to (2.3) will read as follows:

LEMMA 4. *Let $a$ and $b$ be arbitrary complex numbers, $a\neq b$, $a\neq 0$, $b\neq 0$. Then*

$$[\bar{w}w']_a^b - \int |w'|^2 d\bar{x} + \int G(x)|w|^2 dx = 0; \tag{3.6}$$

*the integration is extended over an arbitrary rectifiable curve in the com-*

*plex x-plane joining a and b and avoiding the origin.*

This furnishes immediately a bound for the imaginary parts of the zeros of $L_n^{(\alpha)}(x)$. More exactly we prove:

LEMMA 5. *Let* $\alpha = -\beta$ *be negative,* $\beta$ *not an integer,* $n > \beta > 1$. *For any complex zero* $p+iq$ *of* $L_n^{(-\beta)}(x)$ *we have*

$$|q| \leqq B/A. \tag{3.7}$$

Let $q \neq 0$; we apply (3.6), integrating from $p+iq$ to $\infty$ along a straight line parallel to the real axis with $\Re x \to +\infty$. Then, noting that $\overline{w(x)}w'(x) \to 0$ when $x = u+iq$, $u \to +\infty$,

$$-\int_{p+iq}^{\infty} |w'|^2 dx + \int_{p+iq}^{\infty} G(x)\,|w|^2 dx = 0$$

follows, so that $\Im\{G(x)\}$ must vanish somewhere along this path. Thus we must have $-A(u^2+v^2)+2Bu=0$ for a suitable $u$, $u \geqq p$, $v = q$. This is the equation of a circle with center at $(B/A,\ 0)$ and radius $B/A$. From this (3.7) follows.

LEMMA 6. *Let* $\alpha = -\beta$ *be negative,* $\beta$ *not an integer,* $n > \beta > 1$. *Then for the complex zeros* $p+iq$ *of* $L_n^{(-\beta)}(x)$ *we have*

$$|p| \leqq \begin{cases} 2B/A & \text{if } p > 0, \\ |x_0| & \text{if } p < 0, \end{cases} \tag{3.8}$$

*where* $x_0$ *is the only negative zero of* $L_n^{(-\beta)}(x)$ *or* $L_{n+1}^{(-\beta-1)}(x)$ *depending upon whether* $[\beta]$ *is odd or even.*

The previous argument furnishes the first part of the assertion. As to the second part, we assume that $[\beta]$ is odd; the case in which $[\beta]$ is even can be settled by means of Gauss' theorem on account of the relation

$$(d/dx)L_{n+1}^{(-\beta-1)}(x) = -L_n^{(-\beta)}(x).$$

Let us assume that $p < x_0$, $q > 0$. We apply (3.6) with $a = p$, $b = p+iq$ choosing for the path of integration a vertical segment. Then

$$-\overline{w(p)}w'(p) - \int_p^{p+iq} |w'|^2\overline{dx} + \int_p^{p+iq} G(x)\,|w|^2 dx = 0 \tag{3.9}$$

follows. We write

$$\mathfrak{w}(x) = e^{-x/2}(-x)^{(\alpha+1)/2}L_n^{(\alpha)}(x) \tag{3.10}$$

and take the real parts of the terms in (3.9):

$$(3.11)\quad -\{\mathfrak{w}(p)\}^2\frac{\mathfrak{w}'(p)}{\mathfrak{w}(p)} - \int_p^{p+iq}\{\Im G(x)\}\,|\mathfrak{w}(x)|^2\,|dx| = 0.$$

Now $\mathfrak{w}(x)$ satisfies the differential equation (2.3) and $G(x) < 0$ for $x < 0$. Hence $\mathfrak{w}''(x)$ and $\mathfrak{w}(x)$ are of the same sign, that is $\mathfrak{w}''(x) > 0$ for $x < x_0$. Thus $\mathfrak{w}'(x)$ is increasing for $x < x_0$ and since $\mathfrak{w}'(x_0) = e^{-x_0/2}(-x_0)^{(\alpha+1)/2}L_n^{(\alpha)\prime}(x_0) < 0$ we have $\mathfrak{w}'(x) < 0$ for $x < x_0$. Consequently the first term of (3.11) is positive. But according to (2.4) $\Im\{G(x)\} < 0$ so that (3.11) involves a contradiction.

4. **Conclusion.** From the Lemmas 3, 5 and 6 we deduce Theorem 1 in a familiar way (cf. the end of §2). By using (3.1), (3.7) and (3.8) we find more precisely:

THEOREM 2. *Let $\alpha = -\beta$ be negative, $\beta$ not an integer, $\beta > 1$. The non-positive zeros of the function $z^{\beta/2}J_{-\beta}(2z^{1/2})$ are situated in the rectangle*

$$(4.1)\quad \left.\begin{matrix} -\beta^2/4 \\ -(\beta+1)^2/4 \end{matrix}\right\} \leqq \Re z \leqq (\beta^2-1)/2; \quad |\Im z| \leqq (\beta^2-1)/4.$$

*In the first inequality the upper or lower expression holds depending on whether $[\beta]$ is odd or even.*

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# ON LINEAR SPACES WHICH MAY BE RENDERED COMPLETE NORMED METRIC SPACES

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In this paper, we obtain a characterization of linear spaces which may be normed so as to become complete, linear, normed metric spaces. In this connection, K. Kunugui[1] and M. Fréchet[2] have shown that every metric space $S$ is isometric with a subset of a complete, linear, normed metric space. It follows from our result that if the cardinal number of $S$ is the limit of a denumerable sequence of cardinals, then there is no complete, linear, normed metric space isometric with $S$. Results on topological spaces which may be rendered linear, normed metric spaces and complete, linear, normed metric spaces have been given by A. Kolmogoroff[3] and B. Z. Vulich.[4]

It will be assumed that the reader is familiar with certain elementary portions of the theory of linear and metric spaces, and with transfinite cardinal and ordinal numbers.[5] Using the generalized continuum hypothesis and normal order theorem, we prove the following:

THEOREM. *A necessary and sufficient condition that a linear space may be made a complete, linear, normed metric space by a suitable definition of norm is that the cardinal number of its Hamel basis should not be the limit of any denumerable sequence of cardinals which precede it.*

A Hamel basis of a linear space $S$ is a subset $T$ of $S$ such that every element of $S$ is a linear combination, with real coefficients, of a finite number of elements of $T$, and there is no proper subset of $T$ with this property. The following properties of a Hamel basis will be used in demonstrating the theorem, and are given without proof:

(a) A linear space $S$ has a Hamel basis.[6]

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[1] K. Kunugui, *Applications des espaces à une infinité de dimensions à la théorie des ensembles*, Proc. Imp. Acad. Tokyo vol. 11 no. 9 pp. 351–353.

[2] M. Fréchet, *Sur les espaces distanciés*. Memorial volume dedicated to D. A. Grave. Moscow, 1940, pp. 265–267.

[3] A. Kolmogoroff, *Zur Normierbarkeit eines allgemeinen topologischen Raumes*, Studia Mathematica, vol. 5 (1934) pp. 29–33.

[4] B. Z. Vulich, *Linear spaces with given convergence*, Leningrad State University Annals, Mathematics Series vol. 10 (1940) pp. 40–63.

[5] For the elements of transfinite number theory, see F. Hausdorff, *Mengenlehre*, Berlin, 1927; for the theory of linear spaces, see S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

[6] Proofs of (a) and (b) have been given by H. Lowig, *Über die Dimension linearer Räume*, Studia Mathematica, vol. 5 (1934) pp. 18–24.

(b) $S$ may have many distinct bases, but these must have the same cardinal number.

(c) If the basis $T$ of $S$ is nondenumerable, then $s$ and $t$, the respective cardinals of $S$ and $T$, are identical.

(d) The representation of every element $x$ of $S$ as a linear combination, with nonzero coefficients, of a finite number of elements of a given basis $T$ of $S$ is unique.

1. **Sufficiency.** We note that a linear, normed metric space with a finite Hamel basis, of cardinal $n$, is homeomorphic with euclidean $n$-space; hence, is complete. To prove the sufficiency part of the theorem we need, therefore, consider only the infinite case; that is, we must show that a linear space whose basis is of infinite cardinal not the limit of any denumerable sequence of cardinals which precede it may be rendered complete by a suitable definition of norm. Such cardinals are either non-limiting, infinite cardinals, or limiting cardinals which are not the limit of any denumerable sequence of cardinals. First, let $\alpha$ be a non-limiting, infinite cardinal, and $\beta$ its immediate predecessor in the normal order of cardinals. By the generalized continuum hypothesis, $2^\beta=\alpha$. Let $S$ be the linear space of all bounded, real functions defined on a given set of cardinal $\beta$, the sum of two functions and the product of a function by a real number being defined in the customary way. $S$ is of cardinal $c^\beta=2^\beta=\alpha$. With $\nu(f)=\text{l.u.b.}\ |f(x)|$, the least upper bound of $|f(x)|$, as norm, $S$ is evidently a complete, linear, normed metric space. For $\alpha>c$, as previously noted, every basis of $S$ has the same cardinal $\alpha$ as $S$. For $\alpha=c$, $S$ certainly has no finite basis and, since complete, has no denumerable basis, as will be shown in the proof of the necessity part of the theorem. Hence, by the continuum hypothesis, if $S$ is of cardinal $c$, its basis is also of cardinal $c$. We have thus defined, for every non-limiting, infinite cardinal $\alpha$, a complete, linear, normed metric space with basis of cardinal $\alpha$. Any linear space $S'$ with basis of cardinal $\alpha$ is isomorphic with $S$. For, if $T$ and $T'$ are respective bases of $S$ and $S'$, a biunique correspondence exists between them. It then follows from the uniqueness of the representation of the elements of a space as a linear combination of basal elements, that $S$ and $S'$ are isomorphic linear spaces. Hence, by letting the norm of every element of $S'$ be identical with that of its mate in $S$, the linear space $S'$ is made complete. Now, let $\alpha$ be a limiting, infinite cardinal which is not the limit of any denumerable sequence of cardinals which precede it. For every cardinal $\beta<\alpha$, let $T_\beta$ be a set of cardinal $\beta$ such that, for every $\gamma<\beta$, $T_\gamma$ is a subset of $T_\beta$. Such a class of sets may be obtained, for ex-

ample, by normally ordering a set of cardinal $\alpha$, and letting $T_\beta$ be the proper initial segment of the order which introduces the cardinal $\beta$. Let $T=\sum_{\beta<\alpha}T_\beta$ and let $S_\beta$ be the set of bounded, real functions defined over $T$, arbitrarily on $T_\beta$ and identically zero on $T-T_\beta$. The cardinal number of $S_\beta$ is the immediate successor of $\beta$, by the generalized continuum hypothesis, and the cardinal number of $S=\sum_{\beta<\alpha}S_\beta$ is $\alpha$. As in the preceding case, $S$ may be rendered a linear, normed metric space by defining the sum of two functions of $S$ and the product of a function of $S$ by a real number in the manner customary for functions, and by letting $\nu(f)=\text{l.u.b. } |f(x)|$, for every $f(x)$ in $S$. The space $S$ is a complete metric space. For, let $\{f_n(x)\}$ be any convergent sequence of elements of $S$. Then $\{f_n(x)\}$ is a uniformly convergent sequence of functions. Let $\beta_n$ be the smallest cardinal such that $f_n(x)$ is identically zero on the set $T-T_{\beta_n}$. Then $\beta_n<\alpha$ and, since $\alpha$ is not the limit of any denumerable sequence of cardinals which precede it, there exists a cardinal $\beta<\alpha$ such that $\beta_n<\beta$, for all $n$. The functions of the sequence $\{f_n(x)\}$ are thus identically zero on $T-T_\beta$ and, since the sequence is convergent, its limit, $f(x)$, is also identically zero on $T-T_\beta$. Moreover, $f(x)$ is bounded, for it is the limit of a uniformly convergent sequence of bounded functions. Hence, $f(x)$ is in $S_\beta$ and thus in $S$. The linear, normed metric space $S$ is therefore complete. It then follows from the isomorphism of linear spaces whose bases are of the same cardinal, that every linear space with basis of cardinal $\alpha$ may be normed so as to be complete.

2. **Necessity.** We show that a linear, normed metric space is not a complete metric space if the cardinal number of its basis is the limit of a denumerable sequence of cardinals which precede it. We prove the following:

LEMMA. *If $T$ is a nondense, linear subspace of a linear, normed metric space $S$, then $T$ is nowhere dense in $S$.*

PROOF. Let $\sigma$ be a sphere of elements of $S$. Then $c\sigma$, the set obtained by multiplying the elements of $\sigma$ by a real number $c\neq 0$, and $x+\sigma$, the set obtained by adding to the elements of $\sigma$ a given element $x$ of $S$, are also spheres in $S$. Since $T$ is nondense in $S$, there is a sphere $\sigma$ in $S$, containing no point of $T$. Let $\{c_n\}$ be a sequence of reals converging to zero. Then, for every $x$ in $T$, the sequence $\{x+c_n\sigma\}$ of spheres converges to $x$. But these spheres contain no points of $T$. Hence, $T$ is nowhere dense in $S$.

Suppose $S$ is a linear, normed metric space with a denumerable basis, whose elements are $\xi_1, \xi_2, \cdots, \xi_n, \cdots$. Let $S_n$ be the linear subspace of $S$ with basis $\xi_1, \xi_2, \cdots, \xi_n$. Then $S=\sum_1^\infty S_n$. Since $S_n$ has a finite basis, it is a complete metric space; hence, closed in $S$. But a closed subset of a metric space is nondense in the space. Therefore, by the lemma, $S_n$ is nowhere dense in $S$, and $S$, as the limit of $\aleph_0$ nowhere dense sets, is of the first category, and therefore not complete.

Finally, we construct, for every nondenumerable cardinal $\alpha$ which is the limit of a denumerable sequence of cardinals, a linear space of cardinal $\alpha$, and hence, with basis of cardinal $\alpha$, which cannot be normed so as to be rendered complete. Since, as noted above, two linear spaces are isomorphic if their bases are of the same cardinal, it follows that every linear space of cardinal $\alpha$ has the desired property. Now, let $\{\alpha_n\}$ be the sequence of consecutive, increasing cardinals which converges to $\alpha$, and let $T_n$ be a set of cardinal $\alpha_n$ such that, for every $n$, $T_n$ is a subset of $T_{n+1}$. Let $T=\sum_1^\infty T_n$. Let $S_n$ be the set of bounded, real functions defined over $T$, arbitrarily on $T_n$ and zero elsewhere, and let $S=\sum_1^\infty S_n$. The cardinal number of $S_n$ is $2^{\alpha_n}=\alpha_{n+1}$, by the generalized continuum hypothesis, and the cardinal number of $S$ is $\alpha=\sum_1^\infty \alpha_{n+1}$. $S$ is rendered a linear space, if the sum of two functions and the product of a function by a real number are defined in the usual way. Suppose $S$ is further rendered normed and metric by any given choice of norm for the elements of $S$. $S_n$ is obviously a linear, normed metric proper subspace of $S$. Moreover, $S_n$ is a nondense subset of $S$. For there are only $\alpha_{n+1}^{\aleph_0}=\alpha_{n+1}$ sequences composed of elements of $S_n$ and, since $S$ is of cardinal $\alpha>\alpha_{n+1}$, there exists an element of $S$ not a limit point of $S_n$. By the lemma, $S_n$ is nowhere dense in $S$ and $S=\sum_1^\infty S_n$ is of the first category. Hence $S$ is not complete.

As a consequence of the necessity part of this theorem, it immediately follows that a metric space whose cardinal number is the limit of no denumerable sequence of cardinals is homeomorphic, hence isometric, with no complete, linear, normed metric space.

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# THE STRONGER FORM OF CAUCHY'S INTEGRAL THEOREM

E. F. BECKENBACH

**1. Introduction.** The so-called weaker and stronger forms[1] of Cauchy's integral theorem are the following.

THEOREM 1. CAUCHY'S INTEGRAL THEOREM (WEAKER FORM). *If $f(z)$ is holomorphic on the finite simply-connected open domain $D$, and if $C$ is a closed rectifiable curve in $D$, then*

$$\int_C f(z)dz = 0.$$

THEOREM 2. CAUCHY'S INTEGRAL THEOREM (STRONGER FORM). *If $f(z)$ is holomorphic on the interior $D$ of a simply closed rectifiable curve $C$, and continuous on $D+C$, then*

$$\int_C f(z)dz = 0. \tag{1}$$

Each of these theorems follows readily when it has been established that there is a sequence of closed curves $\{C_n\}$ in $D$, of uniformly bounded length, converging in the sense of Fréchet to $C$, such that

$$\int_{C_n} f(z)dz = 0.$$

In the case of Theorem 2 it appears to be more difficult to establish the existence of the sequence $\{C_n\}$, since the convergence must be from only one side; but the difficulty may be overcome by more or less tedious topological considerations, a program which has been undertaken by a number of authors.[2] Recent excellent proofs by Reid and Hestenes, loc. cit., appear to be about as simple as a valid elementary proof of this result could be.

The above type of proof applies equally well to yield the corresponding stronger form of Green's lemma, as some authors have

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[1] This terminology is taken from M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge, England, 1939, pp. 154, 156.

[2] For bibliographical discussions see S. Pollard, *On the conditions for Cauchy's theorem*, Proc. London Math. Soc. vol. 21 (1923) pp. 456–482; E. Kampe, *Zu dem Integralsatz von Cauchy*, Math. Zeit vol. 35 (1932) pp. 539–543; W. T. Reid, *Green's lemma and related results*, Amer. J. Math. vol. 63 (1941) pp. 563–574, M. R. Hestenes, *An analogue of Green's theorem in the calculus of variations*, Duke Math. J. vol. 8 (1941) pp. 300–311.

pointed out, and also to yield other integral results such as the analogous stronger form of Cauchy's integral formula, and so on.

Walsh and Heilbronn[3] have given brief and simple, but not elementary, proofs of Theorem 2, using results of Walsh and Fejér on polynomial boundary approximations to holomorphic functions. Their methods of proof do not yield the stronger form of Green's lemma.

As an application of Theorem 2, consider Painlevé's theorem in the following form.

PAINLEVÉ'S THEOREM. *If open plane domains $D_1$ and $D_2$ abut along the rectifiable arc $C$, if $f_j(z)$ is holomorphic on $D_j$ and continuous on $D_j+C$, $j=1, 2$, and if $f_1(z)=f_2(z)$ on $C$, then $f_1(z)$ and $f_2(z)$ are analytic continuations of one another across $C$.*

The proofs of this theorem use the finite accessibility of points of $C$ from $D_1$ and from $D_2$, together with the stronger form of Cauchy's integral theorem and sometimes of Cauchy's integral formula; the weaker forms do not suffice.

The following proof of Theorem 2, though not elementary, uses only tools which are usually considered standard. It is brief and simple, and the method has the advantage of being equally applicable to establishing the stronger form of Green's lemma, Cauchy's integral formula, and so on.

## 2. **Lemmas.** We shall use the following two lemmas.

LEMMA 1. BIEBERBACH-CSILLAG THEOREM.[4] *If $z=g(w)$ is holomorphic on $|w|<1$ and continuous on $|w|\leqq 1$, and if $l(r)$ denotes the length of the image of $|w|=r\leqq 1$ on the $z$-plane, then $l(r)$ is a monotonically non-decreasing function of $r$ for $0\leqq r\leqq 1$.*

PROOF. Fix $r<1$ and let $r<R\leqq 1$. Since the length of a curve is the limit of the lengths of a proper sequence of inscribed polygons, for any positive $\epsilon$ there is a set of values $\theta_1, \theta_2, \cdots, \theta_n, \theta_{n+1}=\theta_1$, with $\theta_1<\theta_2<\cdots<\theta_n<\theta_1+2\pi$, such that

$$p(r) \equiv \sum_{j=1}^{n} \left| g(re^{i\theta_{j+1}}) - g(re^{i\theta_j}) \right| > l(r) - \epsilon. \tag{2}$$

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[3] J. L. Walsh, *The Cauchy-Goursat theorem for rectifiable Jordan curves*, Proc. Nat. Acad. Sci. U.S.A. vol. 19 (1933) pp. 540–541; H. Heilbronn, *Zu dem Integralsatz von Cauchy*, Math. Zeit. vol. 37 (1933) pp. 37–38.

[4] L. Bieberbach, *Über die konforme Kreisabbildung nahezu kreisförmiger Bereiche*, Preuss. Akad. Wiss. Sitzungsber. 1924, pp. 181–188.

Consider the function $p(w)=\sum_{j=1}^{n}|g(we^{i\vartheta_{j+1}})-g(we^{i\vartheta_j})|$; now $p(w)$ is continuous for $|w|\leqq 1$; and since the absolute value of a holomorphic function is subharmonic and the sum of a finite number of subharmonic functions is subharmonic, it follows that $p(w)$ is subharmonic for $|w|<1$. Hence $p(w)$ assumes its maximum value on $|w|\leqq R$ at some point $Re^{i\vartheta_0}$ on $|w|=R$; in particular,

$$p(r) \leqq p(Re^{i\vartheta_0}). \tag{3}$$

Clearly $p(Re^{i\vartheta_0})$ is the length of a polygon inscribed in the image of $|w|=R$, so that

$$p(Re^{i\vartheta_0}) \leqq l(R). \tag{4}$$

Now (2), (3) and (4) yield $l(r)-\epsilon<l(R)$, whence $l(r)\leqq l(R)$.

LEMMA 2. COROLLARY OF HELLY-BRAY THEOREM.[5] *In the unit circle $r\leqq 1$, let $g(re^{i\theta})$ and $h(re^{i\theta})$ be continuous complex functions and let $g(re^{i\theta})$, as function of $\theta$, $0\leqq\theta\leqq 2\pi$, be of limited variation, uniform with respect to $r$. Then*

$$\int_0^{2\pi} h(e^{i\theta})dg(e^{i\theta}) = \lim_{r\to 1}\int_0^{2\pi} h(re^{i\theta})dg(re^{i\theta}).$$

PROOF. Let $T$ be the least upper bound of the total variations $T_r$. Since $h(re^{i\theta})$ is uniformly continuous in the unit circle, for any positive $\epsilon$ there is a positive integer $n$ such that for all $r\leqq 1$,

$$| h(re^{i\theta}) - h(re^{i\theta'}) | \leqq \epsilon/(3T)$$

provided $|\theta-\theta'|\leqq 2\pi/n$. Then for all $r\leqq 1$, we have[6]

$$\begin{aligned}\Bigg|\int_0^{2\pi} h(re^{i\theta})dg(re^{i\theta}) - \sum_{j=1}^{n} h(re^{2j\pi i/n})\big[g(re^{2j\pi i/n})& \\ - g(re^{2(j-1)\pi i/n})\big]\Bigg| &\leqq \epsilon/3.\end{aligned} \tag{5}$$

Also, by the continuity of $g(re^{i\theta})$ and $h(re^{i\theta})$ as functions of $r$, there is an $r_0<1$ such that for $r_0<r<1$,

$$\begin{aligned}\Bigg|\sum_{j=1}^{n} h(re^{2j\pi i/n})\big[g(re^{2j\pi i/n}) - g(re^{2(j-1)\pi i/n})\big]& \\ - \sum_{j=1}^{n} h(e^{2j\pi i/n})\big[g(e^{2j\pi i/n}) - g(e^{2(j-1)\pi i/n})\big]\Bigg| &< \epsilon/3.\end{aligned} \tag{6}$$

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[5] See for instance G. C. Evans, *The logarithmic potential*, Amer. Math. Soc. Colloquium Publications, vol. 6, 1927, pp. 14–16.

[6] Cf. K. Knopp, *Funktionentheorie*, vol. 1, Berlin, 1930, p. 44, Hilfssatz 2.

Now (5), taken once for $r_0<r<1$ and again for $r=1$, and (6) yield

$$\left|\int_0^{2\pi} h(e^{i\theta})dg(e^{i\theta}) - \int_0^{2\pi} h(re^{i\theta})dg(re^{i\theta})\right| < \epsilon,$$

and the lemma is established.

3. **Proof of Theorem** 2. By the Riemann mapping theorem, there is a holomorphic function $z=g(w)$ which maps the interior of the unit circle $|w|<1$ on $D$; further, by the Osgood-Carathéodory theorem, $g(w)$ remains continuous on $|w|\leqq 1$ and $z=g(w)$ maps $|w|=1$ topologically on $C$. Let $f(z)=f[g(w)]=h(w)$. Now $g(w)$ and $h(w)$ are continuous on $|w|\leqq 1$ and, by Lemma 1, $g(re^{i\theta})$, as function of $\theta$, $0\leqq\theta\leqq 2\pi$, is of limited variation, uniform with respect to $r$. Consequently, by Lemma 2,

$$\begin{aligned}\int_C f(z)dz &= \int_0^{2\pi} f[g(e^{i\theta})]dg(e^{i\theta}) = \int_0^{2\pi} h(e^{i\theta})dg(e^{i\theta}) \\ &= \lim_{r\to 1}\int_0^{2\pi} h(re^{i\theta})dg(re^{i\theta}) = \lim_{r\to 1}\int_{C_r} f(z)dz,\end{aligned} \tag{7}$$

where $C_r$ is the image of $|w|=r$ on the $z$-plane. But by Theorem 1,

$$\int_{C_r} f(z)dz = 0. \tag{8}$$

Now (1) follows from (7) and (8).

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# ON AUTOMORPHISMS OF COMPACT GROUPS

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**1. Introduction and definitions.** Let $G$ be a compact abelian group and $\alpha$ a continuous automorphism of $G$. We write $G$ multiplicatively and use, accordingly, the exponent notation for automorphisms. Thus the image under $\alpha$ of the element $x \in G$ will be denoted by $x^\alpha$; similarly we shall write for (complex valued) functions $f(x)$, $f^\alpha(x) = f(x^\alpha)$.[1]

If $m$ is Haar measure[2] in $G$ (normalized so that $m(G)=1$) we consider the set function $m'(E) = m(E^\alpha)$. ($E^\alpha$ is the set of all $x^\alpha$, $x \in E$.) Since $m'$ is a measure on $G$ possessing all defining properties of $m$ it follows from the uniqueness of Haar measure[3] that $m'(E) = m(E)$ for every measurable set $E$. In other words $\alpha$ is a measure preserving transformation of $G$; the purpose of this note is to investigate a few simple properties of $\alpha$ from the point of view of measure theory.

We shall make use of the Pontrjagin duality theory,[4] and, in particular, we shall need the fact that the group of automorphisms of $G$ is essentially the same as that of the character group $G^*$. More precisely: if to any $\phi = \phi(x) \in G^*$ we make correspond $\phi^\alpha = \phi^\alpha(x) = \phi(x^\alpha)$, then $\phi^\alpha \in G^*$, and the correspondence $\phi \to \phi^\alpha$ is an automorphism of $G^*$. The duality theory also enables us to reverse this argument: every automorphism of $G^*$ is induced in this way by a continuous automorphism of $G$.

We recall some standard definitions from ergodic theory. A measure preserving transformation $\alpha$ (not necessarily an automorphism) is *ergodic* if the only (complex valued, measurable) solutions $f$ of the equation $f^\alpha = f$ are constant almost everywhere. The transformation $\alpha$ is *mixing* if the only (complex valued, measurable) solutions $f$ of the equation $f^\alpha = \lambda f$, for any constant $\lambda$, are constant almost everywhere.[5] (It is true, though irrelevant, that for $\lambda \neq 1$ even a constant fails to be a solution unless it is zero.) It is well known that the mapping

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[1] This notation dovetails, as usual, with ordinary exponentiation in $G$; thus $x^{3\alpha^2} = (x^3)^{\alpha^2} = (x^{\alpha^2})^3$, and so on.

[2] For a general discussion of measure theory in topological groups see A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris, 1938.

[3] Weil, op. cit., pp. 36–38.

[4] Weil, op. cit., chap. 6.

[5] See E. Hopf, *Ergodentheorie*, Berlin, 1937, chap. 3, for a discussion of the fact that these definitions are equivalent to the ones more commonly given.

$f \to f^{\alpha}$ induced by a measure preserving transformation $\alpha$ on the space of functions over $G$ is a unitary transformation of the Hilbert space $L_2(G)$.[6] Two measure preserving transformations $\alpha$ and $\beta$ are of the *same spectral type* if there is a unitary transformation $\omega$ of $L_2(G)$ ($\omega$ need not be induced by a transformation of $G$) for which $f^{\omega\alpha\omega^{-1}} = f^{\beta}$ for all $f \in L_2(G)$. Given any point $x \in G$ (or function $f$ on $G$) the set of all $x^{\alpha^n}$ (or $f^{\alpha^n}$), $n = 0, \pm 1, \pm 2, \cdots$ is the *orbit* of $x$ (or $f$). If $\alpha$ is an automorphism the orbit of the identity consists of the identity only; if this is the only finite orbit we shall say, for the sake of brevity, that $\alpha$ has no finite orbits.

In terms of the definitions of the preceding paragraph we can state our results quite concisely. In Theorem 1 we obtain a simple characterization of ergodicity and mixing in terms of the orbits of $\alpha$ in $G^*$. Theorem 2 is a statement concerning abstract groups and, prima facie, has nothing to do with measure theory; together with Theorem 1 and the duality theory it yields, however, a complete description of the spectral type of ergodic automorphisms. In the concluding section we state some unsolved problems and emphasize the importance of group automorphisms as a source of many new and simple examples of transformations with properties that were once considered difficult to obtain.[7]

## 2. Ergodic and mixing automorphisms.

We prove the following theorem:

THEOREM 1. *A continuous automorphism $\alpha$ of a compact abelian group $G$ is ergodic (or mixing) if and only if the induced automorphism on the character group $G^*$ has no finite orbits.*

We call attention to the somewhat surprising fact that for continuous automorphisms ergodicity is equivalent to the apparently stronger mixing condition. We shall see later that much more than this is true: if $\alpha$ is ergodic then it automatically has the strongest of the whole known hierarchy of mixing properties.

The similarity of our definitions of ergodicity and mixing to each other enables us to prove both parts of the theorem simultaneously. Suppose that $f \in L_2(G)$ and $\lambda$, $|\lambda| = 1$, are such that $f^{\alpha} = \lambda f$.[8] We may expand $f$ in a Fourier series in the characters $\phi \in G^*$, $f(x) = \sum_{\phi} a(\phi)\phi(x)$.[9] Concerning this series we must make two comments. First, even if $G^*$

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[6] Hopf, op. cit., p. 9.

[7] Hopf, op. cit., p. 42.

[8] The fact that, considered as a transformation of $L_2$, $\alpha$ is a unitary operator implies that if there is any proper value $\lambda$ at all then it must be of modulus one.

[9] Weil, op. cit., p. 76.

is uncountable, at most a countable number of $\phi$'s fail to be orthogonal to $f$, so that at most a countable number of $a$'s are different from zero. Second, the series need converge only in the sense of $L_2$ (mean square convergence); the known fact that a sub-sequence of its partial sums converges almost everywhere is sufficient to justify the simple formal steps that follow. Replacing $x$ by $x^\alpha$ we obtain

$$\sum_\phi \lambda a(\phi)\phi(x) = \lambda f(x) = f^\alpha(x) = f(x^\alpha)$$

$$= \sum_\phi a(\phi)\phi(x^\alpha) = \sum_\phi a(\phi)\phi^\alpha(x) = \sum_\phi a(\phi^{\alpha^{-1}})\phi(x).$$

Hence (using the orthogonality of the characters) we may equate coefficients and obtain $\lambda a(\phi) = a(\phi^{\alpha^{-1}})$, or $|a(\phi)| = |a(\phi^{\alpha^{-1}})|$. Since $\phi$ is arbitrary it follows that all coefficients $a(\phi)$ corresponding to $\phi$'s in the same orbit are equal in modulus. Since $\sum_\phi |a(\phi)|^2 < \infty$ it follows that all $a$'s corresponding to $\phi$'s in the same infinite orbit vanish. This settles the *only if* part of our theorem: a non-constant $f$ can exist only if $\alpha$ (considered as an automorphism of $G^*$) has finite orbits.

The converse is easier. Let $\phi \in G^*$ $(\phi \neq 1)$ have a finite orbit; suppose, for definiteness, that $n$ is the least positive integer for which $\phi^{\alpha^n} = \phi$. It follows that for the function $f = \phi + \phi^\alpha + \cdots + \phi^{\alpha^{n-1}}$ we have $f^\alpha = f$. The orthogonality and, a fortiori, linear independence of the $\phi$'s show that $f$ is not constant. Since this shows that the existence of a finite orbit implies non-ergodicity, the proof of Theorem 1 is complete.

THEOREM 2. *Let $\alpha$ be any automorphism of the discrete abelian group $H$; if $\alpha$ has no finite orbits then it has an infinite number of orbits.*[10]

*Case* I. We assume first that there is in $H$ an element $\phi_0$ of finite order. By raising $\phi_0$ to a suitable power we may assume that the order of $\phi_0$ is a prime $p$. Write $\phi_n = \phi_0^{\alpha^n}$, $n = 0, \pm 1, \pm 2, \cdots$; we shall prove that the $\phi_n$ are independent mod $p$. (It is clear that the order of each $\phi_n$ is $p$.) Suppose, on the contrary, that $\phi_{i_1}^{r_1} \cdots \phi_{i_k}^{r_k} = 1$; it is merely a notational change to write $i_1 = 1, \cdots, i_k = k$, and we may, of course, assume that $r_1$ and $r_k$ are not congruent to zero mod $p$. We have then

$$(1) \qquad \phi_1 = (\phi_2^{-r_2} \cdots \phi_k^{-r_k})^{r_1^{-1}}, \qquad \phi_k = (\phi_1^{-r_1} \cdots \phi_{k-1}^{-r_{k-1}})^{r_k^{-1}}.$$

(The exponents $r_1^{-1}$, $r_k^{-1}$ make sense since we may interpret them in the modular field $GF(p)$.) Consider now the finite subgroup $H_0$ of $H$

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[10] The author's thanks are due to R. Baer, R. H. Fox, and H. Samelson for many valuable discussions of this theorem and its proof.

generated by $\phi_1, \cdots, \phi_k$. It follows from the definition of the $\phi_n$ and the relations (1) that $H_0$ is invariant under both $\alpha$ and $\alpha^{-1}$ and contains consequently the entire orbit of $\phi_0$. Since this contradicts the assumed nonexistence of finite orbits, the $\phi_n$ must indeed be independent. The desired conclusion follows at once: the elements $\psi_n = \phi_1 \cdots \phi_n$, $n = 1, 2, \cdots$, must all lie in different orbits.

*Case* II. If $\phi_0$ is an element of infinite order we write $\phi_i = \phi_0^{\alpha^i}$ as before and we ask for which (positive or negative) integers $i$ it is true that $\phi_i$ is a positive rational power of $\phi_0^r$ (that is, a positive integral power of some root of $\phi_0$). If $\phi_i = \phi_0^r$ then $(\phi_{-i})^r = (\phi_0^r)^{\alpha^{-i}} = (\phi_i)^{\alpha^{-i}} = \phi_0$, so that $\phi_{-i}$ is an $r$th root of $\phi_0$. If also $\phi_j = \phi_0^s$ then $\phi_{i+j} = (\phi_j)^{\alpha^i} = \phi_0^{s\alpha^i} = \phi_0^{\alpha^i s} = \phi_0^{rs}$. In other words the set of $i$'s under consideration is an additive group of integers; let $i_0$ be a generator of this group, $\phi_{i_0} = \phi_0^{r_0}$. Then for any integer $n$

$$\phi_{i_0 n} = (\cdots((\phi_0)^{r_0})^{r_0}\cdots)^{r_0} = \phi_0^{r_0^n};$$

in other words the set of possible $r$'s that may occur as exponents in a relation $\phi_i = \phi_0^r$ consists of all (positive, negative, or zero) integral powers of $r_0$. Hence the set of powers of $\phi_0$ which are in the same orbit as $\phi_0$ consists only of powers of $r_0$, and consequently there is a power of $\phi_0$ which does not lie in this orbit. We may choose this power of $\phi_0$ as a new starting element (of infinite order) and repeat the above argument ad infinitum. This completes the proof of Theorem 2.

We are now prepared to describe the spectral type of ergodic automorphisms. Let $\mathcal{S}$ be a complete orthonormal set in an abstract, not necessarily separable, Hilbert space and denote by $\psi$ any particular element of $\mathcal{S}$. Arrange all remaining elements of $\mathcal{S}$ as an infinite matrix in such a way that each row contains a countably infinite number of elements. Use as row index any set of suitable power and as column index the set of all (positive, negative, or zero) integers. A unique unitary operator $\sigma$ is defined on Hilbert space by the requirement that it send $\psi$ into itself and $\phi_{i,j}$ into $\phi_{i,j+1}$ for $j = 0, \pm 1, \pm 2, \cdots$, and all $i$. The spectral type of $\sigma$ depends only on the number of rows; if we agree to use $\aleph$ for this cardinal number we may write $\sigma = \sigma(\aleph)$. We summarize our result in terms of these $\sigma$'s.

THEOREM 3. *If $\alpha$ is a continuous ergodic automorphism of a compact abelian group $G$ and $\aleph$ is the (necessarily infinite) cardinal number of $G^*$ then $\alpha$ has the spectral type of the unitary operator $\sigma(\aleph)$.*

For the proof we need remember only that the characters of $G$ form a complete orthonormal set of elements of $L_2(G)$. The principal character plays the role of $\psi$ and the orbits of the other characters may be written as the rows of the matrix mentioned above. Theorem 2 shows that there must be an infinite number of rows, and the well known fact that for any infinite cardinal $\aleph \cdot \aleph_0 = \aleph$[11] shows that the number of rows is the same as the total number of elements in $G^*$.[12]

3. **Examples and questions.** Let $H$ be any compact abelian group and let $G$ be the direct product of $H$ with itself a countable number of times. We write $G$ as the set of all sequences $\{x_n \mid n=0, \pm 1, \pm 2, \cdots\}$, and we define a continuous ergodic automorphism $\alpha$ of $G$ by the relations

$$\{x_n\}^\alpha = \{x_n'\}, \quad x_n' = x_{n+1}, \qquad n = 0, \pm 1, \pm 2, \cdots.$$

Transformations isomorphic to such $\alpha$'s (not only in the spectral but even in the stronger, measure theoretic, sense) were among the first known examples of ergodic transformations.

Examples apparently very simple from the algebraic point of view, but very difficult to handle geometrically, are furnished by the solenoids. Consider for instance the multiplicative group of all real numbers of the form $e^r$, where $r$ is a dyadic rational number. The operation of squaring is an automorphism of this group (with no finite orbits), and hence an ergodic automorphism of its compact character group.

More in the classical spirit than either of the last two examples are the (continuous) automorphisms of the $n$-dimensional toral group. In order to retain our multiplicative notation we write the torus as $n$-tuples $(x_1, \cdots, x_n)$ of complex numbers of modulus one; thus the product of $(x_1, \cdots, x_n)$ by $(y_1, \cdots, y_n)$ is $(x_1y_1, \cdots, x_ny_n)$ and the identity element is $(1, \cdots, 1)$. It is well known that the automorphism group of the torus is the unimodular group, in the following sense. Given any $n \times n$ matrix $\alpha = (a_{ij})$ whose elements are integers and whose determinant is $\pm 1$, we consider the mapping $(x_i) \rightarrow (\prod_j x_j^{a_{ij}})$: this mapping is the most general continuous automorphism of the torus. The condition of ergodicity—that is, the condition that, considered as an automorphism of the character group, $\alpha$ have no finite orbits—is equivalent in classical terms to the requirement that no root of unity should be a proper value of $\alpha$. This remark enables us to write down any desired number of quite different looking analytic ergodic (and hence mixing) measure preserving transformations on the finite dimensional torus.

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[11] See F. Hausdorff, ***Mengenlehre***, Berlin, 1935, p. 71.

[12] The first explicit discussion of the spectral form of a measure preserving transformation of type $\sigma(\aleph_0)$ was carried out by J. L. Doob and R. A. Leibler, *On the spectral analysis of a certain transformation*, Amer. J. Math. vol. 65 (1943) pp. 263–272.

This last example and the particular structure of the automorphisms it describes suggests a new, purely measure theoretic, invariant of measure preserving transformations. The Hamilton-Cayley equation says, in our notation, that if $p$ is the characteristic polynomial of $\alpha$ then for every $x$ in the torus $x^{p(\alpha)}=1$. It follows that for every character $f$ of the torus we have, similarly, $f^{p(\alpha)}=1$. The existence or nonexistence of $f$'s thus *annihilated* by certain polynomials in $\alpha$ (with, of course, integer coefficients) and, if they exist, their algebraic and measure theoretic structure, furnishes the invariant to which we referred. To illustrate the possible application of these invariants we mention the following: if it could be proved that the characters are (except for trivial changes on a set of measure zero) the only measurable functions of constant modulus one which are *annihilated* by $p(\alpha)$, it would follow rather easily that two ergodic automorphisms of the torus are measure theoretically isomorphic if and only if they correspond to conjugate elements in the unimodular group. We could thus obtain the first examples of measure theoretically distinct transformations of the spectral type of $\sigma(\aleph_0)$,[13] and the usual proper value theory would then point the way to further, more delicate, invariants.

In conclusion we mention an unsolved problem of purely technical interest, but one whose solution may throw some light on the deeper problems raised above. The question is simply: do there exist measure preserving transformations (on spaces of finite measure) of the spectral type of $\sigma(\aleph)$, where $\aleph$ is a *finite* cardinal?

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[13] The first examples of measure theoretically different but spectrally isomorphic transformations are due to J. von Neumann. These examples (not yet published) are, however, not mixing transformations.

# CAUCHY-STIELTJES AND RIEMANN-STIELTJES INTEGRALS

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1. **Introduction.** This note treats the equivalence of the Riemann-Stieltjes and Cauchy-Stieltjes integrals (abbreviated RS and CS integrals) and conditions for the existence and equality of the latter. The ordinary Riemann, the left Cauchy, and the right Cauchy integrals are defined as limits of the sums $\sum_1^n f(\xi_i)(x_i - x_{i-1})$, $x_{i-1} \leqq \xi_i \leqq x_i$, $\sum_1^n f(x_{i-1})(x_i - x_{i-1})$ and $\sum_1^n f(x_i)(x_i - x_{i-1})$ respectively; it is known [2] that these integrals are equivalent. Corresponding to these integrals, we have the RS and the two CS integrals, defined as limits of the sums $\sum_1^n f(\xi_i)[g(x_i) - g(x_{i-1})]$, $\sum_1^n f(x_{i-1})[g(x_i) - g(x_{i-1})]$, and $\sum_1^n f(x_i)[g(x_i) - g(x_{i-1})]$. The right modified RS integral is obtained from the sums $\sum_1^n f(\xi_i)[g(x_i) - g(x_{i-1})]$, $x_{i-1} \leqq \xi_i < x_i$. Examples in §4 show that the CS integrals may exist, with equal or unequal values, when the RS and the right modified RS do not; that the right modified RS integral may exist when the RS does not; and that one of the CS integrals may exist when the other does not. Thus the RS, the right modified RS, and the two CS integrals are not equivalent. Since all these integrals obviously exist when the RS integral does, it is natural to investigate conditions under which the existence of a CS or right modified RS integral implies the existence of the RS integral. It is shown in this note that if $g$ is non-decreasing, if $f$ and $g$ have no common discontinuities on the same side, and if the left CS integral exists, then the RS integral exists and has the same value, the integrals being limits in the sense of increasing refinement of subdivisions. This result is established in two steps: (a) if $g$ is non-decreasing, if $f$ and $g$ have no common discontinuities on the right, and if the left CS integral exists, then the right modified RS integral exists; (b) if the right modified RS integral exists, if $g$ is non-decreasing, and if $f$ and $g$ have no common discontinuities on the left, then the RS integral exists. This result obviously includes the previously proved equivalence of the Riemann and Cauchy integrals [2] and certain others [3]. Further, it states sufficient conditions for the equality of the two CS integrals; these conditions show that the two ordinary Cauchy integrals are always equal. The note closes with a proof that the CS integrals exist when $f$ has only simple discontinuities and $g$ has bounded variation. We conclude from this result and others stated above that both of the CS integrals properly include the RS integral, and that neither CS integral includes the other. Precise statements of

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theorems are given in §5.

2. **Notation.** Let $f$ and $g$ denote real, single valued, bounded functions defined on $a \leqq x \leqq b$. Let $X$ denote a subdivision $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $a \leqq x \leqq b$, and let $x' \leqq x \leqq x''$ denote a generic interval of $X$. Let $X_2 \geqq X_1$ denote that $X_2$ is a refinement of $X_1$, and let $X_1X_2$ denote the subdivision determined by all the points of division of $X_1$ and $X_2$. Also let $\|X\|$ denote the norm of $X$, that is, the maximum length of a subinterval of $X$. Finally, let $X\Psi(\Delta)$ denote the sum of the values of the interval function $\Psi$ for all the intervals $\Delta$ of $X$. The interval functions to be met below are of the form $f(\xi)[g(x'') - g(x')]$, $x' \leqq \xi \leqq x''$. This notation has been used by Gęöcze [1].

3. **Definitions.** In this section we give definitions of various integrals.

DEFINITION. *We say $f$ has a left* CS $X$ *integral with respect to $g$, denoted by* $(C_L, X)\int_a^b f dg$, *if and only if for each* $\epsilon > 0$ *there exists an* $X_0(\epsilon)$ *such that*

$$(1) \qquad \left| (C_L, X) \int_a^b f dg - Xf(x')[g(x'') - g(x')] \right| < \epsilon, \qquad X \geqq X_0(\epsilon).$$

DEFINITION. *We say $f$ has a left* CS $\|X\|$ *integral with respect to $g$, denoted by* $(C_L, \|X\|)\int_a^b f dg$, *if and only if for each* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that*

$$(2) \qquad \left| (C_L, \|X\|) \int_a^b f dg - Xf(x')[g(x'') - g(x')] \right| < \epsilon, \qquad \|X\| \leqq \delta(\epsilon).$$

The definitions of the corresponding right CS integrals $(C_R, X)\int_a^b f dg$, $(C_R, \|X\|)\int_a^b f dg$ are obtained by replacing $f(x')$ by $f(x'')$ in (1) and (2). Also, if $f(x')$ is replaced by $f(\xi)$, $x' \leqq \xi \leqq x''$, we obtain the RS integrals $(R, X)\int_a^b f dg$, $(R, \|X\|)\int_a^b f dg$. If $f(x')$ in (1) is replaced by $f(\xi)$, $x' \leqq \xi < x''$, we have the right modified RS $X$ integral $(R^*, X)\int_a^b f dg$.

4. **Examples.** A first example shows that the CS integrals may exist, with equal or unequal values, even when the RS and the right modified RS integrals do not, and that the right modified RS integral may exist when the RS does not. Let $f$ have the value 1 when $x = 0$ and the value 0 elsewhere on the interval $-1 \leqq x \leqq 1$. Set $g(0) = c$, $g(x) = -1$ on $-1 \leqq x < 0$, and $g(x) = 1$ on $0 < x \leqq 1$. Then $(C_L, X)\int_{-1}^1 f dg = 1 - c$, $(C_R, X)\int_{-1}^1 f dg = 1 + c$. The $\|X\|$ CS integrals do not exist, and neither do the $X$ and $\|X\|$ RS integrals. If $c = 1$, then $(R^*, X)\int_{-1}^1 f dg$

exists and equals $(C_L, X)\int_{-1}^{1} fdg$.

A second example shows that one CS integral may exist when the other does not. Let $f(x)=\sin(1/x)$ and $g(x)=x-1$ on $-1\leqq x<0$; $f(x)=0$ and $g(x)=x+1$ on $0\leqq x\leqq 1$. Then $(C_R, X)\int_{-1}^{1} fdg$ exists but $(C_L, X)\int_{-1}^{1} fdg$ does not (see a necessary condition for the existence of the latter [3, p. 267, 2.14]).

5. **Theorems.** The results to be established will be stated in this section.

THEOREM 1. *Let f and g, defined on* $a\leqq x\leqq b$, *have the following properties:*

(i) $|f(x)|\leqq M$;
(ii) $g(x)$ *is non-decreasing;*
(iii) *f and g have no common discontinuities on the left;*
(iv) $(R^*, X)\int_a^b fdg$ *exists.*
*Then* $(R, X)\int_a^b fdg$ *exists and equals* $(R^*, X)\int_a^b fdg$.

THEOREM 2. *Let f and g satisfy* (i), (ii) *and the following:*
(v) *f and g have no common discontinuities on the right;*
(vi) $(C_L, X)\int_a^b fdg$ *exists.*
*Then* $(R^*, X)\int_a^b fdg$ *exists and equals* $(C_L, X)\int_a^b fdg$.

COROLLARY 1. *Let f and g satisfy* (i), (ii), (vi), *and the following:*
(vii) *f and g have no common discontinuities on the same side.*
*Then* $(R, X)\int_a^b fdg$, $(C_R, X)\int_a^b fdg$ *exist and equal* $(C_L, X)\int_a^b fdg$.

COROLLARY 2. *If the hypotheses of Corollary* 1 *are strengthened by replacing* (vii) *by*
(viii) *f and g have no common discontinuities,*
*then* $(R, \|X\|)\int_a^b fdg$ *also exists.*

THEOREM 3. *If f satisfies* (i) *and has only simple discontinuities, and if g has bounded variation, then* $(C_L, X)\int_a^b fdg$ *and* $(C_R, X)\int_a^b fdg$ *exist.*

6. **Proofs of the theorems.** The proof of Theorem 1 will be omitted; a more general result is known [3, p. 273].

Consider Theorem 2. Let $\epsilon$ be an arbitrary positive constant, and let $X_0(\epsilon)$ be the subdivision which exists by (1) and (vi). Let $\eta$ be another arbitrary positive number. In each interval $x'\leqq x\leqq x''$ of $X_0$ define a point $t$ as follows: (a) $t=x''$ if $f(x)\geqq f(x')+\eta$ has no solution $x$ satisfying $x'<x<x''$; (b) $t=$ g.l.b. $E_x[f(x)\geqq f(x')+\eta, x'<x<x'']$ in case this set is not empty. With each point $t$ associate a sequence of points $t_1, t_2, \cdots$ as follows: (a) $t_k=t$, $k=1, 2, \cdots$, if $t=x''$; (b) $t_k=t$, $k=1, 2, \cdots$, if $f(t)\geqq f(x')+\eta$; (c) in the remaining case

$\{t_k\}$ is a sequence such that $f(t_k) \geqq f(x')+\eta$, $k=1, 2, \cdots$, and $\lim_k t_k = t$. The third case occurs only when $f$ is discontinuous on the right at $t$; at such a point $g$ is continuous on the right by (v). Then for every interval of $X_0$ $\lim_k [g(x'')-g(t_k)] = [g(x'')-g(t)]$. This fact will be used to prove

$$X_0\eta[g(x'') - g(t)] \leqq 2\epsilon. \tag{3}$$

Let $X_1$ be the subdivision obtained by adding the points of division $t$ to those of $X_0$, and let $X_1^k$ be the subdivision obtained from $X_0$ by subdividing each of its intervals by $t_k$. Since $X_1^k \geqq X_0$, we have

$$\left| X_1^k f(x')[g(x'') - g(x')] - X_0 f(x')[g(x'') - g(x')] \right| < 2\epsilon \tag{4}$$

by (1) and (vi). But since

$$X_1^k f(x')[g(x'') - g(x')] = X_0 f(x')[g(t_k) - g(x')] + X_0 f(t_k)[g(x'') - g(t_k)],$$

we find that the left-hand side of (4) reduces to

$$\left| X_0[f(t_k) - f(x')][g(x'') - g(t_k)] \right| \geqq X_0\eta[g(x'') - g(t_k)]. \tag{5}$$

Thus from (4) and (5) we find $X_0\eta[g(x'')-g(t_k)]<2\epsilon$; by taking the limit as $k\to\infty$, we obtain (3).

Let $U^*(f; x', x'')$ and $L^*(f; x', x'')$ denote the l.u.b. and g.l.b. respectively of $f$ on $x' \leqq x < x''$. Then since

$$\begin{aligned} &X_1 U^*(f; x', x'')[g(x'') - g(x')] \\ &\qquad = X_0 U^*(f; x', t)[g(t) - g(x')] + X_0 U^*(f; t, x'')[g(x'') - g(t)], \end{aligned}$$

we have

$$\begin{aligned} &\left| X_1 U^*(f; x', x'')[g(x'') - g(x')] - X_0 f(x')[g(x'') - g(x')] \right| \\ &\qquad \leqq \left| X_0[U^*(f; x', t) - f(x')][g(t) - g(x')] \right| \\ &\qquad\qquad + \left| X_0[U^*(f; t, x'') - f(x')][g(x'') - g(t)] \right| \\ &\qquad \leqq \eta[g(b) - g(a)] + 2MX_0[g(x'') - g(t)] \leqq \eta[g(b) - g(a)] + 4M\epsilon/\eta \end{aligned}$$

by (i), (ii), (3), and the definition of $t$. Thus

$$\begin{aligned} &\left| X_1 U^*(f; x', x'')[g(x'') - g(x')] - (C_L, X)\int_a^b f dg \right| \\ &\qquad\qquad \leqq \eta[g(b) - g(a)] + 4M\epsilon/\eta + \epsilon \end{aligned}$$

by the result just established and the definition of $X_0$. If $\zeta$ is an arbitrary positive number, it is possible to choose $\eta$ and $\epsilon$ so small that the expression on the right of the last inequality is less than $\zeta$. By a similar argument we can show that there exists a subdivision $X_2$ such that

$$\left|X_2L^*(f;\, x',\, x'')[g(x'') - g(x')] - (C_L,\, X)\int_a^b fdg\right| < \zeta.$$

It follows from these results that

$$\left|Xf(\xi)[g(x'') - g(x')] - (C_L,\, X)\int_a^b fdg\right| < \zeta$$

for $x' \leqq \xi < x''$ and any $X \geqq X_1X_2$. Thus $(R^*, X)\int_a^b fdg$ exists and equals $(C_L, X)\int_a^b fdg$, and the proof of Theorem 2 is complete.

Corollary 1 follows from Theorems 1 and 2. A sufficient condition that $(R, \|X\|)\int_a^b fdg$ exist is that $(R, X)\int_a^b fdg$ exist, and that $f$ and $g$ have no common discontinuities [3, p. 269, 4.14]; Corollary 2 follows from this condition and Corollary 1.

Before we begin the proof of Theorem 3, two preliminary results are needed. Let $g$ be any function of bounded variation on $a \leqq x \leqq b$, and let $x'$ be any point such that $a \leqq x' < b$. Then given any $\zeta > 0$, there exists an $x''$ sufficiently near $x'$ on the right so that the total variation of $g$ on any interval $t \leqq x \leqq x''$, where $x' < t < x''$, is less than $\zeta$. To prove this statement we define $h(x) = g(x)$, $x \neq x'$, and $h(x') = g(x'+0)$ and recall that the total variation of $h(x)$ is continuous on the right at $x'$ [6, p. 356]. The second preliminary result is the following. If $f$ is defined on $\alpha \leqq x \leqq \beta$ and has simple discontinuities only, and if the oscillation of $f$ at each point of the interval $\alpha \leqq x < \beta$ is less than $\lambda$, then $\alpha \leqq x \leqq \beta$ can be subdivided into a finite number of intervals $x' \leqq x \leqq x''$ such that the oscillation of $f$ on $x' \leqq x < x''$ is less than $2\lambda$. The proof follows readily from a theorem of Baire [4, p. 311].

Finally, consider the proof of Theorem 3. Let any $\epsilon > 0$ be given. Form a subdivision $X_0(\epsilon)$ of $a \leqq x \leqq b$ by points $a = x_0 < x_1 < \cdots < x_n = b$ so that (a) each of the $r$ points at which the oscillation of $f$ is equal to or greater than $\lambda$ is a point of division; (b) if $x' \leqq x \leqq x''$ is an interval of $X_0$ such that the oscillation of $f$ at $x'$ is equal to or greater than $\lambda$, the total variation of $g$ on any interval $t \leqq x \leqq x''$, $x' < t < x''$, is less than $\zeta$; (c) if $x' \leqq x \leqq x''$ is not an interval of the type described in (b), the oscillation of $f$ on $x' \leqq x < x''$ is less than $2\lambda$.

Let $X_1$ be any refinement of $X_0$. A straightforward calculation shows that

$$(6)\quad |X_1f(x')[g(x'') - g(x')] - X_0f(x')[g(x'') - g(x')]| < 2Mr\zeta + 2\lambda V,$$

where $V$ is the total variation of $g$ on $a \leqq x \leqq b$. We may suppose that

$\lambda$ was chosen so that $2\lambda V < \epsilon/4$. This choice of $\lambda$ determines the integer $r$. We may suppose further that $\zeta$ was chosen so that $2Mr\zeta < \epsilon/4$. Then the right-hand member of (6) is less than $\epsilon/2$. Similarly, for $X_2 \geqq X_0$, we have

$$\left| X_0 f(x')[g(x'') - g(x')] - X_2 f(x')[g(x'') - g(x')] \right| < \epsilon/2;$$

hence

$$\left| X_1 f(x')[g(x'') - g(x')] - X_2 f(x')[g(x'') - g(x')] \right| < \epsilon, \qquad X_1,\ X_2 \geqq X_0(\epsilon).$$

This is a sufficient condition that $(C_L, X)\int_a^b f dg$ exists [5, p. 106]. A similar proof shows that $(C_R, X)\int_a^b f dg$ also exists.

All proofs are now complete.

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# EXACT $n$TH DERIVATIVES

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Let $y$ be a function of $x$ with derivatives of all orders, and let $\theta$ be a function of $x$, $y$, and a finite number of derivatives of $y$. If, independently of the choice of the function $y$, $\theta$ is the $n$th total derivative of some function $\psi$ of $x$, $y$, and derivatives of $y$, then we shall call $\theta$ an *exact nth derivative*. The problem with which this note is concerned is to determine, for any given function $\theta$ and positive integer $n$, if $\theta$ is an exact $n$th derivative. The case for which $n=1$ is completely covered by the well known Euler differential equation which arises in the simplest problem of the calculus of variations. For a function $\theta$ to be an exact first derivative, it is necessary and sufficient that $\theta$ satisfy the Euler differential equation. The contribution of this paper is the treatment of the cases in which $n$ exceeds unity. A system of $n$ differential equations is developed, satisfaction of which by $\theta$ constitutes a necessary condition that $\theta$ be an exact $n$th derivative. These equations do not yield an altogether satisfactory sufficient condition. It turns out that if $\theta$ satisfies the equations in question, it may still fail to be an exact $n$th derivative. However, under these circumstances, $\theta$ must differ from an exact $n$th derivative by a function of very special character.

*Notation.* Let us suppose $y$ to be an arbitrary function of $x$ possessing derivatives of all orders. We shall denote the $j$th derivative of $y$ with respect to $x$ by $y_j$, and sometimes denote $y$ itself by $y_0$. We suppose $\theta$ to be a function of $x$, $y$, and of finitely many of the $y_j$, possessing partial derivatives of all orders with respect to all its arguments. The operation of differentiation with respect to $x$ will be indicated by the symbol $D$; thus $D=\partial/\partial x+\sum y_{i+1}\partial/\partial y_i$. We shall understand that the range of the subscript $i$ in $D$ extends from zero to plus infinity, recognizing that when $D$ operates on a function of $x$, $y$, and of finitely many of the $y_j$ it reduces to a finite sum. The symbol $D^i$, where $i$ is a positive integer, will denote the operation of taking the $i$th derivative. We shall use the expression $C_{p,q}$ to denote the binomial coefficient $p\cdot(p-1)\cdots(p-q+1)/q!$ where $q$ is a nonnegative integer and $p$ is any integer.

*Summary of results.* Let $n$ be a positive integer. Let operators $E_t$, $t=1, \cdots, n$, be defined as follows. Expand, formally, $E_t=(1+D\partial/\partial y_1)^{-t}\partial/\partial y$ as the product by $\partial/\partial y$ of a power series in $D\partial/\partial y_1$, and replace terms $(D\partial/\partial y_1)^i\partial/\partial y$ by $D^i\partial/\partial y_i$. *Let there be a*

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*function* $\psi$ *of* $x$, $y$, *and of finitely many* $y_i$ *such that* $D^n\psi = \theta$ *identically in* $x$ *and* $y$. *Then*[1]

$$E_i\theta = 0, \qquad i = 1, 2, \cdots, n, \tag{1}$$

*identically in* $x$ *and* $y$.

We also establish a partial converse to this result. *Let* $\theta$ *satisfy equations* (1) *identically in* $x$ *and* $y$. *Then there exist polynomials in* $x$ $\pi_1, \pi_2, \cdots, \pi_{n-1}$, *where the degree of each nonzero* $\pi_i$ *is less than* $i$, *such that* $\theta - \pi_1 y_1 - \cdots - \pi_{n-1}y_{n-1}$ *is an exact nth derivative.* Note that no term in $y_0$ need be subtracted from $\theta$ to render $\theta$ exact, so that for $n$ equal to unity we see that $\theta$ itself is a first derivative if it satisfies (1). The equation $E_1\theta = 0$ is simply the Euler differential equation, so that for $n$ equal to unity our result restates the well known property of the Euler equation.

**Necessity proof.** We use the relations

$$\frac{\partial}{\partial y_r} D^n = \sum_{i=0}^{\min(r,n)} C_{n,i} D^{n-i} \frac{\partial}{\partial y_{r-i}}, \tag{2}$$

$$r = 0, 1, 2, \cdots; n = 1, 2, 3, \cdots,$$

which we proceed to establish. For $r$ equal to zero, this amounts to the statement that $(\partial/\partial y)D^n = D^n\partial/\partial y$, $n = 1, 2, 3, \cdots$. Clearly

$$\frac{\partial}{\partial y} D = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} + \sum y_{i+1}\frac{\partial}{\partial y_i}\right) = \frac{\partial^2}{\partial y \partial x} + \sum y_{i+1}\frac{\partial^2}{\partial y \partial y_i} \cdot$$

Since the order in which the partial derivatives are extracted can be reversed, this last expression equals $D\partial/\partial y$. This disposes of the case $r = 0$, $n = 1$, and the cases $r = 0$, $n$ arbitrary, follow immediately.

We now treat the cases in which $r$ is greater than zero. In evaluating $(\partial/\partial y_r)D$ we must consider that $y_r$ appears in the term $y_r\partial/\partial y_{r-1}$ as a coefficient of a partial derivative, which was not the case for $r$ equal to zero. We obtain then

$$\frac{\partial}{\partial y_r} D = D\frac{\partial}{\partial y_r} + \frac{\partial}{\partial y_{r-1}} \cdot$$

Consequently,

$$\frac{\partial}{\partial y_r} D^n = D\frac{\partial}{\partial y_r} D^{n-1} + \frac{\partial}{\partial y_{r-1}} D^{n-1}.$$

Proceeding by induction, we assume that the right member of this equation has a representation

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[1] These equations are written out explicitly as equations (4) below.

$$D\sum_{i=0}^{\min(r,n-1)} C_{n-1,i}D^{n-1-i}\frac{\partial}{\partial y_{r-i}} + \sum_{i=0}^{\min(r-1,n-1)} C_{n-1,i}D^{n-1-i}\frac{\partial}{\partial y_{r-1-i}}.$$

Collecting similar terms, we see that the coefficient of $D^{n-i}\partial/\partial y_{r-i}$ is $C_{n-1,i}+C_{n-1,i-1}$. It is well known that this sum is $C_{n,i}$. This suffices to prove that if relations (2) hold for values less than $n$ and less than or equal to $r$, they likewise hold for $n$ and $r$. It remains to discuss the transition from $n$ and $r-1$ to $n$ and $r$. This part of the proof is so similar to the foregoing discussion that the details may be omitted.

From now on $n$ is to be a positive integer, arbitrary but fixed.

We multiply both members of the $r$th of equations (2) by $D^r$, $r=1, 2, \cdots$. As a consequence, the power of $D$ which multiplies any given $\partial/\partial y_s$ is the same in all the modified equations (2) in which the expression appears, namely the $(n+s)$th power.

Denote the terms $D^r(\partial/\partial y_r)D^n$ by $\Lambda_r$, $r=0, 1, 2, \cdots$. After the above multiplication, equations (2) furnish us expressions for the $\Lambda_r$ as linear combinations, with numerical coefficients, of terms $D^{n+s}\partial/\partial y_s$. The numerical coefficients can be readily obtained. A term $D^{n+s}\partial/\partial y_s$ appears first in the expression for $\Lambda_s$, and is present, in all, in $\Lambda_s, \Lambda_{s+1}, \cdots, \Lambda_{s+n}$. Its numerical coefficient in each of these $\Lambda_{s+i}$ is $C_{n,i}$.

We now seek numbers $a_0, a_1, a_2, \cdots$, such that $\sum_{i=0}^{\infty} a_i\Lambda_i$ is identically zero; that is, in this sum the numerical coefficient of each term $D^{n+s}\partial/\partial y_s$ is to be zero. This implies that the $a_i$ must satisfy

$$\sum_{i=0}^{n} a_{r+i}C_{n,i} = 0, \qquad r = 0, 1, 2, \cdots. \tag{3}$$

In order to solve these equations with a minimum of computation, we formulate an equivalent problem involving power series. Let $u$ be an indeterminate and let $\sum a_iu^i$ be a formal power series in $u$, with numerical coefficients. Observe that in the product $(1+u)^n\sum a_iu^i$ the coefficient of $u^{n+r}$, $r=0, 1, 2, \cdots$, is precisely the left member of the corresponding equation (3). Thus if we can attribute values to the $a_i$ which make the above product a polynomial of degree less than $n$, these $a_i$ will also be solutions of (3). But the solution of this problem is immediate. Expand any one of the fractions $1/(1+u)^t$, $t=1, \cdots, n$, as a power series in $u$. The coefficients obtained in this way obviously serve as solutions for the polynomial problem, and, consequently, furnish solutions for equations (3).

The proof can now be quickly completed. Let $\theta$ be the $n$th derivative of a function $\psi$; in symbols $\theta = D^n\psi$. Then $D^r\partial\theta/\partial y_r = \Delta_r\psi$. Since $\sum a_r\Delta_r = 0$ we have $\sum a_r D^r\partial\theta/\partial y_r = 0$. Using explicit values for the $a_r$ we have

$$\frac{\partial\theta}{\partial y} + \sum_{r=1}^{\infty} C_{-t,r} D^r \frac{\partial\theta}{\partial y_r} = 0, \qquad t = 1, 2, \cdots, n. \tag{4}$$

The details of the proof make plain the connection between equations (4) and equations (1).

**Sufficiency proof.** We need the following lemma.

*Let $n$ be a positive integer and let $\pi(u)$ be a polynomial in $u$, not identically zero, whose degree is less than $n$. Then in the expansion of the function $\mu(u) = \pi(u)/(1+u)^n$ as a power series in $u$ the coefficients of no $n$ consecutive powers of $u$ are zero.*

This is obviously true for $n$ equal to unity, for the series of $1/(1+u)$ has no missing terms, and multiplication by a nonzero constant cannot introduce any such terms. We prove the lemma by induction on $n$. Representing $\pi(u)$ as a polynomial in $(1+u)$, we have $\pi(u) = b_0 + b_1(1+u) + \cdots + b_{n-2}(1+u)^{n-2} + b_{n-1}(1+u)^{n-1}$ where not all the $b_i$ are zero. After being multiplied by $(1+u)$ our function has the form

$$(1+u)\mu(u) = \left[b_0 + b_1(1+u) + \cdots + b_{n-2}(1+u)^{n-2}\right]/(1+u)^{n-1} + b_{n-1}. \tag{5}$$

The number of missing terms in $\mu(u)$ is related in a simple way to the number of missing terms in the product of this function by $(1+u)$. We distinguish two cases. If the first $n$ coefficients of $\mu(u)$ are zero, then they are likewise zero in the product. If, on the other hand, at least $n$ consecutive coefficients in $\mu(u)$ are zero, and if furthermore this gap is preceded by nonzero terms, then in the product at least $n-1$ consecutive coefficients are zero and this gap must also be preceded by nonzero terms. We show that each of these cases leads to a contradiction. We may suppose that not all of $b_0, b_1, \cdots, b_{n-2}$ are zero, for otherwise the problem reduces to the case of $n$ equal to unity. Thus we see from (5) that $(1+u)\mu(u)$ is the sum of a constant and a function which is subject to our induction hypotheses. This function can have at most $n-2$ consecutive missing terms. We have seen that if $\mu(u)$ is to lack $n$ consecutive terms, $(1+u)\mu(u)$ must lack at least $n-1$ such terms. We have also seen that in (5) the first summand can have at most $n-2$ missing terms. We conclude that it is the addition of $b_{n-1}$ to this summand which contributes another zero coefficient. But this can only be the case if it is the first $n-1$ terms of

$(1+u)\mu(u)$ which are zero. In this case our argument shows that if $\mu(u)$ lacked $n$ consecutive terms, $(1+u)\mu(u)$ would also have $n$ consecutive missing terms. This discrepancy proves the lemma.

We are now in a position to prove our partial converse to the result established above. Let $\theta$ be a function of $x$, $y$, and the $y_n$ such that the relations $E_i\theta=0$, $i=1, 2, \cdots, n$, are identities in $x$ and $y$. If the partial derivatives of $\theta$ with respect to all the $y_j$ vanish, then $\theta$ is a function of $x$ alone, and is certainly an exact $n$th derivative. We assume henceforth that not all these partial derivatives of $\theta$ vanish.

Let $m$ be the largest integer for which $\partial\theta/\partial y_m \neq 0$. Assuming that $m$ is not less than $n$, we are going to show that there exists a function $\psi$ such that $\theta - D^n\psi$ contains effectively no derivative $y_j$ of order greater than $n-1$. For this purpose we prove first that $\theta$ is linear in $y_m$, and that in $\theta$ the coefficient $\partial\theta/\partial y_m$ of $y_m$ is free of $y_{m-1}, y_{m-2}, \cdots, y_{m-n+1}$.

It follows from the theory of determinants that there exist constants $b_1, b_2, \cdots, b_n$, not all zero, such that $b_1E_1+b_2E_2+\cdots+b_nE_n$ is free of $D^i\partial/\partial y_i$ where $i$ has values from $m-n+1$ to $m-1$ inclusive. By virtue of our lemma, this linear combination of the $E_i$ must contain effectively the expression $D^m\partial/\partial y_m$. We suppose the $b_i$ selected in such a way that the coefficient of this expression is unity. Then, because $\theta$ does not contain a $y_s$ with $s$ greater than $m$, we have

$$\sum_{i=1}^{n} b_iE_i\theta = D^m\frac{\partial\theta}{\partial y_m} + cD^{m-n}\frac{\partial\theta}{\partial y_{m-n}} + \cdots + g\frac{\partial\theta}{\partial y}, \tag{6}$$

where $c, \cdots, g$ are constants. The operators $E_i$ are linear, so our assumption that $\theta$ is annulled by the $E_i$ implies that both members of (6) are zero. If the term $D^m\partial\theta/\partial y_m$ is zero, it must be that $\partial\theta/\partial y_m$ is a polynomial in $x$ of degree less than $m$. This certainly agrees with our statement concerning the manner in which $y_m$ is present in $\theta$. Assuming now that this term is not zero, we show that $\partial\theta/\partial y_m$ is free of $y_m, y_{m-1}, \cdots, y_{m-n+1}$. Suppose this is not so. Let $y_r$ be the derivative of greatest order effectively present in $\partial\theta/\partial y_m$. Then $y_{m+r}$ is effectively present in $D^m\partial\theta/\partial y_m$. The other terms of the right member of (6) cannot contain effectively derivatives of $y$ of order greater than $2m-n$, because the partial derivatives of $\theta$ are of order at most $m$ and the differentiations increase this order by at most $m-n$. Thus $m+r$ does not exceed $2m-n$, whence $r$ does not exceed $m-n$. This shows that $\theta$ differs from $y_m\partial\theta/\partial y_m$ by a function of order less than $m$, and that $\partial\theta/\partial y_m$ is free of $y_m, y_{m-1}, \cdots, y_{m-n+1}$.

We now introduce an auxiliary function $\psi_m$ described by the relation

$$\psi_m = \int \frac{\partial\theta}{\partial y_m}\, dy_{m-n},$$

where the integration is performed with respect to $y_{m-n}$, treating $x, y, y_1, \cdots, y_{m-n}$ as independent variables. Because of the use of the indefinite integral in its definition, $\psi_m$ contains an arbitrary additive function of $x, y, y_1, \cdots, y_{m-n-1}$; thus the integration actually leads to many functions, some of which may not possess partial derivatives of all orders. However, the integration also yields functions with partial derivatives of all orders, and it is from these that we suppose a definite $\psi_m$ to be selected. This function also has the property that its partial derivative with respect to $y_{m-n}$ is $\partial\theta/\partial y_m$. Because the order of $\partial\theta/\partial y_m$ is less than $m-n+1$ it follows that $D\psi_m$ differs from $y_{m-n+1}\partial\theta/\partial y_m$ by a function of order less than $m-n+1$. Similarly $D^n\psi_m$ differs from $y_m\partial\theta/\partial y_m$ by a function of order less than $m$. Then the order of $\theta - D^n\psi_m$ is less than $m$. In addition, this difference is annulled by all the operators $E_1, \cdots, E_n$. If the order of $\theta - D^n\psi_m$ exceeds $n-1$ we can follow the same procedure with this new function subtracting from it an exact $n$th derivative and reducing its order still further. After a finite number of steps we obtain a function $\psi$ such that the order of $\theta - D^n\psi$ is less than $n$.

Let us denote the difference $\theta - D^n\psi$ by $\tau$. It may be that $\tau$ is identically zero. In this case $\theta$ is an exact $n$th derivative. Even if $\tau$ is not zero, it must still be annulled by $E_1, \cdots, E_n$. Because the order of $\tau$ is less than $n$, the equations $E_i\tau = 0$, $i = 1, 2, \cdots, n$, constitute a system of $n$ homogeneous linear equations for the $n$ quantities $\partial\tau/\partial y, D\partial\tau/\partial y_1, \cdots, D^{n-1}\partial\tau/\partial y_{n-1}$. It follows from the lemma that the determinant of this system is not zero. We conclude that each of the quantities $\partial\tau/\partial y, D\partial\tau/\partial y_1, \cdots, D^{n-1}\partial\tau/\partial y_{n-1}$ is zero. Thus $\tau$ is a function of the form $\pi_1 y_1 + \pi_2 y_2 + \cdots + \pi_{n-1} y_{n-1}$ where each $\pi_i$ is a polynomial in $x$ of degree less than $i$. We do not overlook the fact that $\tau$ could also contain a term free of the $y_j$, but since such a term, being a function of $x$ alone, would be an exact $n$th derivative, we may suppose it incorporated into $\psi$. This completes the proof of the results enunciated at the beginning of the paper.

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# SPECTRAL THEORY

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Spectral theory has for its origin the classical theory of finite matrices and, in a sense, includes all theories which may be specialized to some particular phase of that classical theory. Such a meaning for the term spectral theory is perhaps too broad, for its acceptance would force one, as is becoming increasingly evident, to admit that spectral theory embraces not a small part of such fields of mathematics as the ergodic theory, the theory of probability, absolutely convergent trigonometric series, Fourier transforms, the theory of continuous functions on a bicompact Hausdorff space, the theory of partially ordered rings and Abelian groups, and perhaps even the well known embedding theorem concerning completely regular topological spaces. In fact, even though by far the greatest part of spectral theory that has been at present developed has for its roots the properties of a very special type of matrix, namely the Hermitian matrix, it is certainly true that modern developments have made significant contacts with all the theories just mentioned.

Although I shall touch briefly upon certain features of these modern developments my chief concern here will not be an attempt to describe the wealth of ideas that have grown out of the properties of an Hermitian matrix, but rather to show how the properties of a general matrix can guide the way to the solution of a special class of problems of vital interest in analysis. The type of problem with which I shall be concerned is illustrated by the following question. How can it be determined whether or not a given sequence of polynomials $f_n(T)$ in a linear operator $T$ on a Banach space converges to a specified type of limit operator? Before making this problem more precise I should like to show how the elementary properties of a finite matrix can, if put in the proper form, give an immediate answer to all such questions of convergence for a finite dimensional space.

Suppose $\mathfrak{X}$ is an $m$-dimensional linear vector space over the field of complex numbers (or over any algebraically closed field). Let $T$ be a linear operator in $\mathfrak{X}$, that is, $T$ is an $m\times m$ matrix whose elements are complex numbers. For every complex number $\lambda$ and every

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The address by Taylor which follows this was delivered on the same day before the Stanford meeting. To some extent the two papers overlap each other in subject matter, but they also complement each other through the divergence of the authors' points of view.

$n=0, 1, 2, \cdots$ define the linear manifolds

$$\mathfrak{N}_\lambda^n = (\lambda I - T)^n \mathfrak{X}, \qquad \mathfrak{M}_\lambda^n = \mathop{\mathcal{E}}_{x}\,[x \in \mathfrak{X}, (\lambda I - T)^n x = 0].$$

It is clear that for a fixed $\lambda$ the manifold $\mathfrak{M}_\lambda^n$ is non-decreasing in $n$ and the manifold $\mathfrak{N}_\lambda^n$ is nonincreasing. It is also clear that each $\lambda$ defines uniquely a positive integer or zero, $\nu=\nu(\lambda)$, called the ***index*** of $\lambda$, such that $\mathfrak{M}_\lambda^\nu=\mathfrak{M}_\lambda^n$ for $n \geqq \nu$ while for $n<\nu$ we have $\mathfrak{M}_\lambda^n$ a proper subset of $\mathfrak{M}_\lambda^{n+1}$. It is also readily shown that $\mathfrak{N}_\lambda^\nu=\mathfrak{N}_\lambda^n$ for $n \geqq \nu$ while for $n<\nu$ we have $\mathfrak{N}_\lambda^{n+1}$ a proper subset of $\mathfrak{N}_\lambda^n$. If $n \geqq \nu(\lambda)$ we always have

$$\mathfrak{X} = \mathfrak{N}_\lambda^n \oplus \mathfrak{M}_\lambda^n, \tag{1}$$

and conversely if (1) holds then $n \geqq \nu(\lambda)$. Let $\lambda_1, \cdots, \lambda_k$ be the distinct characteristic numbers of $T$ and let $\nu_1, \cdots, \nu_k$ be the corresponding indices, then

$$\mathfrak{X} = \mathfrak{M}_{\lambda_1}^{\nu_1} \oplus \cdots \oplus \mathfrak{M}_{\lambda_k}^{\nu_k}. \tag{2}$$

Equation (2) shows that there are uniquely defined projection operators (that is, idempotent matrices) $E_{\lambda_1}, \cdots, E_{\lambda_k}$ such that

$$I = E_{\lambda_1} + \cdots + E_{\lambda_k}, \quad E_{\lambda_i}\mathfrak{X} = \mathfrak{M}_{\lambda_i}^{\nu_i}, \quad E_{\lambda_i} E_{\lambda_j} = 0, \qquad i \neq j. \tag{3}$$

Since, in the space $\mathfrak{M}_{\lambda_i}^{\nu_i}$, any polynomial $f(T)$ in $T$ is equal to its residue modulo $(T-\lambda_i I)^{\nu_i}$ we have from (3)

$$f(T) = \sum_{i=1}^{k} \sum_{j=0}^{\nu_i - 1} \frac{(T - \lambda_i I)^j}{j!} f^{(j)}(\lambda_i) E_{\lambda_i}. \tag{A}$$

While our chief purpose will be to investigate some of the consequences of this formula (A), or rather the formula corresponding to (A) in the infinite case, we should like to point out very briefly the relation of this formula to the special case which has led to so many fruitful investigations. If the indices $\nu_i$ of $T$ are all 1, in particular if $T$ is an Hermitian matrix, then (A) becomes

$$f(T) = \sum_{i=1}^{k} f(\lambda_i) E_{\lambda_i}. \tag{B}$$

The important algebraic difference between cases (A) and (B) is that, as is easily seen, in case (B) the ring of polynomials in $T$ is isomorphic with a direct sum of fields, that is, isomorphic with a linear space of scalar functions, while in case (A) the ring of all polynomials in $T$ is isomorphic with a direct sum of reduced polynomial rings. Another way of seeing the chief difference between the two cases is to observe that in case (B) we have no nilpotent elements, that is, no radical in the ring, while in case (A) we may have $f^n(T)=0$, $f(T)\neq 0$. It is formula (B) which symbolizes the classical theory of reduction of quadratic forms. This theory was extended to infinite forms by Hilbert and his school and by E. H. Moore and his students. It is perhaps just to say that it is F. Riesz to whom we owe the advantages of the concept of a linear operator. To F. Riesz we owe the formula

$$f(T) = \int_{-\infty}^{+\infty} f(\lambda)dE_\lambda \tag{B$'$}$$

which replaces (B) in the case of a bounded self-adjoint operator in Hilbert space. It was J. von Neuman and M. H. Stone who developed the theory symbolized by (B′) in the case where $T$ is an unbounded self-adjoint operator in Hilbert space. In recent years the formula (B′) has been rapidly disappearing from the theory of quadratic forms. In fact, the theory of bounded quadratic forms has become, in the hands of Stone,[1] Gelfand,[2] Kakutani,[3] and others, almost identical with the theory of continuous real functions on a bicompact Hausdorff space. Each of these mathematicians has, I believe, taken roughly the point of view just mentioned, that is, that a commutative family of Hermitian matrices may be thought of as a family of scalar functions and each one has been led to a characterization of the family of *all* real continuous functions on a bicompact Hausdorff space. While their starting points are somewhat different there are, in each case, postulates sufficient to rule out a radical in the system, that is, these mathematicians have all been working with systems generalizing formula (B).

With this very incomplete summary of what has been done with formula (B) let us return to (A) and ask the question:—when does a given sequence $f_n(T)$ of polynomials in a general matrix $T$ converge? We shall describe two ways of answering this question, the first being fairly obvious, the second less so but in many ways more practical than the first. The first method is illustrated by the following theorem.

THEOREM 1. *The sequence $f_n(T)$ converges if and only if $f_n^{(j)}(\lambda_i)$ con-*

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[1] M. H. Stone, *A general theory of spectra* I, Proc. Nat. Acad. Sci. U. S. A. vol. 26 (1940) pp. 280–283.

[2] I. Gelfand, *Normierte Ringe*, Recueil Math. (Mat. Sbornik) N.S. vol. 51 (1941) pp. 3–24.

[3] S. Kakutani, *Concrete representation of abstract (M) spaces*, Ann. of Math. vol. 42 (1942) pp. 994–1024.

*verges for each* $i=1, \cdots, k$ *and* $j=0, \cdots, \nu_i-1$. *Also* $f_n(T)\to f(T)$ *if and only if* $\lim_n f_n^{(j)}(\lambda_i)=f^{(j)}(\lambda_i)$, $i=1, \cdots, k$, $j=0, \cdots, \nu_i-1$.

The sufficiency of the conditions is obvious from (A) while their necessity is an immediate consequence of the definition of $\nu_i$.

We shall reserve any general statement of the second method until later and here merely illustrate it by a typical example.

EXAMPLE. *The sequence* $(\sum_{\nu=0}^{n-1}T^\nu)/n$ *converges if and only if* $T^n/n\to 0$.

The truth of this statement is readily seen by applying Theorem 1. For if $f_n(\lambda)=\lambda^n/n$ then $f_n(\lambda)\to 0$ if and only if $|\lambda|\leq 1$. Whereas all derivatives $f_n^{(j)}(\lambda)\to 0$ if $|\lambda|<1$, not even the first derivative $f^{(1)}(\lambda)\to 0$ if $|\lambda|=1$. Thus by Theorem 1 we can say that $T^n/n\to 0$ if and only if all roots $\lambda_i$ of $T$ have $|\lambda_i|\leq 1$, and also every root $\lambda_i$ of $T$ with $|\lambda_i|=1$ has the corresponding index $\nu_i=1$. A similar application of Theorem 1 shows that $(\sum_{\nu=0}^{n-1}T^\nu)/n$ converges if and only if the spectral points of $T$ satisfy precisely the same conditions.

It is not pure coincidence that these two statements about convergence have precisely the same spectral interpretation. It is a consequence of a general principle which will be discussed more fully later.

The formal difference that we wish to emphasize between the theorem and the example is obvious; namely, in the example there is no mention of the spectrum of $T$. It is for this reason that a theorem of the type of the example, that is, a theorem asserting the equivalence of two statements about convergence, is in many ways more practical than one like Theorem 1. For example if a deck of cards is repeatedly shuffled in such a way that in any given shuffle there is a definite probability $p_{ij}$ that the card in the $i$th place goes into the card in the $j$th place, and this probability is fixed and independent of preceding shuffles, then one knows immediately that the probability of finding in the $j$th place, after $n$ shuffles, the card which was originally in the $i$th place is the element $p_{ij}^{(n)}$ of the $n$th iterate of the matrix $P=(p_{ij})$. Hence it is seen without any spectral considerations that $P^n/n\to 0$. Similarly there are many examples of linear operators $T$ in analysis where for one reason or another (perhaps because $|T^n|\leq M$) it is known that $T^n/n\to 0$, and hence a theorem of the type of the example could be much more easily applied than one like Theorem 1.

Our main objective will now be the description of a method for arriving at a class of theorems similar to the example, that is, theorems asserting the equivalence of two statements about convergence, but in the case where $T$ is a bounded linear operator on a complete normed linear vector space not necessarily of finite dimension. In this general situation there are several different but closely related problems, for there are usually three and sometimes four different meanings for convergence. We always have the notions of convergence in the uniform, strong, and weak topologies, and, in case the space in question consists of measurable functions we have the notion of convergence almost everywhere. I wish to outline a unified theory of convergence, in each of the four types, of a sequence $f_n(T)$ of functions of an operator $T$ to a *specified* type of limit operator.

In brief our method for arriving at this goal consists of two steps. The first and easiest step is the establishment of an operational calculus to replace formula (A). The second and more difficult one is the matter of applying the operational calculus. Here the chief tools are theorems corresponding to (a) the minimal equation theorem for matrices, (b) the theorem of Sylvester concerning the determinant of a polynomial in a matrix and (c) the various decompositions of the whole space into a direct sum of its subspaces determined by the spectrum of a matrix.

Returning our attention for a moment to a finite matrix $T$ and in particular to equation (A) we observe that one formula, and perhaps the only one, which may be used to replace (A) and which does not in any way exhibit the finite character of the space $\mathfrak{X}$ or the operator $T$ is

$$f(T) = \frac{1}{2\pi i}\int_C \frac{f(\lambda)d\lambda}{\lambda I - T}, \tag{A$'$}$$

where $C$ is a set of small circles $C_1, \cdots, C_k$ with $C_i$ surrounding $\lambda_i$. That formula (A′) is valid for any polynomial $f(T)$ may be verified by direct computation or it may be derived from (A) by using the Hermite interpolation formula. Conversely, it is not difficult to pass from (A′) to (A) without going through the preliminary algebraic arguments used to derive (A). The reason for taking $C$ as a set of small circles each one of which surrounds one and only one spectral point of $T$ rather than taking $C$ as one large circle containing the whole spectrum is the following. It is essential for our purposes that we do not have to restrict the operational calculus expressed by (A′) to the class of functions $f(\lambda)$ which are regular and single-valued on a *connected domain* containing the spectrum of $T$. We shall consider the class $\mathcal{F}(T)$ of functions $f(\lambda)$ which are regular and single-valued in the closure of a domain $\mathfrak{D}$ (which may vary with the function $f \in \mathcal{F}(T)$) satisfying the conditions

(i) $\mathfrak{D}$ is a finite sum of connected open sets $\mathfrak{D}_i$ with $\overline{\mathfrak{D}}_i\overline{\mathfrak{D}}_j=0$, $i\neq j$.

(ii) The boundary $C$ of $\mathfrak{D}$ consists of a finite number of disjoint closed rectifiable Jordan curves contained in the resolvent set $\rho(T)$ of $T$.

(iii) The spectrum $\sigma(T)$ of $T$ is contained in $\mathfrak{D}$.

A domain $\mathfrak{D}$ satisfying (i) and (ii) but not necessarily (iii) will be called a *$T$-admissible domain*, in symbols $\mathfrak{D}=\mathfrak{D}(T)$. These are the requirements whether $T$ be a finite matrix, a bounded linear operator on a complex Banach space, or an element of a normed ring. Thus for example in the case of the finite matrix we may take $f(\lambda)\equiv 1$ in and on $C_i$ and $f(\lambda)\equiv 0$ in and on $C_j$, $j\neq i$, and if we do the result is

$$E_{\lambda_i}=\frac{1}{2\pi i}\int_{C_i}\frac{d\lambda}{\lambda I-T}\cdot$$

This result, as well as the general one expressed by (A′), was discovered in 1928 by Fantappié,[4] who gave (A′) as a basis for an operational calculus on finite matrices. Fantappié proved the standard rules for such a calculus, namely,

($\alpha$) $(\alpha f+\beta g)(T)=\alpha f(T)+\beta g(T)$.

($\beta$) $(f\ g)(T)=f(T)\cdot g(T)$.

($\gamma$) If $f(\lambda)=\sum_{n=1}^{m}\alpha_n\lambda^n$ then $f(T)=\sum_{n=1}^{m}\alpha_n T^n$.

It was pointed out by E. Cartan[5] that formula (A′) could be used as a basis for an operational calculus on infinite matrices. In what follows we shall think of $T$ as a bounded linear operator on a complex Banach space and shall write $\rho(T)$ for the resolvent set of $T$, that is, the set of complex numbers for which $R_\lambda(T)=(\lambda I-T)^{-1}$ exists as an everywhere defined and hence bounded linear operator. The symbol $\sigma(T)$ will be used for the spectrum of $T$, that is, the complex numbers not in $\rho(T)$. It should be recalled that the index $\nu=\nu(\lambda)$ (as defined for finite matrices) of a point $\lambda\in\sigma(T)$ may be 0 or $\infty$. Recalling that $R_\lambda(T)$ is analytic on $\rho(T)$ we define for every $f\in\mathfrak{F}(T)$ the operator $f(T)$ by the formula (A′), where the integral is taken around the boundary $C$ of any $T$-admissible domain $\mathfrak{D}$ which contains $\sigma(T)$ and upon which $f(\lambda)$ is regular and single-valued. We shall add to the basic rules $(\alpha)$, $(\beta)$, $(\gamma)$ of Fantappiè two more, namely,

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[4] L. Fantappié, *La calcul des matrices*, C. R. Acad. Sci. Paris vol. 186 (1928) pp. 619–621. In the case where $f(\lambda)$ is a power series and $C$ is a circle, formula (A′) has been given by H. Poincaré, *Sur les groupes continus*, Transactions of the Cambridge Philosophical Society vol. 18 (1900) pp. 220–255.

[5] The suggestion was made in a letter to G. Giorgi. For this reference as well as for others pertaining to an operational calculus see C. C. MacDuffee, *The theory of matrices*, Berlin, 1933.

$(\delta)$ $f(\sigma(T))=\sigma(f(T))$, $f\in\mathcal{F}(T)$.

$(\epsilon)$ If $f\in\mathcal{F}(T)$, $g\in\mathcal{F}(f(T))$, and $F(\xi)=g(f(\xi))$ then $F\in\mathcal{F}(T)$ and $F(T)=g(f(T))$.

A frequently used corollary of $(\beta)$ is the following. If $f\in\mathcal{F}(T)$ is identically 1 in a neighborhood of a spectral set $\sigma$ of $T$ (a spectral set $\sigma$ of $T$ is a subset of $\sigma(T)$ which is open and closed in $\sigma(T)$) and identically zero in a neighborhood of the complement $\sigma'=\sigma(T)-\sigma$ of $\sigma$ then $f(T)$ is a projection. In other words the operator $E_\sigma[T]=(1/2\pi i)\int_C R_\lambda(T)d\lambda$, where $C$ is the boundary of a $T$-admissible domain $\mathfrak{D}$ with $\sigma=\mathfrak{D}\cdot\sigma(T)$, is a projection. This result was proved recently by Lorch[6] who used methods independent of an operational calculus. Lorch also showed that the Boolean algebra of all such projections is isomorphic with the Boolean algebra of all spectral sets of $T$. The fact that the mapping $\sigma\rightarrow E_\sigma[T]$ is a homomorphism is readily seen from $(\alpha)$, $(\beta)$. That it is actually an isomorphism follows from the following fundamental principle.

THEOREM 2. *When $T$ is considered as an operator in the space $\mathfrak{X}_\sigma=E_\sigma[T]\mathfrak{X}$ it has $\sigma$ for its spectrum. Furthermore for $\lambda\in\sigma$ any one of the following statements is true if and only if it is true when $T$ is regarded as an operator in $\mathfrak{X}_\sigma$: $\lambda$ is in the point spectrum of $T$, $\lambda$ is in the residual spectrum of $T$, $\lambda$ is in the continuous spectrum of $T$, $\lambda$ is a pole of order $\nu$ for $R_\xi(T)$, $\lambda$ is an essential singularity for $R_\xi(T)$.*

This is all we shall say about the formal operational calculus. The finite dimensional space has been replaced by a complex Banach space $\mathfrak{X}$, the matrix by a linear operator $T$ in $\mathfrak{X}$, the class of polynomials by the class $\mathcal{F}(T)$, and the formula (A) by the formula (A'). Now it is perfectly evident that the arguments used to derive Theorem 1 from formula (A) can not be applied immediately to formula (A'). Certain elementary facts can, however, be stated immediately. On the one hand, as a necessary condition for the convergence of a sequence $f_n(T)$ where $f_n\in\mathcal{F}(T)$, we have from $(\alpha)$ and $(\delta)$ that $|f_n(\lambda)-f_m(\lambda)|\leq|f_n(T)-f_m(T)|$, $\lambda\in\sigma(T)$, so that if $f_n(T)$ converges in the uniform topology of operators to an operator $U$ we must have $f_n(\lambda)$ converging uniformly for $\lambda\in\sigma(T)$ to a continuous function $f(\lambda)$ defined for $\lambda\in\sigma(T)$. This function $f(\lambda)$ which will be mentioned later is called the *spectral function* of $U$ and depends only on $U$ and not on the particular sequence $f_n(T)$ with $f_n \in \mathcal{J}(T)$ used to approach $U$. On the other hand it is clear from (A′) that the uniform convergence of $f_n(\lambda)$ on a *fixed* (that is, independent of $n$) $T$-admissible domain containing $\sigma(T)$ is a sufficient condition for the convergence of $f_n(T)$. Unfortunately, however, most of the interesting problems in analysis of this type are concerned with a sequence $f_n(\lambda)$ whose points of divergence have a spectral point of $T$ as a limiting point. Take for example the ergodic theorem and suppose that $\lambda = 1$ is in the spectrum $\sigma(T)$ (this is the only non-trivial case). The sequence of polynomials $(\sum_{v=0}^{n-1} \lambda^v)/n$ clearly diverges at points in every neighborhood of $\lambda = 1$. Thus it seems necessary to seek for more refined methods. To this end we shall virtually eliminate formula (A′) by deriving from it certain fundamental results of an algebraic nature which can be used as a basis for the convergence theory. We shall describe here three of the principal theorems of this type.

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[6] E. R. Lorch, *The spectrum of linear transformations*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 238–248.

**The minimal equation theorem.** For a finite matrix $T$ this theorem is well known and asserts that a polynomial $f(T)$ in $T$ vanishes if and only if $f(\lambda)$ contains $\prod_{i=1}^{k} (\lambda - \lambda_i)^{\nu_i}$ as a factor. The statement for a general linear operator is almost identical and reads as follows.

THEOREM 3 (THE MINIMAL EQUATION THEOREM). *If* $f \in \mathcal{J}(T)$ *then* $f(T) = 0$ *if and only if*

(i) *For every pole* $\lambda$ *of* $R_\xi(T)$ *of order* $\nu$, $f^{(j)}(\lambda) = 0$, $j = 0, 1, \cdots, \nu - 1$.

(ii) $f(\lambda) = 0$ *on a neighborhood containing all the spectrum* $\sigma(T)$ *except perhaps poles of* $R_\xi(T)$.

A corollary of this is

COROLLARY 1. *If* $f, g \in \mathcal{J}(T)$ *then* $f(T) = g(T)$ *if and only if*

(i) *For every pole* $\lambda$ *of* $R_\xi(T)$ *of order* $\nu$, $f^{(j)}(\lambda) = g^{(j)}(\lambda)$, $j = 0, \cdots, \nu - 1$.

(ii) $f(\lambda) = g(\lambda)$ *for every* $\lambda$ *in a neighborhood containing all the spectrum* $\sigma(T)$ *except perhaps poles of* $R_\xi(T)$.

COROLLARY 2. *Let* $\lambda_1, \cdots, \lambda_k$ *be poles of* $R_\xi(T)$ *of orders* $\nu_1, \cdots, \nu_k$ *respectively. Let* $\sigma'$ *be the complement of the spectral set* $\sigma = (\lambda_1, \cdots, \lambda_k)$. *Then for every* $f \in \mathcal{J}(T)$ *we have*

$$f(T) = f(T)E_{\sigma'}[T] + \sum_{i=1}^{k} \sum_{j=0}^{\nu_i - 1} \frac{(T - \lambda_i I)^j}{j!} f^{(j)}(\lambda_i) E_{\lambda_i}.$$

Corollary 2 follows from Corollary 1 as follows. If $\psi(\lambda)$ is the characteristic function of a small neighborhood of $\lambda_i$ then by Corollary 1 and ($\beta$) we have $\psi(T)f(T) = \psi(T)P(T) = E_{\lambda_i}[T]P(T)$ where $P(\lambda) = \sum_{j=0}^{\nu_i - 1} (\lambda - \lambda_i)^j f^{(j)}(\lambda_i)/j!$

As is suggested by these theorems we have also the fact that if $\lambda$ is a pole of order $\nu$ for $R_\xi(T)$ then $\lambda$ has index $\nu$ and also the manifold $\mathfrak{N}_\lambda^\nu$ is closed and $\mathfrak{N}_\lambda^\nu = \mathfrak{N}_\lambda^n$ for $n \geqq \nu$, while $\mathfrak{N}_\lambda^{n+1}$ is a proper subset of $\mathfrak{N}_\lambda^n$ for $n < \nu$. Just as in the case of finite matrices (see equation (1)) there is the decomposition[7]

$$(1') \qquad \mathfrak{X} = \mathfrak{N}_\lambda^n \oplus \mathfrak{M}_\lambda^n, \qquad n \geqq \nu,$$

and *conversely if for some $n$ we have $\mathfrak{N}_\lambda^n$ a closed complement of $\mathfrak{M}_\lambda^n$ then $\lambda$ is a pole of order $\nu \leqq n$.*

When $\lambda$ is a pole of order $\nu$ we always have $\mathfrak{M}_\lambda^\nu = \mathfrak{X}_\lambda = E_\lambda[T]\mathfrak{X}$. When $\lambda$ is not a pole but has index $\nu$ we frequently have the decomposition

$$(1'') \qquad \mathfrak{X} = \overline{\mathfrak{N}_\lambda^\nu} \oplus \mathfrak{M}_\lambda^\nu.$$

**The Sylvester theorem.** This theorem generalizes the formula $\sigma(f(T)) = f(\sigma(T))$. For finite matrices this formula merely means that if $\lambda_1, \cdots, \lambda_k$ are the distinct roots of an $n \times n$ matrix $T$ then $f(\lambda_1), \cdots, f(\lambda_k)$ are the roots of $f(T)$. A theorem due to Sylvester asserts more, namely, if $\lambda_1, \cdots, \lambda_k$ are the distinct roots of $T$ and if $m_i$ is the multiplicity of $\lambda_i$ then

$$\text{(i)} \qquad \operatorname{Det}(f(T) - \lambda I) = \prod_{i=1}^{k} (f(\lambda_i) - \lambda)^{m_i}.$$

This means that if $\lambda_1, \cdots, \lambda_n$ are the roots of $T$, each repeated according to its multiplicity, then $f(\lambda_1), \cdots, f(\lambda_n)$ are the roots of $f(T)$, where the number of repetitions of a given number in this array is its multiplicity as a root of $f(T)$. Recalling that the multiplicity $m(\lambda, T)$ of a root $\lambda$ of $T$ of index $\nu$ is the number of linearly independent solutions of $(\lambda I - T)^\nu x = 0$, that is, $m(\lambda, T)$ is the dimension of $\mathfrak{M}_\lambda^\nu = \mathfrak{X}_\lambda[T] = E_\lambda[T]\mathfrak{X}$, we see that the Sylvester Theorem, that is, the formula

$$\text{(ii)} \qquad m(\mu, f(T)) = \sum_{\lambda \in \sigma} m(\lambda, T), \qquad \sigma = f^{-1}(\mu),$$

is an immediate consequence of

$$\text{(iii)} \qquad \mathfrak{X}_\mu[f(T)] = \sum_{\lambda \in \sigma} \mathfrak{X}_\lambda[T], \qquad \sigma = f^{-1}(\mu),$$

(where the sum on the right is a vector sum). Whereas formula (iii) states the equality of two manifolds, formula (ii) merely states the equality of their dimensions. A more desirable form of this theorem than (iii) is

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[7] At this point reference should be made to F. Riesz, *Über lineare Funktionalgleichungen*, Acta Math. vol. 41 (1918) pp. 71–98, who proved similar theorems for a compact operator.

$$E_\tau[f(T)] = \sum_{\lambda\in\sigma} E_\lambda[T] = E_\sigma[T], \tag{iv}$$

which states the equality of two projections rather than, as (iii) does, the equality of the manifolds upon which they project. It is, roughly speaking, the form (iv) that the Sylvester theorem takes in the infinite case. To be more precise we have

THEOREM 4 (THE SYLVESTER THEOREM). *Let $f\in\mathfrak{F}(T)$ and let $\tau$ be a spectral set of $f(T)$. Then $\sigma=\sigma(T)\cdot f^{-1}(\tau)$ is a spectral set of $T$ and $E_\tau[f(T)]=E_\sigma[T]$.*

This theorem has a further generalization which will be needed later. Let $\mathfrak{R}(T)$ be the ring of all operators $f(T)$ where $f\in\mathfrak{F}(T)$ and let $\overline{\mathfrak{R}(T)}$ be the ring of all operators which are limits in the uniform topology of operators of elements in $\mathfrak{R}(T)$. Then, as we have seen earlier, with each $U\in\overline{\mathfrak{R}(T)}$ is associated a unique continuous function $f(\lambda)$ defined on $\sigma(T)$. This spectral function also has the property that $f(\sigma(T))\subset\sigma(U)$. Of course if $U=g(T)\in\mathfrak{R}(T)$ then $f(\lambda)=g(\lambda)$ on $\sigma(T)$. In terms of these notions we have

THEOREM 4′ (THE GENERAL SYLVESTER THEOREM). *Let $f$ be the spectral function of an operator $U\in\overline{\mathfrak{R}(T)}$ and let $\tau$ be a spectral set of $U$. Then $\sigma=f^{-1}(\tau)$ is a spectral set of $T$ and $E_\sigma[T]=E_\tau[U]$.*

**A typical decomposition theorem.** There are many types of decomposition theorems that seem to be needed for a general theory of convergence. We shall content ourselves here with one illustration. This theorem to follow generalizes the theorem mentioned for finite matrices in connection with equation (1), and its comprehension needs the following concepts. Let $f\in\mathfrak{F}(T)$ and define the manifolds

$$\mathfrak{N}[f] = f(T)\mathfrak{X}, \qquad \mathfrak{M}[f] = \mathcal{E}\,[x\in\mathfrak{X}, f(T)x = 0].$$

THEOREM 5. *Let $f\in\mathfrak{F}(T)$ and suppose that $f(\lambda)$ is not identically zero on any of the domains in which it is regular. Let $\lambda_1, \cdots, \lambda_k$ be the roots of $f(\lambda)$ and $m_1, \cdots, m_k$ their multiplicities. Then the following statements are equivalent.*

(1) $\mathfrak{X}=\mathfrak{M}[f]\oplus\mathfrak{N}[f]$, $\mathfrak{N}[f]$ *is closed.*

(2) *For $i=1, \cdots, k$, $\lambda_i$ is either in the resolvent set $\rho(T)$ or else a pole of order $\nu_i\leq m_i$ of $R_\xi(T)$.*

(3) *The finite set* $\sigma=(\lambda_1, \cdots, \lambda_k)\sigma(T)$ *is a spectral set of* $T$ *and* $\mathfrak{X}_\sigma=\mathfrak{M}[f]$, $\mathfrak{X}_{\sigma'}=\mathfrak{N}[f]$.

When the roots $\lambda_i$ of $f(\lambda)$ are not poles of $R_\xi(T)$, perhaps not even isolated in the spectrum $\sigma(T)$, but have indices $\nu_i \leqq m_i$ we often have a decomposition of the form $\mathfrak{X}=\mathfrak{M}[f]\oplus\overline{\mathfrak{N}[f]}$.

**Some convergence theorems.** While it is clear that a certain type of convergence theorem may be obtained from Corollary 2 of the minimal equation theorem just as Theorem 1 was obtained from formula (A), we wish to restrict our attention here to a special case which presents unexpected features not found in the general situation. We shall be interested in when a given sequence $f_n(T)$, $f\in F(T)$, that is, a given sequence $f_n(T)\in R(T)$, converges to a special type of projection $E$. We shall first discuss the case of convergence in the uniform topology of operators. As simple examples show we do not in general have $\mathfrak{R}(T)=\overline{\mathfrak{R}(T)}$, that is, in general $U=\lim_n f_n(T)$ is not in $\mathfrak{R}(T)$. However in case $U$ is a projection $E$ it is necessarily in $\mathfrak{R}(T)$. This is the first of the two fundamental necessity theorems, and it is the necessity of the conditions in the case of convergence in the uniform topology of operators that presents the most difficulty.

THEOREM 6 (THE FIRST NECESSITY THEOREM). *Every projection* $E\in\overline{\mathfrak{R}(T)}$ *is in* $\mathfrak{R}(T)$ *and* $E=E_\sigma[T]$ *where* $\sigma$ *is a spectral set of* $T$ *and consists of all* $\lambda\in\sigma(T)$ *where the spectral function* $f(\lambda)$ *of* $E$ *has the value* 1. *Thus* $E=0$ *if and only if there are no such* $\lambda$.

This theorem is an immediate consequence of the general Sylvester theorem (Theorem 4′). To see this we observe the readily verified fact that the resolvent $R_\xi(E)$ of a projection $E$ is given by the equation $R_\xi(E)=E/(\xi-1)+(I-E)/\xi$, $\xi\neq 0, 1$. Thus $E_1[E]=E$, and if we take $\tau$ in Theorem 4′ to be the point $\lambda=1$ then $E=E_1[E]=E_\sigma[T]$ where $\sigma$ consists of all $\lambda\in\sigma(T)$ with $f(\lambda)=1$.

THEOREM 7 (THE SECOND NECESSITY THEOREM). *Let* $P$ *be a polynomial not identically zero and let* $f_n\in\mathfrak{J}(T)$. *If* $f_n(T)$ *converges to a projection* $E$ *with* $E\mathfrak{X}\subset\mathfrak{M}[P]$ *then either there are no* $\lambda\in\sigma(T)$ *where* $f_n(\lambda)\rightarrow 1$ *in which case* $E=0$ *or else*

(1) *The set* $\sigma$ *of points* $\lambda\in\sigma(T)$ *where* $f_n(\lambda)\rightarrow 1$ *consists of a finite number of poles* $\lambda_1, \cdots, \lambda_k$ *of* $R_\xi(T)$.

(2) *If* $\nu_1, \cdots, \nu_k$ *are the orders of the poles* $\lambda_1, \cdots, \lambda_k$ *then* $f_n(\lambda_i)\rightarrow 1$, $f_n^{(j)}(\lambda_i)\rightarrow 0$, $j=1, \cdots, \nu_i-1$, $i=1, \cdots, k$.

(3) *For* $i=1, \cdots, k$, $\lambda_i$ *is a root of* $P(\lambda)$ *and its multiplicity is at least* $\nu_i$.
(4) $\prod_{i=1}^{k}(\lambda_i I-T)^{\nu_i}E=0$.

This theorem is a consequence of the minimal equation theorem together with the preceding theorem. The proof goes as follows. Let $\sigma$ be the set of $\lambda\in\sigma(T)$ where $f_n(\lambda)\to 1$. Then, by Theorem 7, $\sigma$ is a spectral set of $T$ and $E=E_\sigma[T]$. Since $E\mathfrak{X}\subset\mathfrak{M}[P]$ we have $P(T)E_\sigma[T]=0$ and so (1), (3) and (4) follow immediately from the minimal equation theorem whereas (2) follows from its Corollary 2.

We have now outlined all of the difficult steps encountered in the proof of the following basic theorem. We shall omit the rest of the details.[8]

THEOREM 8. *Let* $P(\lambda)=\prod_{i=1}^{k}(\lambda_i-\lambda)^{\nu_i}$ *be a polynomial whose distinct roots are* $\lambda_1, \cdots, \lambda_k$. *Let* $f_n\in\mathcal{J}(T)$ *satisfy*
(1) $f_n(\lambda_i)\to 1$, $f_n^{(j)}(\lambda_i)\to 0$, $i=1, \cdots, k$, $j=1, \cdots, \nu_i-1$.
(2) $P(T)f_n(T)\to 0$.
*Then the following assertions are equivalent.*
(3) $f_n(T)\to E$, $E^2=E$, $E\mathfrak{X}=\mathfrak{M}[P]$.
(4) *For each* $i=1, \cdots, k$, $\lambda_i$ *is either in* $\rho(T)$ *or else a pole of* $R_\xi(T)$.
(5) *For* $i=1, \cdots, k$, $\lambda_i$ *is either in* $\rho(T)$ *or else a pole of* $R_\xi(T)$ *of order at most* $\nu_i$.
(6) $\mathfrak{X}=\mathfrak{N}[P]\oplus\mathfrak{M}[P]$, $\mathfrak{N}[P]$ *is closed.*
(7) $\mathfrak{N}_{\lambda_i}^{\nu_i+1}$, $i=1, \cdots, k$, *is closed.*

It is the two assertions "$P(T)f_n(T)\to 0$" and "$f_n(T)$ converges" (for if $f_n(T)$ converges it must, in view of (1), converge to a projection on $\mathfrak{M}(P)$) which in this situation (that is, in view of (1) and (4) for example) have precisely the same spectral significance. This theorem serves to illustrate the general principle mentioned in connection with the example following Theorem 1, for if $P(T)=I-T$ and $f_n(T)=(\sum_{\nu=0}^{n-1}T^\nu)/n$ then the condition $P(T)f_n(T)\to 0$ is precisely $T^n/n\to 0$. Thus for a transformation $T$ which satisfies either

(4′) The point $\lambda=1$ is either in the resolvent set $\rho(T)$ or a pole of $R_\xi(T)$,

or

(7′) $(I-T)^2\mathfrak{X}$ is closed,

we can say that the uniform ergodic theorem holds if and only if $T^n/n\to 0$, and conversely if the uniform ergodic theorem holds then so do (4′) and (7′). In fact if the uniform theorem holds we have

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[8] The details of many of the results discussed here appear in the Trans. Amer. Math. Soc. vol. 54 (1943) pp. 185–217.

(6′) $(I-T)\mathfrak{X}$ is closed,

and, as we shall see, this condition is also necessary and sufficient for the uniform theorem providing the strong (that is, mean) theorem holds and $T^n/n \to 0$.

We shall now pass to the case of strong convergence and state

THEOREM 9. *Let* $P(\lambda)=\prod_{i=1}^{k}(\lambda_i-\lambda)^{\nu_i}$ *be a polynomial whose distinct roots are* $\lambda_1, \cdots, \lambda_k$. *Let* $f_n \in \mathcal{J}(T)$ *satisfy*

(1) $f_n(\lambda_i)\to 1, f_n^{(j)}(\lambda_i)\to 0, i=1, \cdots, k, j=1, \cdots, \nu_i-1.$

(2) $P(T)f_n(T)\to 0$ *strongly.*

*Then the following assertions are equivalent.*

(3) $f_n(T)\to E$ *strongly*, $E^2=E$, $E\mathfrak{X}=\mathfrak{M}[P]$.

(4) $f_n(T)x$ *is weakly compact*, $x \in \mathfrak{X}$.

(5) $\mathfrak{X}=\overline{\mathfrak{N}[P]}\oplus\mathfrak{M}[P]$, $|f_n(T)| \leqq M$.

Thus in a reflexive space if (1) and $|f_n(T)| \leqq M$ are satisfied we can say that (2) and (3) are equivalent. This is another example where two statements about convergence have the same spectral significance. Taking $P(T)=I-T$, $f_n(T)=(\sum_{\nu=0}^{n-1}T^\nu)/n$, as before, we can say that in a reflexive space if $|(\sum_{\nu=0}^{n-1}T^\nu)/n| \leqq M$ then $(\sum_{\nu=0}^{n-1}T^\nu)/n$ converges strongly if and only if $T^n/n\to 0$ strongly. The usual condition $|T^n| \leqq M$ assumed in ergodic theorems is not a necessary one. Hille and Szegö[9] have recently constructed operators satisfying the strong (that is, mean) as well as the almost everywhere ergodic theorem, and for which $|T^n|$ is of the order of $n^{1/4}$. The essential algebraic difference between the strong and the uniform theorem is seen by a comparison of (6) of Theorem 8 and (5) of Theorem 9.

We shall omit any discussion of weak convergence and pass to the case of almost everywhere convergence. Here we have as a basis a space $S$ of points $t$. The space $\mathfrak{X}$ consists of scalar functions which are defined on $S$ and which are measurable with respect to a completely additive non-negative measure function $|e|$ defined for $e$ in a Borel field including $S$. It is assumed that $x=0$ if and only if $x(t)=0$ almost everywhere, and that addition and scalar multiplication of the functions correspond to the similar operations in $\mathfrak{X}$. It is assumed that if $x_n\to x$ in $\mathfrak{X}$ and $x_n(t)\to y(t)$ almost everywhere then $x(t)=y(t)$ almost everywhere. It is also assumed in case $|S|=\infty$ that $S$ is the denumerable union of sets of finite measure. For a linear operator $U$ in $\mathfrak{X}$ we shall write $U(x, t)$ for the value of the function $Ux$ at the point $t$.

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[9] E. Hille and G. Szegö, *Remarks on ergodic theorems*, to appear in the Trans. Amer. Math. Soc.

A fundamental theorem of Banach[10] enables us to state the following theorem as an immediate corollary of Theorem 9.

THEOREM 10. *Let* $P, f_n, E$ *be as in the preceding theorem and satisfy* (1), (2), (4) *of that theorem. Then*

(1) *For every* $x \in \mathfrak{X}$ *we have* $\lim_n f_n(T)(x, t) = E(x, t)$ *almost everywhere if and only if*

(2) *For every* $x \in \mathfrak{X}$ $\limsup_n |f_n(T)(x, t)| < \infty$ *almost everywhere and*

(3) *For every* $x$ *in a fundamental set in* $\mathfrak{X}$ *the* $\lim_n f_n(T)P(T)(x, t)$ *exists almost everywhere.*

Again let us interpret this theorem by taking $P(T) = I - T$, $f_n(T) = (\sum_{\nu=0}^{n-1} T^\nu)/n$ where $T$ is now a linear operator in the Lebesgue space $L(S)$, $|S| < \infty$, defined by a 1-1 point map $\phi$ of $S$ into itself according to the formula $Tf = g$, $g(t) = f(\phi t)$. Clearly (3) is trivially satisfied with no restrictions on $\phi$ by merely taking the fundamental set to be the set of bounded measurable functions. Thus (3) is redundant and (2) becomes the necessary and sufficient condition for almost everywhere convergence. But it has been proved[11] that $T$ satisfies the strong (that is, mean) ergodic theorem if and only if

$$\frac{1}{n} \sum_{\nu=0}^{n-1} |\phi^{-\nu} e| \leq M |e|, \tag{4}$$

and that this condition (4) implies (2). *Thus* (2) *is also redundant and* (4) *alone is the single condition which is equivalent to the strong theorem and which implies the almost everywhere theorem.* In the case where $\phi$ is measure preserving (2) has been established by G. D. Birkhoff,[12] N. Wiener,[13] K. Yosida and S. Kakutani,[14] and H. R. Pitt[15] by proving an inequality of the type

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[10] S. Banach, *Sur la convergence presque partout des fonctionelles linéaires*, Bull. Sci. Math. (2) vol. 50 (1926) pp. 36–43. This theorem of Banach has also been used by Yosida in generalizing the ergodic theorem of G. D. Birkhoff, see K. Yosida, *Ergodic theorems of the Birkhoff-Khintchine's type*, Jap. J. Math. vol. 17 (1940) pp. 31–36.

[11] This was discovered by D. S. Miller and the author who will publish the proof elsewhere.

[12] G. D. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. U. S. A. vol. 17 (1931) pp. 656–660.

[13] N. Wiener, *The ergodic theorem*, Duke Math. J. vol. 5 (1929) pp. 1–18.

[14] K. Yosida and S. Kakutani, *Birkhoff's ergodic theorem and the maximal ergodic theorem*, Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 165–168.

[15] H. R. Pitt, *Some generalizations of the ergodic theorem*, Proceedings of the Cambridge Philosophical Society vol. 38 (1942) pp. 325–343.

$$|e_\alpha| \leq \frac{1}{\alpha}\int_S f(t)dt, \tag{5}$$

where $e_\alpha$ is the set where $\sum_{\nu=0}^{n-1} f(\phi^\nu t)/n > \alpha$ and $f$ is positive. While the works of these authors make it clear that an inequality similar to (5) is valid providing $|\phi^{-\nu}e| \leq M|e|$ it is not at all clear whether or not (4) will give anything like (5). Nevertheless (4) implies (2) which is all that is needed for the almost everywhere ergodic theorem. I believe that the ergodic theorems outlined in these pages contain most of the known[16] ergodic theorems in the cyclic case.

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[16] K. Yosida and S. Kakutani, *Operator theoretical treatment of Markoff's process and mean ergodic theorem*, Ann. of Math. (2) vol. 42 (1941) pp. 188–228.

# ANALYSIS IN COMPLEX BANACH SPACES

ANGUS E. TAYLOR

1. **Introduction.** Abstract spaces, and Banach spaces in particular, have played a prominent rôle in recent years in connection with many problems of analysis. There has also been a notable tendency for the concepts and tools of analysis to take a place alongside the algebraic and topological notions which are characteristic of the whole subject of abstract spaces. Thus analysis becomes more algebraic, and at the same time its range is broadened.

For a great many purposes it is immaterial whether a Banach space be real or complex. It is well known that a large portion of the theory of linear operations, as developed in Banach's book (Banach [1][1]), is equally valid for complex or real spaces. There are, however, situations in which the complex number system plays a crucial rôle. The theory of analytic functions in Banach spaces is a case in point. There are two large divisions of this subject: the theory of functions of a complex variable, the values of the functions lying in a Banach space, and the theory of analytic functions of an abstract variable. Our principal concern in this paper will be the first of these two theories. A brief survey of the second theory, and references to the literature, are given in §8.

Henceforth, except as otherwise stated, we shall use the term Banach space to mean a complex Banach space. The algebraic structure of such a space is that of an additive Abelian group with the complex numbers as operators. The topology of the space is defined by a *norm*; the norm of an element $x$ is written $\|x\|$. It has the properties of an absolute value. Then $\|x-y\|$ is the distance between $x$ and $y$, and the space is assumed to be a complete metric space.

2. **Analytic functions.** The basic development of the theory of analytic functions with values in a complex Banach space $E$, the independent variable being a complex number, follows the pattern of classical analysis. It seems to have been pointed out first by Wiener [1] that Cauchy's integral theorem is valid in this general setting. The usual consequences, such as the Cauchy integral formula, Liouville's theorem, and the Taylor and Laurent expansions, then follow. It is only when we come to theorems that deal in some way with di-

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

vision of function values that significant differences appear.

3. **Weak, uniform, and strong analyticity.** If $E$ is an abstract Banach space, the space of all complex linear functionals defined on $E$ is also a Banach space, denoted by $E^*$. Let us consider a single-valued function $x(z)$ with values in $E$, where $z$ ranges over a region $D$ of the complex plane. If $f$ is an element of $E^*$, $fx(z)$ is a numerical function of $z$. If $fx(z)$ is regular in $D$ for each $f$ in $E^*$ we shall say that $x(z)$ is *weakly regular* in $D$.

THEOREM 1. *The function $x(z)$ is regular in $D$ if and only if it is weakly regular there.*

This theorem, in a slightly more general form, is due to Nelson Dunford (Dunford [1, Theorem 76]). Instead of requiring $fx(z)$ to be regular for each $f$ in $E^*$, it is sufficient to impose the requirement for each $f$ in a *determining manifold* in $E^*$. A subset $S$ of $E^*$ is called a determining manifold if it is a closed linear subspace of $E^*$, and if there is a constant $M$ such that for each $x$ in $E$ the least upper bound of $|f(x)|/\|f\|$, as $f$ ranges over $S$, is not less than $M\|x\|$.

The theorem depends upon the following lemma.

LEMMA 1. *If $T$ is an arbitrary range, and $x(t)$ is a function on $T$ into $E$ such that $fx(t)$ is bounded on $T$ for each $f$ in $E^*$ (or, more generally, for each $f$ in a determining manifold), then $\|x(t)\|$ is bounded on $T$.*

The proof of the lemma (Dunford [1, p. 354]) is of the category type. We shall give a new, brief proof of Theorem 1. Let us write

$$Q(z, h) = (x(z + h) - x(z))/h.$$

Suppose that $x(z)$ is weakly regular in $D$. Let $C$ be a circle of radius $r$, with center at $z$, which, together with its interior, lies in $D$. By Cauchy's integral formula

$$fQ(z, h) - fQ(z, k) = \frac{1}{2\pi i}\int_C fx(t)\,\frac{(h - k)dt}{(t - z - h)(t - z - k)(t - z)},$$

provided that $|h|$ and $|k|$ are less than $r$. Now, for each $f$, $fx(t)$ is bounded on $C$; hence, by the lemma, there is a constant $M$ such that $\|x(t)\| \leq M$ on $C$. Thus $|fx(t)| \leq M\|f\|$, and

$$|f[Q(z, h) - Q(z, k)]| \leq \frac{M|h - k|}{(r - |h|)(r - |k|)}\|f\|.$$

But then

$$\|Q(z, h) - Q(z, k)\| \leq \frac{M|h - k|}{(r - |h|)(r - |k|)}.$$

It follows, since $E$ is complete, that $Q(z, h)$ approaches a limit as $h \to 0$. This proves that $x(z)$ is regular in $D$. That weak regularity is a consequence of ordinary (strong) regularity is, of course, trivial.

The next theorem, discovered by the writer (Taylor [6, Theorem 2.1]), is closely related to Theorem 1. Let $E_1$ and $E_2$ be Banach spaces, and let $[E_1, E_2]$ denote the Banach space of all linear transformations of $E_1$ into $E_2$. The norm $\|T\|$ of a transformation $T$ is defined as the least upper bound of $\|Tx\|$ for all elements $x$ of $E_1$ such that $\|x\| = 1$. Let $A_\lambda$ be a family of linear transformations of $E_1$ into $E_2$, the parameter $\lambda$ ranging over an open set $D$ of the complex plane. We say that $A_\lambda$ is *strongly* analytic and regular in $D$ if for each $x$ in $E_1$ the function $A_\lambda x$ is regular as a function with values in $E_2$. We say that $A_\lambda$ is *uniformly* analytic and regular in $D$ if it is regular as a function with values in $[E_1, E_2]$.

THEOREM 2. *The family $A_\lambda$ is uniformly regular in $D$ if and only if it is strongly regular there.*

The proof of this theorem may be modeled very closely after the above proof of Theorem 1. Instead of Lemma 1 we use an analogous lemma about linear transformations (cf. Banach [1, p. 80, Theorem 5]). It is a corollary of Theorems 1 and 2 that the regularity of $\phi A_\lambda x$ for each $x$ in $E_1$ and each $\phi$ in $E_2^*$ implies the strong, and hence the uniform, regularity of $A_\lambda$. It is to be emphasized that Theorems 1 and 2 are not true if we substitute continuity for analyticity.

4. **Concrete Banach spaces.** A space $E$ is said to possess a denumerable basis $\{x_n\}$ if to each $x$ in $E$ there corresponds a unique sequence of numbers $\{u_n\}$ such that

$$x = \sum_{k=1}^{\infty} u_k x_k.$$

If we define $f_n(x) = u_n$, $f_n$ is an element of $E^*$ (Banach [1, p. 111]). A number of familiar spaces have bases with the property that a series $\sum_{k=1}^{\infty} a_k x_k$ converges if the norms of the partial sums $\sum_{k=1}^{n} a_k x_k$ are bounded (Dunford and Morse [1, p. 415]). A space with a basis of this sort will be said to have property $(A)$. The function spaces $L^p$ $(p > 1)$, and the sequence spaces $l^p$ $(p \geqq 1)$ are examples. The space $(c)$ of convergent sequences has a basis, but it does not have property $(A)$.

Theorem 3. *Let $E$ be a space with a denumerable basis. Let $\{u_n(z)\}$ be a sequence of numerical functions of the complex variable $z$, each analytic and regular in a region $D$, and such that*

(a) *to each compact*[3] *subset $S$ of $D$ and each $f$ in $E^*$ there corresponds a constant $M$ such that $|f(\sum_{k=1}^{n} u_k(z)x_k)| \leqq M$ if $z$ is in $S$ and $n=1, 2, 3, \cdots$;*

(b) *the series $\sum_{k=1}^{\infty} u_k(z)x_k$ converges if $z$ is in $D$.*

*Then the function $x(z)=\sum_{k=1}^{\infty} u_k(z)x_k$ is analytic and regular in $D$. Its derivative is $x'(z)=\sum_{k=1}^{\infty} u_k'(z)x_k$.*

We first observe that if the space has property $(A)$, condition (b) is automatically satisfied when (a) is, by virtue of Lemma 1. Because of Theorem 1, it is enough to prove that $x(z)$ is weakly regular. This, however, follows directly from (a) and (b), by Montel's theorem. We leave it for the reader to show that $x'(z)$ is given by the series indicated.

Theorem 3 furnishes a practical method for testing analyticity. For example, it shows at once that a sequence $\{u_n(z)\}$ of functions with a common region of regularity $D$ may be regarded as an analytic function on $D$ into $l^p$ if the partial sums $\sum_{k=1}^{n} |u_k(z)|^p$ are uniformly bounded in each compact subset of $D$. For other results of the same type, see Taylor [6, Theorem 3.1].

Consider next the space $C$ whose elements are complex functions $x(t)$ continuous on the closed interval $[a, b]$, with the norm

$$\|x\| = \max_{a \leqq t \leqq b} |x(t)|.$$

If $x(t, z)$ is a function of $t$ and $z$ which belongs to $C$ for each value of $z$, we shall use the notation $x(\cdot, z)$ when we wish to regard $x(t, z)$ as a function of $z$ with values in $C$. Bôcher [1] used the term *semi-analytic* to describe functions $x(t, z)$ which are continuous in the pair $t$, $z$ and analytic as functions of $z$. Bôcher's results concerning such functions, insofar as the dependence upon $z$ is concerned, may be regarded as corollaries of the abstract theory with which we are concerned, as soon as it is shown that the hypothesis of semi-analyticity is sufficient to make $x(\cdot, z)$ an analytic function of $z$. The following theorem proves this and slightly more: instead of continuity jointly in $t$ and $z$ it is enough to require boundedness.

Theorem 4. *A function $x(t, z)$ defines a function $x(\cdot, z)$ regular in a region $D$ if and only if*

(a) *$x(t, z)$ is continuous on $[a, b]$ for each $z$ in $D$;*

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[3] Compact sets, as we use the term, are closed.

(b) *$x(t, s)$ is regular in $D$ for each $t$ on $[a, b]$;*

(c) *to each compact set $S$ in $D$ there corresponds a constant $M$ such that $|x(t, s)| \leq M$ if $t$ is on $[a, b]$ and $s$ is in $S$.*

By employing Cauchy's integral formula much as in the proof of Theorem 1 we can show directly that the difference quotient

$$(x(t, s + h) - x(t, s))/h$$

converges to its limit as $h \to 0$, uniformly with respect to $t$. But this means that

$$(x(\cdot, s + h) - x(\cdot, s))/h$$

converges to a limit in the topology of $C$. The conditions (a)–(c) are thus seen to be sufficient. Their necessity is trivial.

The continuity of $x(t, s)$ and $\partial x/\partial s$ jointly in $t$ and $s$ is a consequence of the regularity of $x(\cdot, s)$.

The situation in the space $L^p$ is much more complicated. Wiener [1] studied power series in $s$ whose coefficients are functions in $L^2$. Taylor [6, Theorem 3.2] showed that a function $F(s)$ which is analytic in $D$, with values in $L^p$, can be represented by a function $x(t, s)$ which is defined when $s$ is in $D$ and $t$ is on the interval $[a, b]$, but not in a certain fixed set $N$ of measure zero. The significant thing is that this exceptional set is independent of $s$. The function $x(t, s)$ is regular as a function of $s$, and the integral $\int_a^b |x(t, s)|^p dt$ is bounded in each compact subset of $D$. The latter conditions are, conversely, sufficient to determine $x(\cdot, s)$ as an analytic function on $D$ into $L^p$.

In the paper referred to above, Taylor studied abstract linear differential systems of the forms

$$dx/d\lambda = A_\lambda x \quad \text{and} \quad dx/d\lambda = A_\lambda x + s(\lambda),$$

where $A_\lambda$ is a linear transformation depending analytically on $\lambda$, and $s(\lambda)$ is a given abstract function of $\lambda$. In concrete realizations of the abstract theory such systems include a variety of types of equations, such as integro-differential equations, differential equations in an infinity of unknowns, and so forth.

5. **Singularities.** Isolated singularities may be classified, as in classical theory, under the headings: (1) removable singularities, (2) poles, (3) essential singularities. An isolated singularity is removable if it disappears when a proper definition is given to the function at the point in question. We can prove Riemann's theorem: an isolated singularity $s_0$ is removable if $\|x(s)\|$ is bounded in the neighborhood of $s_0$. An isolated singularity $s_0$ is a pole if, for a suitable positive

integer $m$, $(z-z_0)^m x(z)$ has a removable singularity at $z_0$. The smallest admissible integer $m$ is called the order of the pole. An equivalent definition may be based upon the nature of the Laurent expansion about the point $z_0$. By Lemma 1 and Riemann's theorem we may show that if, for each $f$ in $E^*$, $fx(z)$ has a pole of order not greater than $m$ at $z_0$, then $x(z)$ has a pole of order not greater than $m$; and if the pole is of order exactly $m$, then there is at least one $f$ in $E^*$ such that $fx(z)$ has a pole of order $m$ at $z_0$.

Let us consider spaces with denumerable bases. It appears from Theorem 3 that in such a space the singularity of a function $x(z)$ may arise either from a singularity of one of the component functions $u_n(z)$ or from the behavior of the series

$$x(z) = \sum_{k=1}^{\infty} u_k(z) x_k$$

as a whole. As an example, consider the Hilbert space $l^2$, and the function $x(z)$ with components $u_n(z) = z^n$. Since

$$\|x(z)\|^2 = \sum_{k=1}^{\infty} |z|^{2k},$$

it is clear that $x(z)$ is regular if $|z| < 1$, but not defined if $|z| \geqq 1$. The component functions, individually, give no indication of the fact that $|z| = 1$ is a natural boundary.

We can prove, however, that an isolated singularity is always traceable to one of the components.

THEOREM 5. *Suppose that the conditions of Theorem* 3 *obtain in a region* $D$: $0 < |z-z_0| < R$, *and suppose that each of the functions* $u_n(z)$ *is, in addition, regular at* $z_0$. *Then the series* $\sum_{k=1}^{\infty} u_k(z) x_k$ *converges if* $|z-z_0| < R$, *and defines a function regular, not only in* $D$, *but at* $z_0$.

To prove the theorem, consider an element $f$ of $E^*$, and the series

$$fx(z) = \sum_{k=1}^{\infty} u_k(z) f(x_k).$$

The terms of the series are regular if $0 \leqq |z-z_0| < R$. If $0 < R_1 < R$, there exists a constant $M$ such that

$$\left| \sum_{k=1}^{n} u_k(z) f(x_k) \right| \leqq M$$

if $|z-z_0| = R_1$ and $n = 1, 2, \cdots$. The same inequality must hold if

$0 \leq |z-z_0| < R_1$, by the maximum modulus theorem. Hence, letting $n \to \infty$, we deduce that $|fx(z)| \leq M$ if $0 < |z-z_0| \leq R_1$. By Lemma 1 we know that $x(z)$ is bounded in the same range of $z$. By Riemann's theorem, then, the possible singularity at $z_0$ is removable. The conclusion of the theorem is now easily proved.

6. **Resolvents.** It has long been recognized that the methods of the theory of functions of a complex variable are important in the theory of linear transformations. F. Riesz, in his monograph on the subject of infinite systems of linear equations (F. Riesz [1, pp. 117–121]) indicates briefly the potentialities of the calculus of residues as a means of studying the spectrum and the characteristic manifolds of a linear transformation.

Let $T$ be a linear transformation of the Banach space $E$ into itself. If the transformation $T_\lambda \equiv T - \lambda I$ ($I$ the identity) maps $E$ into itself in a one-to-one fashion, the inverse transformation, denoted by $R_\lambda$, will also be linear. We shall call $R_\lambda$ the *resolvent* of $T$, and the set of values of $\lambda$ for which $R_\lambda$ is defined the *resolvent set* $\mathfrak{R}$. All values of $\lambda$ not in $\mathfrak{R}$ belong to the *spectrum* of $T$. A number $\lambda$ is said to be in the *point spectrum* of $T$ if it is a characteristic value, that is, if there is an $x \neq 0$ such that $Tx = \lambda x$.

Our discussion will center around $R_\lambda$ as a function of $\lambda$. As far as questions of analyticity are concerned, it is immaterial whether we study $R_\lambda$ or $R_\lambda x$, because of Theorem 2. We shall study $R_\lambda$ directly.

THEOREM 6. *The resolvent* $R_\lambda$ *is analytic and regular in the resolvent set (which is open, but not necessarily connected). If* $\lambda$ *and* $\mu$ *are in* $\mathfrak{R}$,

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu, \tag{1}$$

$$(d^n/d\lambda^n) R_\lambda = n! R_\lambda^{n+1}. \tag{2}$$

*The set* $\mathfrak{R}$ *contains every* $\lambda$ *such that* $|\lambda| \geq \|T\|$, *and for these values*

$$R_\lambda = -\sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}. \tag{3}$$

*If* $C$ *is a contour enclosing the spectrum of* $T$, *then*

$$T = -\frac{1}{2\pi i}\int_C \lambda R_\lambda d\lambda. \tag{4}$$

Formula (1) is the well known functional equation of the resolvent, familiar in the theory of integral equations. Formula (3) is the Neumann expansion of the resolvent, in abstract form. Formula (4) appears to have been overlooked until now. For the proof of (1)–(3), see Taylor [2]. To obtain (4), multiply (3) by $\lambda$ and integrate, after deforming the contour, if necessary. Other references are: Stone [1, pp. 139–141]; Hildebrandt [1, pp. 195–199]; Nagumo [1]; Taylor [6, pp. 590–593]; Lorch [1].

The resolvent set is the maximal domain of regularity of $R_\lambda$. All the singular points of $R_\lambda$ belong to the spectrum of $T$, and a point of the spectrum is a singularity of $R_\lambda$, provided that it is a limit point of the resolvent set. The resolvent $R_\lambda$ is regular at infinity, and vanishes there, by (3). Hence, by Liouville's theorem, it is either identically zero (in the trivial case that the space $E$ consists of a single element), or the spectrum of $T$ contains at least one point.

With each simple closed contour $C$ lying in $\mathfrak{R}$ we may associate a transformation $P$ defined as follows:

$$P = -\frac{1}{2\pi i}\int_C R_\lambda d\lambda. \tag{5}$$

These transformations play a fundamental rôle in the study of the singularities of $R_\lambda$. Their use in studying the spectrum has recently been indicated by Lorch. We summarize the salient properties of these transformations in a theorem (Lorch [1, Theorems 5 and 6]).

THEOREM 7. *The transformation $P$ is a projection, that is, $P^2=P$. It is permutable with $T$. If $C_1$ and $C_2$ are two contours in $\mathfrak{R}$, and if $P_1$, $P_2$ are the corresponding projections, $P_1P_2=P_2P_1$, and $P_1P_2=P_1$ if $C_1$ lies inside $C_2$, while $P_1P_2=0$ if $C_1$ lies outside $C_2$.*

In proving this theorem, and in many other calculations involving contour integrals, equation (1) provides a valuable key. It may be shown that $P=I$ if and only if the spectrum of $T$ lies within $C$, and that $P=0$ if and only if the spectrum of $T$ lies outside $C$.

A partial analysis of the structure of $T$ and $R_\lambda$ is possible with the aid of $P$. There is not space in this paper for a detailed account. Let us consider merely the case of an isolated point $\lambda_0$ of the spectrum of $T$, that is to say, an isolated singularity of $R_\lambda$. Let $C$ be a small contour inclosing $\lambda_0$ but no other point of the spectrum. Then the Laurent expansion of $R_\lambda$ is

$$R_\lambda = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n A_n + \sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} B_n, \tag{6}$$

where

$$B_n = \frac{1}{2\pi i}\int_C (\lambda - \lambda_0)^{n-1} R_\lambda d\lambda, \qquad n = 0, \pm 1, \pm 2, \cdots,$$

and we understand that $A_n = B_{-n}$. One can show that $B_{n+1} = T_{\lambda_0} B_n$, $n = 1, 2, 3, \cdots$. Also, $B_1 = -P$, and so

$$B_n = -T_{\lambda_0}^{n-1} P, \qquad n = 1, 2, 3, \cdots. \tag{7}$$

The simplest case is that in which $\lambda_0$ is a pole of $R_\lambda$.

THEOREM 8. *If $\lambda_0$ is a pole of $R_\lambda$, it is in the point spectrum of $T$. If $x$ is a corresponding characteristic element, $Px = x$. A sufficient, but not necessary, condition for $\lambda_0$ to be a pole is that $P$ project $E$ into a finite dimensional linear manifold.*

To prove the first statement, let $m$ be the order of the pole. Then from (6) we infer that $B_m \neq 0$ and $B_n = 0$ if $n > m$. Let $y$ be chosen so that $x = B_m y \neq 0$. Then $T_{\lambda_0} x = B_{m+1} y = 0$, by (7). We may also prove that $A_0 T_{\lambda_0} = I - P$. Hence we see that $Px = x$.

To prove the second part of the theorem, we define

$$S_\lambda = \sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} B_n. \tag{8}$$

It is easily proved that $S_\lambda$ is regular if $\lambda \neq \lambda_0$, and that $S_\lambda P = S_\lambda$. Furthermore,

$$S_\lambda T_\lambda = P. \tag{9}$$

Now suppose that the range $E_0$ of $P$ is finite dimensional. Within $E_0$, $P$ coincides with the identity. From the above remarks we therefore conclude that $S_\lambda$ is essentially a transformation of $E_0$ into itself, and that it is, within $E_0$, the resolvent of $T$. Its singularity at $\lambda_0$ is therefore that of the resolvent of a finite matrix, namely, a pole.

There is a large class of transformations $T$ for which the only singularities of $R_\lambda$, aside from $\lambda = 0$, are poles. The most familiar case is that when $T$ is completely continuous. It is easy, in this case, to show that the only possible limit point of the spectrum is the origin. For an isolated singularity $\lambda_0 \neq 0$ it is readily proved that $P$ is completely continuous and that its range $E_0$ is finite dimensional. Theorem 8 then shows that $\lambda_0$ is a pole of the resolvent. This argument has been pointed out by several people (Schauder [1, p. 193]; Hildebrandt [1, p. 198]; Nagumo [1, pp. 79–80]). An extension to transformations which are in a sense "nearly" completely continuous has been indicated in various places (Hildebrandt, loc. cit.; Nikolskij [1]; Yoshida [1, 2]).

In case the only singularities of $R_\lambda$ aside from $\lambda=0$ are poles, the transformation $s=\lambda^{-1}$ carries $R_\lambda$ into a meromorphic function which may be expanded by Mittag-Leffler's theorem. If the number of poles is finite, $R_\lambda$ is rational, and we get an explicit partial fraction representation. In fact, since $R_\lambda$ vanishes at infinity, it is equal to the sum of its principal parts at the poles $\lambda_1, \cdots, \lambda_n$. If we enclose each pole in a small contour $C_j$, and denote the corresponding projection (5) by $P_j$, we find, from (6), that

$$R_\lambda = -\sum_{j=1}^{n}\sum_{k=1}^{m_j}(\lambda-\lambda_j)^{-k}T_{\lambda_j}^{k-1}P_j, \tag{10}$$

where $m_j$ is the order of the pole $\lambda_j$. By integrating around a contour enclosing all the poles we find the relation

$$P_1 + \cdots + P_n = I. \tag{11}$$

Finally, by combining (4) and (10), we find that if $\lambda_1, \cdots, \lambda_r$ are simple poles, while $\lambda_{r+1}, \cdots, \lambda_n$ are multiple poles,

$$T = \sum_{j=1}^{r}\lambda_j P_j + T\sum_{j=r+1}^{n} P_j. \tag{12}$$

Formulas (10)–(12) generalize certain results in the theory of finite matrices (Frobenius [1]; Schwerdtfeger [1, 2, 3]). In case the poles are all simple, $T$ is completely reducible. This is the abstract generalization of the reduction of a matrix to diagonal form.

It is of some interest to record that if $E$ is the Hilbert space $l^2$, and if the resolvent is defined by an infinite matrix with elements $R_{ij}(\lambda)$, then if $T$ is either Hermitian or unitary, the spectrum of $T$ consists of the singular points of the functions $R_{ij}(\lambda)$, together with the limit points of such singularities. This does not hold true for an arbitrary $T$, however (Wintner [1, pp. 178 and 213–214]).

7. **Functions of linear transformations.** Formula (4) of §6 may be generalized as follows: Let $D$ be a fixed simply connected open set containing the spectrum of $T$. Let $\mathcal{J}$ be the class of all single-valued numerical functions $F(\lambda)$ which are holomorphic in $D$. This class $\mathcal{J}$ is a commutative ring. Let us define

$$F(T) = -\frac{1}{2\pi i}\int_C F(\lambda)R_\lambda d\lambda, \tag{1}$$

where $C$ is a contour lying in $D$ and surrounding the spectrum of $T$.

If $F(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$ is an entire function, formula (3) of §6 enables us to show that $F(T)=\sum_{n=0}^{\infty}a_nT^n$.

THEOREM 9. *The class of all linear transformations $F(T)$ generated by (1) is a commutative sub-ring of the ring of all linear transformations of E into itself. The mapping defined by (1) is a homomorphism.*

The proof of the theorem consists in showing that the correspondence $F(\lambda)\rightarrow F(T)$ preserves sums and products. The preservation of sums is trivial. Suppose that $F(\lambda)$ and $G(\lambda)$ are in $\mathcal{J}$. If $\Gamma$ is a contour in $D$, enclosing $C$, and $\lambda$ is on $C$,

$$G(\lambda) = \frac{1}{2\pi i}\int_{\Gamma}\frac{G(\mu)}{\mu-\lambda}\,d\mu.$$

If we substitute the above formula into the integral

$$-\frac{1}{2\pi i}\int_{C}F(\lambda)G(\lambda)R_{\lambda}d\lambda,$$

reverse the order of integration, and use formula (1) of §6, the above integral around $C$ is found to have the value $F(T)G(T)$. This proves that products are preserved by (1).

An immediate and important consequence of Theorem 9 is that if $F(\lambda)$ belongs to $\mathcal{J}$ and has no zeros in $D$, then $F(T)$ has a multiplicative inverse $[F(T)]^{-1}$; the inverse corresponds to $1/F(\lambda)$, and may be written $I/F(T)$.

Formula (1) may be written in the equivalent form

$$F(T) = \frac{1}{2\pi i}\int_{C}\frac{F(\lambda)}{\lambda I - T}\,d\lambda, \tag{2}$$

by recalling that $R_{\lambda}=(T-\lambda I)^{-1}=-(\lambda I-T)^{-1}$. In the finite dimensional case this formula seems to be due to Poincaré, who used it in the study of continuous groups (Poincaré [1, 2, 3]). It has appeared elsewhere (Buhl [1, pp. 37–38]; Schwerdtfeger [1, p. 311]; Giorgi [1, p. 7]; Wedderburn [1, p. 130]). A special case of the formula was used by Hille [1, pp. 3 and 22] in studying semi-groups of transformations in Banach spaces. Theorem 9 seems to be new, and apparently formula (1) itself is also new in the generality with which it is presented here.

As an interesting application, we shall use the above theory to solve the linear differential equation

$$d^nx/dt^n + a_1d^{n-1}x/dt^{n-1} + \cdots + a_nx = y(t) \tag{3}$$

with the initial conditions

$$x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0. \tag{4}$$

The coefficients $a_1, \cdots, a_n$ are assumed to be constants, and $y(t)$ is a given function, continuous on the interval $[0, 1]$.

If $C$ is the Banach space of functions $x(t)$ continuous on $[0, 1]$, let $Tx(\cdot)=y(\cdot)$ mean that

$$y(t) = \int_0^t x(s)ds. \tag{5}$$

One may show without trouble that the solution of the integral equation

$$x(t) - \lambda \int_0^t x(s)ds = y(t)$$

is

$$x(t) = -\frac{1}{\lambda} y(t) - \frac{1}{\lambda^2}\int_0^t e^{(t-s)/\lambda} y(s)ds. \tag{6}$$

The spectrum of $T$ consists of the single point $\lambda=0$, and the resolvent of $T$ is defined by (6).

The differential system (3) and (4) is equivalent to the equation

$$x + a_1 Tx + \cdots + a_n T^n x = T^n y. \tag{7}$$

If $F(\lambda)=1+a_1\lambda+\cdots+a_n\lambda^n$, (7) becomes $F(T)x=T^n y$. Since $F(\lambda)$ and $1/F(\lambda)$ are regular at $\lambda=0$, and hence in the neighborhood of the spectrum of $T$, the inverse of $F(T)$ exists, and the solution of (7) is

$$x = (T^n/F(T))y.$$

Thus, using formula (1),

$$x = -\frac{1}{2\pi i}\int_C \frac{\lambda^n}{F(\lambda)} R_\lambda y \, d\lambda, \tag{8}$$

where $C$ is a contour enclosing $\lambda=0$ and excluding the zeros of $F(\lambda)$. Since the resolvent is defined by (6), (8) gives us an explicit means of calculating $x(t)$. It is convenient to make the substitution $z=\lambda^{-1}$. We then obtain

$$x(t) = \frac{1}{2\pi i}\int_0^t y(s)ds \int_\Gamma \frac{e^{z(t-s)}}{z^n + a_1 z^{n-1} + \cdots + a_n} dz, \tag{9}$$

where $\Gamma$ is a contour enclosing the roots of the polynomial $z^n+a_1 z^{n-1}$

$+\cdots+a_n$. Formula (9) is due originally to Cauchy (see Darboux [1]).

8. **Analytic functions of an abstract variable.** In this section we shall survey briefly the theory of analytic functions where both the domain and range of the functions are Banach spaces. This theory has its roots in ideas which go back to Volterra and other pioneers in functional analysis. The most important early impetus was given by Fréchet, who in a series of papers developed the notions of functional polynomials and power series [1, 2, 4] and pointed the way to a successful calculus based upon his definition of a differential [3]. The work of Gateaux [1, 2] sketched the outlines of a theory of analytic functionals. The first systematic abstract theory of analytic functions was developed by Martin [1]. He was a pupil of A. D. Michal, under whose guidance an intensive study of the whole field of analysis and geometry in abstract spaces has been carried on for more than a decade (see the interpretation of the abstract point of view in Michal [2], and the references to the literature in Michal [1]). Various applications of the theory, in the shape given it by Martin, have appeared (Michal and Clifford [1]; Michal and Martin [1]).

An abstract theory of analytic functions, along the lines indicated by Gateaux, was announced by Graves [1], and independently by Taylor ([1 and 5]; see especially the footnote on p. 466 of [5]). In Martin's theory the basic notion is that of a power series expansion. The terms of a power series are homogeneous polynomials. For the literature on polynomials, see Fréchet [1, 4]; Martin [1]; Mazur and Orlicz [1]; Highberg [1, 2]; Taylor [3]; Banach [2]. In the theory of Graves and Taylor, $F(x)$ is said to be holomorphic in a region $D$ if it is continuous there and if the limit

$$\delta F(x;\, y) = \lim_{\lambda \to 0} (F(x + \lambda y) - F(x))/\lambda$$

exists for each $y$ in the Banach space and each $x$ in the region $D$. It can be shown that $\delta F(x; y)$ is the Fréchet differential, and that $F(x)$ is analytic in the sense of Martin. The two different approaches result in the same general theory. There are certain connections between the above ideas and the work of Fantappié [1, 2] on analytic functionals. Wintner's theory of regular power series [2] is closely allied to the abstract theory described above.

One illustration of the general theory is to be found in the study of resolvents. If we regard the linear transformation $T$ as a variable, and denote the resolvent by $R(\lambda, T)$, the resolvent is an analytic function, not only of $\lambda$, but of $T$ as well.[3] The Fredholm theory provides an explicit exhibition of the dependence of the resolvent upon $\lambda$ and $T$. An abstract treatment of the Fredholm determinant and first minor was given by Michal and Martin [1]. Closely allied results, for transformations of finite norm in Hilbert space, were given by Smithies [1], though without any explicit use of a theory of abstract analytic functions. It would seem to be worth while to pursue the study of $R(\lambda, T)$ and of the functions $F(T)$ defined in §7 from this general point of view.

A theory of analytic functions from Riemann's point of view may be built up by starting from a real Banach space $B$, and constructing a complex Banach space in much the same way that the complex numbers are formed from the real numbers. The only part of the construction which offers the least difficulty is the defining of a suitable norm of the complex element $x+iy$, where $x$ and $y$ are in the real space $B$. That $(\|x\|^2+\|y\|^2)^{1/2}=\|x+iy\|$ will not do, in general, was pointed out in a paper by Michal, Davis, and Wyman [1]. A satisfactory definition is the following, due to Taylor (Michal and Wyman [1, p. 249]):

$$\|x + iy\| = \sup_{\|f\|=1} (|f(x)|^2 + |f(y)|^2)^{1/2}, \qquad f \in B^*.$$

With the complex Banach space so constructed it is possible to develop analogues of the Cauchy-Riemann equations, and a corresponding notion of biharmonic functions (Taylor [4]). It is also possible to extend a "real" analytic function, defined by a series of homogeneous polynomials, into the complex Banach space (Taylor [3, pp. 312–315]). This process utilizes certain relations between abstract polynomials and their polars. There are various unsolved problems connected with bounds and inequalities which make the present state of this work unsatisfactory.

There is a converse problem: can every complex Banach space $E$ be decomposed into a real and an imaginary part? The question may be put in the following form (cf. Michal and Wyman [1, Theorem 3]): does there exist a continuous transformation $T$ of $E$ into itself, having the properties $T^2x=x$, $T(x+y)=Tx+Ty$, $T(ax)=\bar{a}Tx$? We may call such a transformation a *conjugation*. Let us call a subset $M$ of $E$ a *maximal essentially real* linear manifold if

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[3] In this connection we note that a function $F(\lambda, x)$ of a complex variable $\lambda$ and a Banach variable $x$ is analytic in the pair $\lambda$, $x$ if it is analytic in each variable separately (Taylor [5, Theorem 3.2]). Whether the like is true of functions of two abstract variables is at present unknown.

(a) it is closed under addition and under multiplication by real numbers;

(b) $x$ and $ix$ are both in $M$ only if $x=0$;

(c) $M$ is not a proper subset of any set with properties (a) and (b). Then it is not difficult to show that there exists a conjugation of $E$ if and only if there exists in $E$ a closed and maximal essentially real linear manifold. This whole problem has been studied by Taylor [7], but no complete solution has been reached.

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2. *Upon a theory of infinite systems of nonlinear implicit and differential equations*, Amer. J. Math. vol. 53 (1931) pp. 241–257.

K. YOSHIDA

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# HENRY SEELY WHITE—IN MEMORIAM

Henry Seely White, emeritus professor of mathematics at Vassar College, died in Poughkeepsie, New York, on May 20, 1943, his eighty-second birthday. In his death mathematics has lost a distinguished geometer, the college a wise and helpful guide, and his colleagues and the community a much loved friend.

Professor White was born in Cazenovia, New York, on May 20, 1861, the son of Aaron White, principal and teacher of mathematics in Cazenovia Seminary. Of English stock, his first American ancestor was John White who came to the United States in 1632, one of the original settlers of Hartford. The schools and seminary of Cazenovia gave Mr. White his early training and preparation for college. He entered Wesleyan University and graduated with honors in 1882 at the age of twenty-one. During the following year he assisted in astronomy and physics at Wesleyan and returned there again in 1884 after a year at Centenary College where he taught mathematics and chemistry. On completion of three years as registrar and tutor in mathematics in Wesleyan he went to Göttingen to work with the unusual group of mathematicians in the university at that time, and completed his doctorate there in 1891.

It was during this period that he received stimulus and training which gave shape and scope for his mathematical development, strengthening of his independence in research, and a beginning in reading in a variety of fields which is no longer such a common attribute of our great specialists. While his publications were in the field of his specialized geometry, his studies ranged over the entire field. It was this wide scope that made him such a valued adviser and guide both to his own generation and to younger mathematicians.

On his return to the United States, Professor White spent two years in the department of mathematics at Clark University and then was called to Northwestern, one of a large group of unusually able young men. It was from Northwestern that he came to Vassar, in 1905, to be head and guide of the department of mathematics until the age of 75, and close friend and frequent lecturer for the remaining years of his life. Students as well as faculty members prized each opportunity to enjoy his mind. In the week of his death he was scheduled to lecture to a class in the geometry of space curves.

In the mathematics organizations Professor White took an active part as organizer, editor and officer. He was on the editorial staff of the Bulletin of the American Mathematical Society for a number of years, was editor of the Annals of Mathematics from 1899 to 1906, and editor of the Transactions of the American Mathematical Society from 1907 to 1914. He served the American Mathematical Society as president from 1906 to 1908 and the American Association for the Advancement of Science as vice president in 1915. In the same year he was elected Fellow of the National Academy of Sciences. Honors conferred upon him included an LL.D. from Northwestern, 1915, and D.Sc. conferred by Wesleyan at the time of its 100th anniversary in June, 1932.

Among Professor White's publications were contributions to the Bulletin of the American Mathematical Society, the American Journal of Mathematics, and a text of *Plane curves of the third order*. While his studies ranged over a large field in the subject and far outside it, his main interest was in the theory of invariants, the geometry of curves and surfaces, correspondences, algebraic plane and twisted curves, homeomorphic sets of lines in a plane, and relativity in mechanics.

Professor White's life in the community was broad and rich. He and Mrs. White had wide connections through church and city organizations, especially through the University Club, the Yacht Club, and the Vassar Institute. Many in all these varied groups mourn the loss of an acute mind, a friendly, courteous wit, and a great heart.

MARY EVELYN WELLS

# BOOK REVIEWS

*Analytic topology.* By G. T. Whyburn. (American Mathematical Society Colloquium Publications, vol. 28.) New York, American Mathematical Society, 1942. 10+278 pp. $4.75.

In an effort to place the subject matter of this book within the larger domain covered by the title, we take up two illustrations that are in some sense typical of the contents. The first selection is for those who prefer to think in terms of invariants. One finds many-to-one transformations each of which preserves, a priori, certain topological properties, and a subsequent search for new invariants, rather than a study of isometric or other invariants that might seem to fall within the title. Let $F$ be a class of transformations and $S$ be a class of spaces. It is desired to choose $S$ so that any element of $S$ maps onto another element of $S$ under every mapping from $F$, and that, furthermore, there be in $S$ a particular "generator" $E$, such that, given any element $S^*$ of $S$, there is an element $f$ in $F$ for which $f(E)=S^*$. For the case of locally connected continua $S$, and the continuous mappings $F$, this result is well known; here $E$ may be taken as an arc. Other examples are: boundary curves $S$, simple closed curve $E$, and non-alternating transformations $F$; cactoids $S$, sphere $E$, and monotone mappings $F$; or hemi-cactoids, 2-cell, and monotone mappings. In many instances $f(S^*)=E$ is also possible for some $f$ in $F$. The composition of two elements of $F$ is always in $F$, at least on spaces from $S$. A similar situation in which $F$ is unknown and $S$ contains only one element, that is, when the original and image spaces are homeomorphic, is of considerable importance and largely an open question.

The second selection serves to emphasize the highly natural relationship between the older structural ideas and the analytic concepts that form the major part of this work. In order that a continuum $M$ should be a simple continuous arc it is necessary and sufficient that $M$ contain at most two points that do not separate $M$. This characterization is due to R. L. Moore. By its use of an array of cut points between the two end points, it strongly suggests a powerful technical use of the notion of a cutting for singling out linear arrays in continua. One might say that the theories of dimension and regularity given by Menger and Urysohn are based on a similar, but not the same, approach. When the cuttings are points this technique, roughly speaking, culminates in the cyclic element theory for certain special continua. For non-separated cuttings of more than one point this technique yields other results that are unfortunately not as well known (Chapter III). It also affords a simple and natural transition from the structural point of view to the analytic in the form of the non-alternating transformation, since this transformation has (in addition to continuity) the property that those counterimages $f^{-1}(b)$ which are cuttings form a non-separated collection. When considered in conjunction with cyclic elements this mapping acts in a way that suggests a common background with these elements. But the non-alternating transformation also has properties in common with the strongly analytic (Chapter X) interior transformation which preserves open sets. These relations become more evident when non-alternating is specialized to monotone $-f^{-1}(b)$ connected for all $b$ in $B$. In this connection see quasi-monotone. Finally, as was shown by Moore, the monotone image $B=f(E)$ of a topological sphere is again a sphere provided no continuum $f^{-1}(b)$ separates $E$. Since the general properties of these mappings are thoroughly treated it is evident that some parts of this book provide a highly appropriate setting for certain results from the work of Moore, Menger and Urysohn. Various other selections would show a strong representation of other mathematicians.

The author, apparently motivated by a desire to select a nucleus of basic material, has left out a considerable portion of his own work, that of his students, and others, particularly work dealing with local separating points and with property $S$. There is no extensive use of metric methods, combinatorial topology, nor group theoretic material. After brief study the book permits considerable flexibility, but the various chapters can not be read independently.

Since the contents are self-contained and carefully integrated, this volume will have considerable interest for students and certain possibilities as a text. In this connection the various exercises in the first three chapters deserve mention; but it must also be said that it moves swiftly into the chosen course and does not contain many examples.

Roughly, the purely analytic topology occupies the latter half of the book, but one finds a growing use of analytic methods beginning early (Chapter II). In the following outline by chapters, some notices of errors or omissions are given, together with additional cross references and other comments, but these remarks are neither systematic nor complete.

The first half of Chapter I leads from a set of six axioms to the metrization theorem. It is then assumed that all spaces are separable and metric. Compactness is used in the sense of self-compact (subsets conditionally compact). The student will probably wish to read supplementary material concerning the idea of compactness. A locally compact and connected set is a generalized continuum. The chapter continues with connectivity for sets and their limits superior, and then takes up continua, the Brouwer reduction theorem, continua of convergence, and local connectivity. The concluding section on property $S$ and uniformly locally connected sets makes a careful comparison of these important properties. Some of the logical niceties of definition are absent, and on page 4, $q \subset K$ should be $q \in K$.

The second chapter will probably overshadow many other parts of the book as an item of general interest because of its classical results, the interesting nature of the proofs, and the way it gives emphasis to the title of the book. Using the familiar existence-extension technique for uniformly continuous mappings, Whyburn reduces the existence part to a flexible central result which is subsequently used to characterize locally connected continua as continuous images of the unit interval, and compact sets as mappings of the Cantor discontinuum. Furthermore this central result yields an imbedding theorem which, paired with arcwise connectivity for locally connected continua, gives the Moore-Menger arcwise connectivity for complete connected and locally connected sets, and other results. This imbedding theorem furnishes insight into "arcless" continua, but, due to the arrangement of the material, this comment (furnished by the original paper) was omitted here. An altered form of the central result established by Harrold is included, although many of his related results are not. The chapter begins with the necessary preliminaries on transformations, complete sets, and uniform continuity; it concludes with the arcwise connectivity theorem for continua. This final result is established in three ways, including the Kelley proof, so short that it once enjoyed considerable circulation in oral form. The symbols for closure and complete closure (dash and tilde, page 28) could be confused, and the waves over certain letters on page 30 are poorly done. Otherwise the typography is excellent through the whole book.

The fundamental importance of the treatment of non-separated cuttings in Chapter III has been indicated earlier in this review. In addition to the author's order (Menger-Urysohn) theorems for various separating points, the Moore characterizations of the arc and simple closed curve, and the theorem of W. H. Young, there is much basic material on (linear) order, Borel classes, and the nature and existence of cuttings. In particular the set $E(a, b)$ of all points separating two given points has a natural order, is closed for some continua, and—the author used the Wilder proof—compact in locally connected continua. Various examples might be given; the book has none. In line 5 from bottom of page 41, "points" should be "sets."

That the order is "natural," top of page 43, must be shown.

Chapters IV and V might well have been prefaced by the author's paper, *Concerning maximal sets* (Bull. Amer. Math. Soc. 1934), which does not appear in the bibliography. In every other sense these chapters are a polished presentation of the cyclic element theory, incorporating the recent extension to semi-locally-connected continua, the work of Ayres on cyclic connectedness and the Whyburn contributions to multicoherence (Eilenberg) and the classification of continua among curves. The results in section 7, Chapter IV, are particularly important; see also the theorems at the top of page 107. The lemma page 175, top, should be mentioned on pages 72 and 73. Those readers seeking a casual knowledge of cyclic elements may reach this end by studying dendrites (Chapter V), provided they use the pairings: cyclic element—point; cyclic chain—arc; $H$-set—connected subset; $A$-set—subcontinuum. On page 73, part (c) of (7.1), omit the vertical bar before $H$. The references for Chapter IV are particularly rich in worthwhile material; it would have pleased the reviewer to find the work of Kelley, Harry, and Youngs included in the text. Additional reference: Jones, *Aposyndetic continua and certain boundary problems* (Amer. J. Math. 1941).

The special properties that inhere in continua, their complements, and their boundaries, when the containing space is a plane or a sphere, are developed in Chapter VI. This chapter is fundamental—see, for example, the characterization of the sphere—for readers of the book as a whole, and is second to none in its appeal to the casual reader. It is of interest to note that boundary curves take their name from the property exhibited on p. 107. Separation theorems and accessibility are covered, but all the work on primitive skew curves is omitted.

The theory of semi-continuous decompositions is developed and integrated with that of continuous transformations early in Chapter VII. Certain types of transformations are then defined and characterized. One type, $f(A)=B$, with $f^{-1}(b)$ totally disconnected, called "light" becomes increasingly important in the next few chapters. Semiclosed sets, null collections, and related factorizations constitute the main material. However, the brief treatment of the question of finding an $f(A)=B$ such that $A$ and $B$ are homeomorphic should not be overlooked. In this connection a marginal note on page 135 referring to page 171 will prove valuable. In Chapter VIII retractions, quasimonotone transformations and the relative distance mapping are added to the list. The $k$ to 1 mappings are noticeably absent; the 0-regular and local homeomorphism are only briefly considered later. This chapter is nevertheless full of material from the heart of the analytic theory. It is highlighted by the monotone-light factorization theorem, the Wallace quasi-monotone theory (middle ground between interior and monotone), some unpublished work on non-alternating mappings of simple links and $A$-sets, and new proofs in the work with the relative distance transformation. See the application page 206. In the references Wallace [4] should be [3].

Chapter IX deals with applications of monotone and non-alternating transformations. It presents a pleasing blend of general results and particular applications which fix and enrich the concepts which are involved. The author adds materially to the interest by pausing to discuss the general characteristics of certain theorems. The last section is entirely new and the work of Morrey and Moore on the various cactoids is treated using new proofs. These invariants together with some striking characterizations of non-alternating transformations on boundary curves form the major part of the chapter. The work of Steenrod and Roberts on the monotone images of two-dimensional manifolds is not included. Correction: last line page 181 Lemma (3.41) should be (3.31).

Chapter X is devoted exclusively to interior transformations: on linear graphs the one-dimensional Betti group maps homomorphically; on simple closed curves the mapping is like $w=z^k$ on $|z|=1$, or the image is an arc. For the light interior transformation on a compact space, there is an arc in the original set that maps topologically onto a given arc in the image space. This inversion of arcs recalls an open question. However, see the example in the book. The treatment of light interior mappings on a 2-cell virtually establishes the invariance of a 2-cell, but is conceived and used as a central tool in proving the invariance of the 2-manifold property, and the action of these mappings in the small on such manifolds. The manifolds are not required to be compact, and thus this new work further supplants pioneer results of Stoïlow. The local analysis—$w=z^n$ on $|z| \leqq 1$ in one instance—is used, together with some new results on local homeomorphisms, to introduce an analysis in the large. This culminates in a simple numerical relationship between the Euler characteristics of the manifolds and leads to a considerable number of examples and applications—closed surfaces retain orientability under the inverse, spheres map onto spheres, projective planes, or 2-cells, and so on. Certain results are then extended to pseudo-manifolds using the relative distance transformation. Some outstanding new items are the proofs by direct analysis of the action of interior mappings. Open questions are: interior (non-light) action on manifolds; the special properties of the monotone factor of an interior transformation.

References to Puckett at end of the previous chapter, and his work on the inversion of local connectivity, are pertinent to this chapter. The latter is not in the bibliography.

The first part of Chapter XI deals with locally connected continua, non-alternating interior transformations and retractions, and the existence of these mappings. The second half of the chapter treats mappings onto the circle using the methods of Eilenberg. A second major type of factorization for transformations arises from the concept "equivalence to one." Each mapping of one simple closed curve into another is homotopic to an interior mapping. Results such as these help to orient the student in the latter part of this chapter. Corrections: "onto" should be "into" page 219, line 6 from the bottom; 0 should be 1 in line 2 of proof (5.1) page 221; $|t+(1-t)f(0)|$ should be a factor of the right member of the equation in (6.1) page 225, and in the line above, $f_0(X)$ should be $f_0(x)$.

The last chapter opens with theorems concerning the fixed point property and fixed cyclic elements; it continues with characterizations, necessary conditions, and other relations between pointwise almost periodic, regularly almost periodic, pointwise periodic, and periodic properties together with the newly defined Whyburn orbit. The central concepts of this chapter may be traced to many sources, but in general these developments are in the spirit of the work of Ayres. Nearly every page contains unpublished results and evidence of considerable improvement in the methods of proof. The full force of certain combinations of theorems becomes apparent only under systematic study, but the beginner should not find it difficult to progress using the individual theorems and numerous examples. There are numerous corrections. The third line in the proof of (1.11) page 240 should contain $S$ and not $X$; $p_{i_n}$ should be $p_{i_k}$ in line 3, Lemma 2, page 244; and in line 1, §6, page 253, $\subset$ should be $=$. Three others, all on page 225, are: line 1 of (6.2), $\subset$ should be $=$; lines 8 and 9, $-n+1$ instead of $-n$; lines 12 and 15, $n+1$ instead of $n$. Finally, page 264, lines 14 and 25, $z^n$ should be

$$e^{2\pi i/n}z \text{ or } z \exp\left(\frac{2\pi i}{n}\right).$$

This volume is obviously no catalogue of chance items of research. Some of its most striking general features are the new proofs, the integration of the material, and the way it stirs one's interest in innumerable allied topics without violating unity of form and flow of thought. To these virtues may be added a clear pithy style; many theorems are stated in one or two lines. References at the end of each chapter, a large bibliography, a table of contents and an index, give flexible coverage of all items without being cumbersome. It should enjoy a long life and constant use by all who find interest in this type of work.

G. E. SCHWEIGERT

*Poisson's exponential binomial limit*; *Table* I—*Individual Terms*; *Table* II—*Cumulated Terms*. By E. C. Molina. New York, Van Nostrand, 1942. viii+46+ii+47 pp. $2.75.

If $p$ is the probability of a "success" in a single trial, it is well known that the probability of $x$ "successes" in $n$ independent trials is given by

$$C_x^n p^x(1-p)^{n-x} \tag{1}$$

which is the $(x+1)$st term in the expansion of the binomial $[p+(1-p)]^n$. If the limit of (1) is taken as $p \to 0$ and $n \to \infty$ in such a way that $np=a$, one obtains

$$a^x e^{-a}/x! \tag{2}$$

which is the $(x+1)$st term of a distribution originally published by Poisson in 1837. This function not only arises as an approximation to the binomial term (1) for large $n$ and small $p$, but also arises in other problems, as for example in the integration of the chi-square distribution.

Table I of the present book is a tabulation of values of (2) to six places of decimals for $a=0.001(0.001)0.01(0.01)0.3(0.1)15(1)100$ and $x=0(1)150$; $x$, of course, being carried far enough for each given value of $a$ to cover values of (2) to six places of decimals, not all zero. Table II gives the values of $P(c, a)=\sum_{x=c}^{\infty} a^x e^{-a}/x!$ to six places of decimals for the same range of values of $a$ and for $c=0(1)153$.

The book has been lithoprinted by Edwards Brothers and is bound with a flexible paper cover.

Various parts of the tables have appeared in earlier publications. For example, L. v. Bortkiewicz (*Das Gesetz der kleinen Zahlen*, Leipzig, 1898) published tables of (2) to four places of decimals for $a=0.1(0.1)10.0$ and $x=0(1)24$. H. E. Soper (Biometrika, vol. 10 (1914)) published a table of (2) to six places of decimals for $a=0.1(0.1)15.0$ and $x=0(1)37$, which was reprinted in Karl Pearson's *Tables for statisticians and biometricians*, Cambridge, 1914. E. C. Molina (Amer. Math. Monthly, 1913) published tables of $c$ for $P(c, a)=0.0001$, 0.001 and 0.01; for $a=0.0001$ to 928, and similar, but more extensive, tables were published by G. A. Campbell in the Bell System Technical Journal for January, 1923. The present tables are much more extensive than any of these earlier tables.

These tables have great value in probability and statistical problems. They have been used very extensively, for example, by mass production quality engineers in the development of sampling inspection plans. The sampling inspection tables published by Dodge and Romig in the January, 1941, issue of Bell System Technical Journal were calculated primarily from Molina's tables. Because of the frequent occurrence of the term $a^x e^{-a}/x!$ in various functions, the tables will probably also be found very useful in the tabulation of such functions.

S. S. WILKS

*Non-euclidean geometry.* By H. S. M. Coxeter. (Mathematical Expositions, no. 2.) University of Toronto Press, 1942. 15+281 pp. $3.25.

There seems to be a well established pattern for books on the non-euclidean geometries, according to which a more or less elaborate historical sketch is followed by a development of the foundations of the geometries. There is usually little space left available for developing the geometries much beyond the foundations. Thus it not infrequently happens that many interesting results not intimately connected with the beginnings of the subject are declared "beyond the scope of this book."

Though the plan of the book under review presents no radical departure from such a pattern it does offer somewhat more of the subject proper than is usual, while the manner in which it accomplishes its aims sets a new high standard for such texts.

The historical survey is confined to a short first chapter (which the author observes "can be omitted without impairing the main development"). The emphasis being on the projective approach, there follow three chapters concerning the foundations of real projective geometry. Chapters V, VI, and VII are devoted to elliptic geometry of one, two, and three dimensions. Introducing congruence axiomatically (IX) in a "descriptive" geometry (VIII), the author obtains an absolute geometry from which the euclidean and hyperbolic geometries follow upon adjoining the appropriate parallel axiom. Two-dimensional hyperbolic geometry is treated in Chapter X. The next three chapters, entitled Circles and Triangles, The Use of a General Triangle of Reference, and Area, deal, for the most part, with matters apart from the foundations, while the concluding chapter of the book presents the usual euclidean models of the non-euclidean geometries.

The book is heartily recommended as furnishing a very well written account of the fundamental principles of hyperbolic and elliptic geometry from the classical point of view. This was doubtless the author's aim. The reviewer feels constrained to observe, however, that as a modern treatment of an old subject, the book might have recognized some of the contributions to its field made available during the last few years. Thus to the three traditional avenues of approach to the non-euclidean geometries there has been added a fourth—the abstract metric approach—which injects into the rigor of the classical axiomatic methods the stimulus of a rapid development. Still more within the spirit of the book, it seems, would have been a notice of the foundations of hyperbolic geometry due to Menger, Jenks, and Abbott based upon the sole operations of joining and intersecting.

The typography is excellent, and the figures (for the most part simple) are carefully drawn. Those contemplating using the book as a classroom text are advised that it contains no exercises and that many proofs, especially concerning material in the latter part of the book, would have to be supplied.

L. M. BLUMENTHAL

## NOTES

The Navy is urgently in need of women with mathematical training for the Aerology and Radar programs which are sponsored by the WAVES. For both programs it is expected that the candidates be college graduates. For the Radar program, the candidate must have completed with at least a "C" average not less than one year of college physics and one year of college mathematics. For the Aerology program, she must have had at least one year of Calculus and one year of Physics. Interested women should apply to any office of Naval Officer Procurement.

Professor J. L. Synge, formerly of the University of Toronto now of the Ohio State University, has been awarded the Henry Marshall Tory medal of the Royal Society of Canada for achievement in scientific research.

Professor R. C. Archibald of Brown University has retired with the title emeritus.

Professor H. F. Baker of the University of Cambridge, Professor G. D. Birkhoff of Harvard University and Professor Oswald Veblen of the Institute for Advanced Study have been elected honorary fellows of the Royal Society of Edinburgh.

Professor P. W. Bridgman and Professor Harlow Shapley of Harvard University have been elected corresponding members of the Mexican National Academy of Sciences.

Professor D. R. Curtiss of Northwestern University has retired with the title emeritus. He plans to make his home in California.

Professor H. W. Kuhn of the Ohio State University, whose retirement has recently been announced, has been given the title emeritus.

Professor S. E. Rasor of the Ohio State University has retired with the title emeritus.

Assistant Professor Harriet W. Allen of Hollins College, Hollins College, Virginia, has been promoted to an associate professorship.

Assistant Professor D. B. Ames of Rensselaer Polytechnic Institute has been promoted to an associate professorship.

Dr. K. J. Arnold of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor Theodore Bennett of Marietta College, Marietta, Ohio, has been promoted to a professorship.

Assistant Professor M. T. Bird of Utah State Agricultural College has been promoted to an associate professorship.

Dr. R. P. Boas has been appointed visiting lecturer at Harvard University.

Assistant Professor W. M. Borgman of Wayne University has been promoted to an associate professorship.

Assistant Professor Richard Brauer of the University of Toronto has been promoted to an associate professorship.

Assistant Professor F. J. H. Burkett of Union College has been promoted to an associate professorship.

Dr. Herbert Busemann of the Illinois Institute of Technology has been promoted to an assistant professorship.

Dr. Nathaniel Coburn of the University of Texas has been promoted to an assistant professorship.

Associate Professor A. H. Copeland of the University of Michigan has been promoted to a professorship.

Assistant Professor H. S. M. Coxeter of the University of Toronto has been promoted to an associate professorship.

Dr. John DeCicco of the Illinois Institute of Technology has been promoted to an assistant professorship.

Dr. R. P. Dilworth of Yale University has been appointed to an assistant professorship at the California Institute of Technology.

Associate Professor Nelson Dunford of Yale University has been promoted to a professorship.

Dr. Paul Erdös of the University of Pennsylvania has been appointed visiting lecturer at Purdue University.

Assistant Professor D. G. Fulton of Ohio Northern University has been appointed to an assistant professorship at the University of New Hampshire.

Associate Professor W. H. Gage of the University of British Columbia has been promoted to a professorship.

Dr. Abe Gelbart has been appointed to an assistant professorship at Syracuse University.

Sister Agnes A. Green of the Immaculate Heart College, Los Angeles, California, has been promoted to an associate professorship.

Dr. P. R. Halmos of the University of Illinois has been appointed to an assistant professorship at Syracuse University.

Assistant Professor O. G. Harrold of the Louisiana State University has been appointed to an assistant professorship at Pomona College.

Assistant Professor G. G. Harvey of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Dr. F. B. Hildebrand of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. A. S. Householder of the University of Chicago has been promoted to an assistant professorship of mathematical biophysics.

Assistant Professor G. B. Huff of Southern Methodist University has been promoted to an associate professorship.

Professor R. D. James of the University of Saskatchewan has been appointed to a professorship at the University of British Columbia.

Assistant Professor Marie M. Johnson of Oberlin College has been promoted to an associate professorship.

Dr. Mark Kac of Cornell University has been promoted to an assistant professorship.

Associate Professor Claribel Kendall of the University of Colorado has been promoted to a professorship.

Dr. J. P. LaSalle of the Massachusetts Institute of Technology has been appointed research associate at Princeton University.

Professor C. C. MacDuffee of Hunter College has been appointed to a professorship at the University of Wisconsin.

Assistant Professor M. G. Malti of Cornell University has been promoted to a professorship of electrical engineering.

Dr. Rhoda Manning of Oregon State College has been promoted to an assistant professorship.

Professor Deane Montgomery of Smith College is on leave at Princeton University.

Associate Professor C. A. Nelson of New Jersey College for Women, Rutgers University, has been promoted to a professorship.

Dr. H. E. Newell of the University of Maryland has been promoted to an assistant professorship.

Professor Abba V. Newton of Hartwick College, Oneonta, New York, has been appointed to an assistant professorship at Smith College.

Dr. Ivan Niven of Purdue University has been promoted to an assistant professorship.

Dr. E. P. Northrop of Hotchkiss School, Lakeville, Connecticut, has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor Gordon Pall of McGill University has been promoted to an associate professorship.

Assistant Professor P. M. Pepper of the University of Notre Dame has been promoted to an associate professorship.

Dr. Harry Pollard of Harvard University has been appointed to an assistant professorship at Kenyon College, Gambier, Ohio.

Dr. A. E. Poole of Montana School of Mines has been appointed to an assistant professorship at the University of Oregon.

Associate Professor G. B. Price of the University of Kansas has been promoted to a professorship.

Dr. S. E. Rauch of Santa Barbara Teachers College has been appointed to an assistant professorship at the University of Oregon.

Dr. J. K. Riess of Brown University has been appointed to an assistant professorship at Tulane University.

Professor L. D. Rodabaugh of Shurtleff College, Alton, Illinois, has been appointed visiting lecturer at Oberlin College.

Dr. Raphael Salem of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. Hans Samelson of the University of Wyoming has been appointed to an assistant professorship at Syracuse University.

Dr. O. F. G. Schilling of the University of Chicago has been promoted to an assistant professorship.

Dr. Edith R. Schneckenburger of Michigan State Normal College has been promoted to an assistant professorship.

Assistant Professor A. J. Smith of the College of William and Mary has been appointed to an assistant professorship at the Montana School of Mines.

Assistant Professor Domina E. Spencer of American University, Washington, D. C., has been appointed to an assistant professorship in physics at Tufts College.

Professor C. C. Spooner of the Northern Michigan College of Education has retired.

Assistant Professor F. H. Steen of Allegheny College has been promoted to an associate professorship and has been appointed head of the department of mathematics.

Assistant Professor B. M. Stewart of Michigan State College has been appointed to an assistant professorship at Denison University.

Assistant Professor I. D. Stewart of Whitman College, Walla Walla, Washington, has been promoted to an associate professorship.

Assistant Professor A. H. Taub of the University of Washington has been promoted to an associate professorship.

Assistant Professor E. P. Vance of the University of Nevada has been appointed visiting lecturer at Oberlin College.

Assistant Professor G. C. Vedova of the University of Maryland has been appointed to a visiting associate professorship at Haverford College.

Mr. M. L. Vest of West Virginia University has been promoted to an assistant professorship.

Assistant Professor D. L. Webb of the Texas Technological College has been promoted to an associate professorship.

Dr. Alexander Weinstein of the University of Toronto has been promoted to an assistant professorship.

Assistant Professor F. M. Wood of McGill University has been promoted to an associate professorship.

Dr. H. S. Zuckerman of the University of Washington has been promoted to an assistant professorship.

The following appointments to instructorships are announced: University of Alberta: Dr. Max Wyman; Brown University: Dr. W. H. Pell; University of Buffalo: Miss Ruth A. Brendel; University of Chicago: Mr. Daniel Zelinsky; Clark College, Atlanta, Georgia: Dr. David Blackwell; Colorado College: Mr. E. J. Zadina; Columbia University: Dr. Howard Levi; Johns Hopkins University: Mr. W. L. Doyle; Harvard University: Dr. I. S. Cohen, Dr. H. A. Jordan, Dr. G. W. Mackey, Dr. W. J. Thron; Illinois Institute of Technology: Dr. Louis Garfin; Indiana University: Dr. George Whaples; Iowa State Teachers College: Miss Ethel M. Howe; University of Iowa: Mr. Edwin Halfar; Massachusetts Institute of Technology: Dr. Glynn Owens, Mr. D. L. Thomsen; University of Minnesota: Mr. Ernest Johnston; Muhlenberg College, Allentown, Pennsylvania: Mr. J. L. Howell; New Mexico State College: Mr. N. W. Wells; New York University: Mr. P. B. Norman, Mr. L. G. Peck; University of Pennsylvania: Dr. H. D. Huskey; Princeton University: Mr. L. L. Rauch; Purdue University: Mr. Paul Irick, Dr. W. P. Reid; Rice Institute: Mr. George Piranian; University of Saskatchewan: Dr. Peter Scherk; Tufts College: Mrs. Mary L. Boas; University of Virginia: Mr. C. L. Clark; University of Washington: Dr. J. M. Kingston; University of Wisconsin: Mr. R. A. Good; Yale University: Dr. S. P. Avann, Mr. William H. Durfee, Dr. C. E. Rickart.

Associate Professor J. R. Bender of the University of Idaho died March 14, 1943. He had been a member of the Society since 1923.

Dr. Guido Fubini, formerly of the University of Turin and recently of the Institute for Advanced Study and New York University, died June 6, 1943. He had been a member of the Society since 1940.

Professor Emeritus G. A. Harter of the University of Delaware died July 22, 1943. He had been a member of the Society since 1922.

Professor James McGiffert of Rensselaer Polytechnic Institute died June 18, 1943, at the age of seventy-five years. He was a member of the Society for fifteen years.

Professor Emeritus W. F. Osgood of Harvard University died July 22, 1943. He had been a member of the Society since 1895.

The editors of the Bulletin wish to correct an error which appeared on page 346 of the current volume. In the second paragraph the words "the late" should not appear before the name C. A. Noble.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

## ALGEBRA AND THEORY OF NUMBERS

### 199. A. A. Albert: *Quasigroups*. I.

Associate with every quasigroup $\mathfrak{G}$ the group $\mathfrak{G}_T$ of nonsingular transformations on $\mathfrak{G}$ generated by the multiplications of $\mathfrak{G}$, and say that a multiplicative system $\mathfrak{G}'$ is isotopic to $\mathfrak{G}$ if there exist nonsingular mappings $A$, $B$, $C$ on $\mathfrak{G}$ to $\mathfrak{G}'$ such that $xA \cdot yB = (x \cdot y)C$ for every $x$ and $y$ of $\mathfrak{G}$. Every quasigroup is isotopic to a loop (that is, a quasigroup with a two-sided identity element $e$). The normal divisors of a loop $\mathfrak{G}$ are then shown to be the subsets $eW$ defined for normal divisors $W$ of $\mathfrak{G}_T$, $\mathfrak{G}$ is simple if and only if the only intransitive normal divisor of $\mathfrak{G}_T$ is the identity group. All loops isotopic to simple loops are simple. A system $\mathfrak{G}$ is homotopic to $\mathfrak{G}'$ if $xA \cdot yB = (x \cdot y)C$ for equivalent mappings $A$, $B$, $C$ on $\mathfrak{G}$ to $\mathfrak{G}'$ which may be singular. Then a loop $\mathfrak{G}$ is *homotopic* to a loop $\mathfrak{G}'$ if and only if $\mathfrak{G}$ is homomorphic to an isotope of $\mathfrak{G}'$. (Received June 7, 1943.)

### 200. William H. Durfee: *Congruence of quadratic forms over valuation rings.*

Let $R$ be a complete valuation ring whose associated residue-class field has characteristic not two. An equivalent diagonal form for an arbitrary quadratic form over $R$ is obtained, and it is shown that two such nonsingular diagonal forms are equivalent if and only if their corresponding subforms composed of those terms having the same value are equivalent. Using this the author proves for forms over $R$ a theorem stated by Witt for fields and extended by Jones to the $p$-adic integers, namely, if $f$, $g$, and $h$ are nonsingular quadratic forms such that $g$ and $h$ each has no variables in common with $f$, then $f+g$ and $f+h$ are equivalent if and only if $g$ and $h$ are equivalent. (Received August 2, 1943.)

### 201. H. L. Lee: *The sum of the kth power of polynomials of degree m in a Galois field*. Preliminary report.

Let $M = c_0x^m + c_1x^{m-1} + \cdots + c_{m-1}x + c_m$ be a polynomial in the Galois field $GF(p^n)$. If $c_0 = 1$, $M$ is called primary and if $c_0 \neq 0$, write deg $M = m$. Let $S_m^k$ and $R_m^k$ denote respectively the sum of the $k$th power of polynomials $M$ of degree $m$, and of all $M$ of degree less than $m$. By the use of two functions $\psi_m(t) - F_m$, $\psi_m(t)$, which vanish when $M$ is primary and of degree $m$ in one case and when deg $M < m$ in the other, the sum may be made to depend on an exponent less than $k$. Then $S_m^k$ and $R_m^k$

depend on the remainder in the division of $t^L$ by $\psi_m(t)-F_m$ and $\psi_m(t)$ respectively. (Cf. H. L. Lee, Duke Math. J. vol. 9 (1943) pp. 277–292.) (Received July 19, 1943.)

### 202. W. V. Parker: *Limits to the characteristic roots of a matrix.*

Let $A=(a_{ij})$ be a square matrix of order $n$ with elements in the field of complex numbers; and define $S_i=\sum_{j=1}^{n}|a_{ij}|$, $T_j=\sum_{i=1}^{n}|a_{ij}|$, $U_i=2|a_{ii}|-S_i$, and $V_j=2|a_{jj}|-T_j$. Let $S$, $T$ be the greatest of the $S_i$, $T_j$, respectively; and let $U$, $V$ be the least of the $U_i$, $V_j$, respectively. It is shown that the absolute value of each characteristic root of $A$ is not less than the greater of the numbers $U$ and $V$ and is not greater than the smaller of the numbers $S$ and $T$. Similar bounds are also found for the real and imaginary parts of the characteristic roots. (Received July 23, 1943.)

### 203. H. E. Salzer: *Table of first two hundred squares expressed as a sum of four tetrahedral numbers.*

The following empirical theorem is conjectured: Every square integer is expressible as the sum of four positive (including zero) tetrahedral numbers $(n^3-n)/6$. It has been verified by a table prepared for the first 200 squares. This empirical theorem is a partial improvement of the statement that five non-negative tetrahedrals suffice for any integer. (See F. Pollock, Proc. Roy. Soc. London Ser. A. vol. 5 (1850).) (Received June 4, 1943.)

## ANALYSIS

### 204. R. H. Cameron and W. T. Martin: *Transformations of Wiener integrals under translations.*

Let $F[y]$ be a functional defined and Wiener summable over the space $C$ consisting of all functions $x(t)$ continuous in $0\leqq t\leqq 1$ and vanishing at $t=0$. In addition, let $F$ be continuous and let it be bounded over every bounded set $x(\cdot)$ of $C$. ($F$ is called continuous if $F[y^{(n)}]\to F[y^{(0)}]$ whenever $y^{(n)}(t)\to y^{(0)}(t)$ uniformly in $0\leqq t\leqq 1$, and $F$ is bounded over every bounded set $x(\cdot)$ of $C$ if for every positive constant $B$ there exists a constant $K=K_B$ such that $|F[y]|\leqq K$ for all $y(\cdot)$ of $C$ for which $|y(t)|\leqq B$, $0\leqq t\leqq 1$.) Under these conditions on the functional $F$ the authors obtain a transformation formula for Wiener integrals under translations of the form $y(t)=x(t)+x_0(t)$ where $x_0(t)$ is a given function of $C$ with a first derivative $x_0'(t)$ of bounded variation in $0\leqq t\leqq 1$. The transformation formula is $\int_C^w F[y]d_wy=\int_C^w F[x+x_0]\exp\{-\int_0^1[x_0'(t)]^2dt-2\int_0^1x_0'(t)dx(t)\}d_wx$. The formula forms a basis for the calculation of various types of Wiener integrals. (Received July 30, 1943.)

### 205. M. M. Day: *Uniform convexity.* IV.

In this paper relationships between uniform convexity, factor spaces, and conjugate spaces are discussed. Theorem 1: A normed vector space $B$ is uniformly convex if and only if all the two dimensional factor spaces of $B$ are uniformly convex with a common modulus of convexity. The concept of uniform flattening is suggested by a description of a "sharp edge" on the unit sphere in terms of the norm of the space. It is shown [Theorem 2] that this is dual to uniform convexity; that is, $B[B^*]$ is uniformly flattened if and only if $B^*[B]$ is uniformly convex. It follows that a complete uniformly flattened $B$ is reflexive. The proof of Theorem 2 uses a computation for two dimensional spaces of the flattening in $B$ in terms of the convexity in $B^*$ and vice versa; Theorem 1 is then applied to complete the proof. (Received June 16, 1943.)

### 206. G. M. Ewing: *Existence theorems for multiple integral variation problems*. Preliminary report.

This is a sequel to a previous paper by the author (abstract 49-1-30) shortly to appear in the Duke Math. J. The methods used there for the single integral case are extended to obtain existence theorems under fairly general hypotheses for the problem of minimizing a certain $k$-tuple integral, $\int f(x, X)du$, on a class of continuous $k$-surfaces in euclidean $n$-space. Metric methods are used to obtain a convergent minimizing sequence but the final theorems closely resemble the Tonelli pattern. (Received July 23, 1943.)

### 207. Tomlinson Fort: *The weighted vibrating string and its limit.*

In this paper the problem of the vibrating string weighted with discrete particles is studied following lectures given by Bôcher years ago at Harvard. In addition to an outline of the work of Bôcher, an interesting special case is studied. A careful passage to the limit is carried through, thus obtaining the solution of the problem of the vibrating nonhomogeneous string in terms of Sturm-Liouville functions. It is believed that not only is the method new but the final form somewhat different from any previous presentation. Incidentally, under proper restrictions, the solution of a differential equation system is proved the limit of the solution of a recurrence system. The method of proof is the method of successive approximations. (Received August 4, 1943.)

### 208. Szolem Mandelbrojt: *Quasi-analyticity and analytic continuation. A general principle.*

The author proves a theorem which leads to results in different branches of the theory of functions. Results on quasi-analyticity on one hand, and results on the distribution of the singularities of a Dirichlet series and the distribution of the values taken by its analytic continuation, on the other hand, may be considered as particular cases of this theorem. (Received July 29, 1943.)

### 209. D. S. Miller: *On Carathéodory and Gillespie linear measure.*

In this paper the relation between the Carathéodory linear measure $L(A)$ and the Gillespie linear measure $G(A)$ of a plane measurable point set will be considered. The inequality $L(A) \leqq G(A) \leqq (\pi/2)L(A)$ will be shown to be the best possible by presenting sets $E$ and $F$ such that $L(E) = G(E)$ and $G(F) = (\pi/2)L(F)$. (Received August 2, 1943.)

### 210. P. C. Rosenbloom: *Sequences of polynomials, especially sections of power series.*

The domain of boundedness $B$ of a sequence of polynomials is defined as the set of points $z$ such that the sequence is uniformly bounded in some neighborhood of $z$. Relations are obtained between $B$, the gap-like structure of the polynomials, and the distribution of their zeros. The connection between the moduli of the zeros and the coefficients is studied in detail. It is proved that under certain rather general conditions the arguments of the zeros are everywhere dense. If $f(z)=a_0+a_1z+\cdots$, then the order of magnitude of the zeros of $s_n(z)$ is roughly $r_n=|a_n|^{-1/n}$ for a certain subsequence of $n$'s. The arguments of the zeros are everywhere dense except if $f(z)$ is an entire function of zero order. Fairly complete results for the case of a finite or zero radius of convergence, or of an entire function of infinite order, are obtained. For entire functions of finite positive order the author obtains essentially all the results announced by Carlson (C. R. Acad. Sci. Paris vol. 179 (1924) pp. 1583–1585), whose proofs have never been published so far as is known to the author. A connection between the zeros of $s_n(z)$ and the conformal mapping $w=z^{-n}f(r_nz)$ is also obtained. These results are applied to special power series. (Received July 31, 1943.)

### 211. Raphael Salem: *A singularity of the Fourier series of continuous functions.*

The following theorems are proved: I. There exists a continuous function $f(x)$ of period $2\pi$ whose Fourier series converges everywhere uniformly, and such that the Fourier series of $f^2(x)$ diverges at a point. II. There exists a continuous function $f(x)$ of period $2\pi$ whose Fourier series converges everywhere uniformly, and such that the Fourier series of $f^2(x)$ diverges at an everywhere dense set of points having the power of the continuum. (Received July 29, 1943.)

### 212. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* I.

The author outlines a heuristic method of detecting functional equations (mainly of the iterative compositional type) whose solutions involve arbitrary functions, by means of generalization of linear functions of variables which are solutions and which contain arbitrary constants as coefficients. It is assumed hypothetically that if a functional equation has a solution of the indicated type then the solution persists if that part of the linear function containing arbitrary constants is suitably replaced by arbitrary functions of the corresponding variables. The method indicated is heuristic in the sense defined in the author's article, *Revue de métaphysique et de morale*, 1914 (Le principe de continuation). If the method does not succeed, other functional equations frequently result whose solutions approach the solution of the original equation. The preceding effects a gradual transition from functional equations previously solved by the author by differentiation (such as certain of his quasi-transitive equations) to equations at least not readily solved in the latter manner. The preceding type of solutions is contrasted with solutions of equations obtained analogously by generalizing linear functions of particular solutions containing arbitrary constants as coefficients. Reference is made to papers previously reported in this Bulletin (vol. 23 (1917) pp. 300, 393; vol. 24 (1918) p. 279). (Received July 31, 1943.)

### 213. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* II.

The author applies the method of the preceding paper to the solution of functional equations previously defined by him and equations relevant to number systems (such as distributive and generalized associative equations) and in suitable instances, their formal inverses relatively to equations of the type: $\phi[f(x, y_1, y_2, \cdots, y_n), y_1, y_2, \cdots,$

$y_n]=x$. Additional hypotheses are of the type: If $f(x_1, x_2, \cdots, x_{n+1})$ is a solution, then $\psi^{-1}f[\psi(x_1), \psi(x_2), \cdots, \psi(x_{n+1})]$ is also a solution where $\psi(x)$ is an arbitrary function with an inverse. Example: $\zeta(u, y_1, y_2, \cdots, y_n)=\tau(v_1, v_2, \cdots, v_{n+1})$ where $u=\tau(x_1, x_2, \cdots, x_{n+1})$ and $v_i=\zeta(x_i, y_1, y_2, \cdots, y_n)$. Solution: $\zeta(x_1, x_2, \cdots, x_{n+1})=x_1 \cdot\alpha(x_2, x_3, \cdots, x_{n+1})$ and $\tau(x_1, x_2, \cdots, x_{n+1})=\beta(w_1, w_2, \cdots, w_n)\cdot x_{n+1}$ where $w_i=x_i/x_{n+1}$ and $\alpha$ and $\beta$ are arbitrary functions. The preceding equation is abstractly self-inverse. Other examples with solutions are given. The solutions indicated are of course not exhaustive. Application is made in suitable instances to the domain of abstract groups; for instance, for $n=1$ the preceding equation has the solution: $\zeta(x_1, x_2)=x_1\cdot x_2^p$, $\tau(x_1, x_2)=(x_1\cdot x_2^{-1})^q\, x_2$ where $p$ and $q$ are arbitrary integers; and analogously for $n>1$. Reference is made to papers previously reported in this Bulletin (vol. 25 (1919) pp. 250, 257). (Received July 31, 1943.)

### 214. Andrew Sobczyk: *On the extension of linear transformations.*

By the Hahn-Banach theorem, any linear transformation $T$ on a subspace $X$ of a normed linear space $Z$, having a one-dimensional range, may be extended to the whole space with preservation of the norm. It is shown that the same is true for any linear transformation, provided that the range $Y$ may be enlarged to a normed linear space $W\supset Y$. Specifically, a norm-preserving extension is always possible with range a subspace of the quotient space $Z/X'$, where $X'\subset X$ is the zero subspace of $T$, and the norm in $Z/X'$ is an extension of the norm on $Y\cong X/X'$ with respect to the Banach norm $\inf_{x'\epsilon X'}\|z+X'\|$. Results are obtained on the question of when the existence of an extension requires the existence of a projection. A classification of linear transformations into four fundamental types is given, with examples showing the existence of each type. (Received July 29, 1943.)

### 215. Otto Szász: *On uniform convergence of Fourier series.*

The main object of this paper is to establish conditions for uniform convergence of a Fourier series at a given point. A necessary condition is that the function be continuous at that point; the additional condition restricts the coefficients to a certain type of slow oscillation, such as were used in Tauberian problems. The result yields Gibbs' phenomenon for such series at points of discontinuity. A theorem of a converse nature is: If $f(\theta)\sim\sum b_n \sin n\theta$ is continuous at $\theta=0$, and if for some positive constants $K$ and $C$, $0\leqq(n+1)b_{n+1}+K\leqq(1+C/n)(nb_n+K)$, $n\geqq 1$; then $nb_n\to 0$. (Received July 26, 1943.)

### 216. H. S. Wall and Marion D. Wetzel: *Positive definite forms and convergence theorems for continued fractions.*

The form (1) $\sum_{p=1}^{n}(\beta_p+y)x_p^2-2\sum_{p=1}^{n-1}\alpha_p x_p x_{p+1}$ is positive definite for all $y>0$ if and only if (2) $\beta_p\geqq 0$, $\alpha_p^2=\beta_p\beta_{p+1}(1-g_{p-1})g_p$, $0\leqq g_p\leqq 1$. The $J$-fraction (3) $1/(b_1+z)-a_1^2/(b_2+z)-a_2^2/(b_3+z)-\cdots$ is called positive definite if $\beta_p=\Im(b_p)$, $\alpha_p^2=\Im(a_p)$ satisfy (2). There exists a nest of circles $K_p(z)\supset K_{p+1}(z)$ such that the $p$th approximant of (3) is upon $K_p(z)$, $(\Im(z)>0)$. The distinction between the two cases $r_p(z)\to 0$, $r_p(z)\to r(z)>0$ ($r_p(z)=$radius of $K_p(z)$) is invariant under change in the value of $z$ in $\Im(z)>0$. If $\liminf |a_p|$ is finite, then $r_p(z)\to 0$. This, along with (2), leads to the theorem that the continued fraction (4) $1/1+c_1/1+c_2/1+\cdots$ converges if $\liminf |c_p|<\infty$ and $|c_p|-\Re(c_p)\cos(\phi_p+\phi_{p+1})+\Im(c_p)\sin(\phi_p+\phi_{p+1})\leqq[2\cos\phi_p\cos\phi_{p+1}(1-g_{p-1})g_p]/$

$[(1+\gamma \sec \phi_p)(1+\gamma \sec \phi_{p+1})]$, $\gamma>0$, $-(\pi/2)<\phi_p<+(\pi/2)$, $0\leqq\xi_p\leqq 1$. Also, (4) converges if $\liminf |c_p| < \infty$ and $\sum [|c_p| - \Re(c_p)] < 2$. (Received July 2, 1943.)

### 217. F. T. Wang: *On Riesz summability of Fourier series.* III.

Denote by $\phi_\alpha(t)$ the fractional integral of order $\alpha$ of an even integrable function $\phi(t)$, periodic, with period $2\pi$. The following two results are proved: (i) If $\phi_\alpha(t)=o(t^\gamma)$, $\gamma>\alpha>0$, then the Fourier series of $\phi(t)$ is summable $(R, e^{n^{1-\alpha/\gamma}}, \gamma+\delta)$, $\delta>0$, at $t=0$. (ii) If $\phi_\alpha(t)=o(t^\alpha(\log t)^{-1})$, $\alpha>0$, then the Fourier series of $\phi(t)$ is summable $(R, n^{(\log n)^{1/\alpha}}, \alpha+1)$ at $t=0$. (Received June 11, 1943.)

## APPLIED MATHEMATICS

### 218. Stefan Bergman: *Solutions of linear partial differential equations of fourth order.*

Generalizing the known formula $u=\mathrm{Re}[f_1(z)+\bar z f_2(z)]$, $z=x+iy$, $\bar z=x-iy$, for the biharmonic functions (that is, functions $u$ for which $\Delta\Delta u=16u_{z\bar z z\bar z}=0$), the author proves that for every equation $L(u)=U_{z\bar z z\bar z}+a_1u_{zz}+a_2u_{z\bar z}+a_3u_{\bar z\bar z}+a_4u_z+a_5u_{\bar z}+a_6u=0$ where $a_\nu=a_\nu(z,\bar z)$ are analytic functions of $z$ and $\bar z$ there exist two functions of $z$, $\bar z$ and a real variable $t$, $E_k(z,\bar z,t)$, $k=1,2$, such that every solution of $L(u)=0$ which is regular in a star domain $\mathfrak{D}$ can be represented in $\mathfrak{D}$ in the form $u(\bar z)=\mathrm{Re}\{\int_{-1}^{+1}[\sum_{k=1}^{2}E_k(z,\bar z,t)f_k \cdot (z(1-t^2)/2)dt/(1-t^2)^{1/2}]\}$. The methods and results of the paper *Linear operators in the theory of partial differential equations* (Trans. Amer. Math. Soc. vol. 53 pp. 130–155) can be applied to the functions $u$, satisfying $L(u)=0$. (Received July 30, 1943.)

### 219. G. E. Forsythe: *Note on equivalent potential temperature.*

Let a parcel of moist air be saturated with $w_s=w_s(p,T)$ tons of water vapor per ton of dry air. Let the following quantities be measured in meter-ton-second-absolute degree mechanical units: $T$ = temperature; $p$ = total air pressure; $L=L(T)$ = latent heat of evaporation of water; $c_p=c_p(T)$ $[c_v=c_v(T)]$ = specific heat of dry air at constant pressure [volume]; $e_s=e_s(T)$ = vapor pressure of saturated water vapor; $p_d=p-e_s$; $k=(c_p-c_v)/c_p$; $\theta_d=T(100/p_d)^k$. Let $\lambda=\lambda(p,T)=Lw_s/(c_pT)$. Rossby (Massachusetts Institute of Technology Meteorological Papers, vol. I, no. 3 (1932)) defines the equivalent potential temperature $\theta_e$ by the relation $\theta_e=\theta_d\exp\lambda$. He asserts without proof that, as $p\to 0$ in a process for which $\theta_e$ is constant, one has $\theta_d\to\theta_e$. The present note uses elementary estimates to prove Rossby's assertion: (i) under the oversimplifying assumption that $L$ and $c_p$ are bounded away from 0, as $T\to 0$; (ii) under weaker but physically artificial assumptions about $L$ and $c_p$. The note includes a further discussion of the important meteorological quantity $\lambda$. (Received July 26, 1943.)

### 220. S. H. Gould: *The Rayleigh-Ritz method for higher eigenvalues.*

An elementary proof is given of the following theorem, fundamental in applications of the Rayleigh-Ritz method for calculating the eigenvalues of a given variational eigenvalue problem: the $n$ roots, arranged in order of magnitude, of the determinantal equation obtained by using $n$ coordinate functions are upper bounds respectively for the first $n$ eigenvalues of the original problem. (Received July 14, 1943.)

### 221. G. E. Hay and Willy Prager: *On plane rigid frames containing curved members.*

In a recent paper (Quarterly of Applied Mathematics vol. 1 (1943) pp. 49–60), the authors have extended the method of the conjugate beam to the case of plane rigid frames consisting of straight members and loaded perpendicularly to the plane of the frame. In the present paper, the method is again extended to include frames containing curved members, with loading both in the plane of the frame and perpendicular to it. There is introduced a conjugate frame similar in shape to the original frame, but with different supports in general. This conjugate frame is subjected to a load which is a certain function of the distribution in the original frame of bending moment, Young's modulus, and the moment of inertia of the cross section. The equilibrium conditions of this conjugate frame afford a simple determination of redundant reactions in the original frame. Also, the distribution of bending moment and shear in the conjugate frame are equal respectively to the distribution of displacement and rotation in the original frame, which fact affords a simple determination of the distribution of displacement and rotation in the original frame. (Received July 31, 1943.)

### 222. F. B. Hildebrand: *On the stress distribution in cantilever beams.*

The problem of determining the stress distribution in a flanged cantilever beam of narrow rectangular cross section, supported in such a way that no distortion of the cross section takes place at one end of the beam, is treated as a problem in the theory of plane stress. A limiting case of anisotropy is considered, in which the material of the beam is perfectly rigid in the direction of the prescribed transverse loading which acts in the plane of the web plate. The exact solution is found in terms of a rapidly convergent series of almost periodic functions and a corresponding approximate solution is obtained in closed form by a direct application of the calculus of variations. The results are evaluated numerically in representative cases and, on the basis of comparisons with earlier solutions of related problems (F. B Hildebrand and E. Reissner, Journal of Applied Mechanics vol. 9 (1942) pp. A-108–A-116), conclusions regarding the validity of certain assumptions and approximations are drawn. (Received July 27, 1943.)

### 223. E. G. Kogbetliantz: *Detailed quantitative interpretation of maps of gravitational anomalies with the aid of mathematical analysis.* Preliminary report.

The importance of gravific methods in geophysical surveying, especially in the prospecting for new oil fields, is very well known. The interpretation of gravity and other gravific maps is today limited to only qualitative conclusions. All attempts made for their qualitative interpretation, that is, for the numerical determination of important geological parameters such as depth below the surface, thickness of the disturbing layer, slope of an anticlinal flank, were unsuccessful. The reason for these systematic failures is the fact that the observed values, mapped after all usual corrections—correction for the regional anomaly included—reflect not only the effect due to the studied underground structure or body but also the gravific action of the immediate neighborhood under the station. This punctual anomaly due to the nonhomogeneity of the ground and affecting only one point-station cannot be evaluated and corrected. To eliminate from the interpretation the errors due to these unknown punctual anomalies, it is necessary to use in the computations of geological parameters only the average values. In these average values are melted all observed values and the effect of punctual anomalies is eliminated because their average values are negligible. The importance of this new interpretation-method is illustrated by an anomaly map due to an anticlinal for which the numerical values of all its geological parameters are expressed by definite integrals. These integrals are easily computed with the aid of a planimeter if the gravific map is given. (Received August 2, 1943.)

### 224. A. N. Lowan and H. E. Salzer: *Table of coefficients for inverse interpolation.*

Although many tables have been published to facilitate direct interpolation, there has been to date no table of coefficients to handle the more cumbersome task of inverse interpolation. To meet this need, the Mathematical Tables Project has computed the polynomial coefficients of the ratios of differences of various orders in the formula obtained by the inversion of the Everett formula for direct interpolation (H. T. Davis, *Tables of the higher mathematical functions*, vol. 1, pp. 82–83). The coefficients of the five fourth order terms were calculated to ten decimal places at intervals of 0.001 of the argument $m=(u-u_0)/(u_1-u_0)$. Also a short table gives the values of the coefficients of the ten sixth order terms. The values of the coefficients were computed at intervals of 0.1. It was not necessary to compute the coefficients of the second order terms, since these coefficients are tabulated at intervals of 0.0001 in the table of *Lagrangian interpolation coefficients* to be published by the Columbia University Press. The table here described will be of particular value whenever inverse interpolation is to be carried out in a table with a fairly large tabular interval. It is proposed to designate the coefficients here discussed as "inversolants." (Received June 11, 1943.)

### 225. Morris Marden: *On the stream function of axially symmetric flows.*

In his papers (Math. Zeit. vol. 27 (1926) p. 641 and Math. Ann. vol. 99 (1928) p. 629) Bergman studies the behavior of harmonic functions in the neighborhood of point and line singularities. Using the fact that the functions $\Phi=\Phi_1+i\Phi_2=(1/2\pi)\cdot\int_0^{2\pi}f(u)dt$ and $\Psi=\Psi_1+i\Psi_2=(1/2\pi)\int_0^{2\pi}f(u)(x-u)dt$, where $u=x+iy\cos t+iz\sin t$, are the potential and stream function of two three-dimensional axial-symmetric flows of an incompressible fluid, the author investigates the behavior of the stream function in the neighborhood of the above mentioned singularities. The author constructs certain functions possessing infinitely many line singularities (periodic stream functions) and proves certain properties of these functions. (Received July 30, 1943.)

### 226. Seymour Sherman: *Finite aspect ratio effect in the Glauert-Prandtl regime.*

The usual simplified development of the Prandtl integral equation for the circulation about a wing of finite aspect ratio is not readily extended to the case of compressible flow because the expression for the downwash due to a vortex line becomes more complicated. By first making the usual assumptions of thin airfoil theory and linearising the equations of compressible flow and then appealing to the work of E. Reissner (*The equations of lifting strip theory for tapered wings*, Curtiss Research Laboratory Report SB-76-S-1) on the flutter of finite aspect ratio it is shown that the compressibility effect can be included in the Prandtl equation by substituting for $\partial C_{L\,\mathrm{inc}}(y)/\partial\alpha$ the quantity $(1/(1-\beta^2)^{1/2})\partial C_{L\,\mathrm{inc}}(y)/\partial\alpha$. This is done by setting up the problem as a boundary value problem for the appropriate elliptic partial differential equation. (Received July 26, 1943.)

### 227. Alexander Weinstein: *On the bending of a clamped plate.*

Let $\{p_i(x, y)\}$ denote a complete, but not necessarily orthogonal, sequence of harmonic functions of integrable square in a domain $S$ with the boundary $C$. Consider the equation $\Delta\Delta w = f$ with the boundary conditions $w = dw/dn = 0$. Let $w_m = GGf + \sum_{i=1}^{m} a_{mi} G p_i$, where $Gf$ is the Green's potential of $f$, and the constants $a_{mi}$ are determined by the equations $\sum_{i=1}^{m} a_{mi}(p_i, p_k) = -(f, Gp_k)$, $k = 1, 2, \cdots, m$. It is proved in this paper that $w_m$ converges uniformly to $w$ in $S+C$ and that $\partial w_m/\partial x$, $\partial w_m/\partial y$ converge uniformly to $\partial w/\partial x$, $\partial w/\partial y$ in every domain interior to $S$. The approximating functions $w_m$ can be computed explicitly in the case when $S$ is a rectangle. The proofs are based on the results of a paper by N. Aronszajn and A. Weinstein, Amer. J. Math. vol. 44 (1942) p. 625. (Received July 10, 1943.)

## GEOMETRY

### 228. Herbert Busemann: *On spaces in which two points determine a geodesic.*

Let $R$ be a finitely compact, convex, and locally strictly externally convex, metric space. A geodesic in $R$ is defined as a continuous curve which is locally isometric with the real axis. (A symmetric variational problem in parametric form will satisfy these conditions, when the extremals are considered as geodesics). There is at least one geodesic through any two distinct points of $R$. If this geodesic is unique, then $R$ is either simply connected and all geodesics are isometric with a euclidean straight line, or $R$ has a two-sheeted universal covering space and all geodesics of $R$ are congruent to one euclidean circle. (Received July 26, 1943.)

### 229. John DeCicco: *Kasner's pseudo-angle.*

The theory of functions of a single complex variable is identical with plane conformal geometry. This is not the case in the theory of functions of two or more complex variables. A set of $n$ functions of $n$ complex variables induces a correspondence between the points of a real $2n$-dimensional space $R_{2n}$. The infinite group $G$ of these correspondences has been termed the pseudo-conformal group by Kasner. In 1908, Kasner showed that for $n = 2$ a transformation of $R_4$ is pseudo-conformal if and only if it preserves the pseudo-angle between a curve and a three-dimensional variety. This pseudo-angle theorem can be carried over to $2n$ dimensions almost without change. A system of $(2n-1)$-dimensional hypersurfaces is said to be bi-isothermal if it is pseudo-conformally equivalent to $\infty^1$ parallel $(2n-1)$-dimensional flats. Any system of $(2n-1)$-dimensional hypersurfaces is bi-isothermal if and only if the pseudo-angle between the given hypersurfaces and any system of parallel lines is a biharmonic function. Many other results are obtained by using Kasner's pseudo-angle. (Received July 27, 1943.)

### 230. V. G. Grove: *A general theory of surfaces and conjugate nets.*

This paper studies the projective differential geometry of surfaces and conjugate nets by means of a tensor analysis based on a connection appearing naturally in the theory. All congruences harmonic or conjugate to a surface are found. A simple geometric construction for reciprocal congruences is given independent of the quadrics of Darboux. A simple tensor method of studying conjugate nets is used. Among other applications, a certain interesting involution on each line protruding from the surface is studied. Finally some of the results are specialized to the metric geometry of surfaces in euclidian space of three dimensions. (Received July 23, 1943.)

### 231. C. C. Hsiung: *An invariant of intersection of two surfaces.*

Let two surfaces $S_1$, $S_2$ in ordinary space intersect at an ordinary point $O$ with distinct tangent planes $\tau_1$, $\tau_2$; assume that the common tangent is distinct from the asymptotic tangents of the surfaces $S_1$, $S_2$. The author proves the existence of a projective invariant determined by the second order terms of the surfaces $S_1$, $S_2$ at the point $O$, and he gives the invariant a projective as well as a metric characterization. The metric characterization takes the following simple form. Let $K_1$, $K_2$ be the total curvatures of the surfaces $S_1$, $S_2$ at the point $O$; and $R_1$, $R_2$ the radii of curvature at $O$ of the curves in which the tangent planes $\tau_2$, $\tau_1$ intersect the surfaces $S_1$, $S_2$ respectively. Then $R_2^4K_2/R_1^4K_1$ is the projective invariant under consideration. (Received July 1, 1943.)

### 232. C. C. Hsiung: *Projective invariants of some particular pairs of space curves.*

It is known that if two plane curves have contact of order $k-1$ at a point $P$, and if nonhomogeneous projective coordinates are chosen so that the expansions representing these two curves in the neighborhood of $P$ are, respectively, $y=ax^k+\cdots$, $y=\alpha x^k+\cdots$, $a\neq\alpha$, then the ratio $a/\alpha$ is a projective invariant studied for $k=2$ by H. J. S. Smith (Proc. London Math. Soc. (1) vol. 2 (1867) pp. 196–248) and R. Mehmke (Schlömilchs Zeitschrift für Mathematik und Physik vol. 36 (1891) pp. 56–60, 206–213). Their results were interpreted and extended by C. Segre (Rendiconti dei Lincei (5) vol. 6 (1897) pp. 168–175 and vol. 33 (1924) pp. 325–329). The present paper supplements these investigations by studying pairs of curves in 3-space: (a) tangent at a general point with distinct osculating planes, (b) intersecting at a general point with distinct osculating planes, (c) intersecting at a general point with common osculating plane. The existence of a projective invariant determined by the third order terms is shown for each of these cases; these invariants are characterized geometrically in terms of certain double ratios. (Received June 11, 1943.)

### 233. C. C. Hsiung: *Theory of intersection of two plane curves.*

This paper is concerned with the study of the projective differential geometry of two plane curves intersecting at an ordinary point. The method used is similar to the one given by the author in a previous paper (*Projective differential geometry of a pair of plane curves*, to appear in Duke Math. J.). Here two projective invariants determined by the fourth order terms in the Taylor expansions of the functions representing the two curves, at the point of intersection, are obtained and characterized geometrically. Further, on the basis of the vanishing or nonvanishing of these two invariants, the author arrives at four different types of canonical representation for the two curves at the point of intersection; the absolute invariants in the expansions of each type are interpreted geometrically in terms of certain double ratios. (Received June 11, 1943.)

### 234. Edward Kasner: *Motion in a resisting medium.*

Consider the motion of a particle moving in the plane under a generalized field of force and influenced by a resisting medium, the resistance $R$ acting in the direction of the motion and varying as some function of the position $(x, y)$, the direction $y'$, and the speed $v$ of the particle. Through a given lineal element $E$, there pass $\infty^1$ trajectories. If the osculating parabolas are constructed to these trajectories at $E$, the locus of the foci varies in shape with the nature of the resistance $R$. If the focal locus is a circle through the point of $E$, it is found that $R$ must be of the form $A(x, y, y')v^2 + B(x, y, y')$. In the final part of the paper, this result is extended to space. Construct the osculating spheres at $E$ to the $\infty^1$ trajectories passing through the lineal element $E$. If the locus of the centers of these spheres is a straight line, the resistance $R$ is of the form $A(x, y, z, y', z')v^2+B(x, y, z, y', z')$. Finally it is shown that if a single trajectory is known in the field of gravity with a resisting medium $R(v)$, then the law of resistance $R(v)$ can be completely determined. (Received July 27, 1943.)

### 235. Edward Kasner and John DeCicco: *A generalized theory of contact transformations.*

The authors present a generalized theory of contact transformations in the plane. Any union-preserving transformation of differential elements of order $n$ into lineal elements is obtained by considering the osculating (up to and including contact of order $n$) of a given parameterized family of $\infty^{n+1}$ curves to an arbitrary curve. If a union-preserving transformation $T$ is such that any two unions which possess $n \geqq 2$ as the order of contact are converted by $T$ into two unions which have at least second order contact, then $T$ must be a contact transformation between lineal elements. As a consequence of this work, the authors find a general theory of evolutes and involutes which contains the osculating circle theory of Huygens and Bernoulli as a special case. (Received July 27, 1943.)

## LOGIC AND FOUNDATIONS

### 236. Theodore Hailperin: *A set of axioms for logic.*

Two well known logical systems claiming adequacy for mathematics are currently studied. These are appellatively described as "Principia Mathematica" and "set-theory." A third and stronger system, called "New Foundations," has been proposed by W. V. Quine. (This system is not to be confused with his *Mathematical logic*, 1940.) Quine's system uses the usual logical primitives for the propositional calculus and the theory of quantifiers, and class membership, but makes no restrictions on the ranges of the variables. To avoid the contradictions, Quine has a metalogical axiom to the effect that only "stratified" formulas determine classes. The purpose of the present paper is the presentation of nine formal axioms and a demonstration of their equivalence with Quine's metalogical axiom in the presence of the restricted predicate calculus and the axiom for identity. (Received July 29, 1943.)

## NUMERICAL COMPUTATION

### 237. H. E. Salzer: *Coefficients for numerical integration with central differences.*

The numbers $M_{2s}=B_{2s}^{(2s)}(s)/(2s)!$, where $B_{2s}^{(2s)}(s)$ denotes the $(2s)$th Bernoulli polynomial of order $2s$ for argument equal to $s$, were computed for $s=1, 2, \cdots, 10$, the result being exact values in lowest terms. Then these quantities were all checked by a cumulative recursion formula. (Previous calculations went only as far as $s=4$.) The numbers $M_{2s}$ are coefficients of central differences of order $2s-1$ in the well known formula for numerical integration, sometimes called the Gauss-Encke formula or the second Gaussian summation formula. All important formulas for and references to these coefficients are included. This calculation was performed in the course of work done for the Mathematical Tables Project, National Bureau of Standards. (Received July 7, 1943.)

## STATISTICS AND PROBABILITY

### 238. E. J. Gumbel: *On the plotting of statistical observations.*

It is well known that there exist two stepfunctions corresponding to a continuous variate. We may attribute to the $m$th observation the ranks $m$ or $m-1$. To obtain one and only one serial number $m$, which will, in general, not be integer, we attribute to $x_m$ an adjusted frequency $m-\Delta$, namely the probability of the most probable $m$th value. The correction $\Delta$ for the rank is unlimited and possesses a mode, $\Delta$ increases for increasing value of the variate from zero up to unity. The correction is important for small numbers of observations. For large numbers of observations and for the ogive it is sufficient to choose $\Delta=1/2$. The calculation of $\Delta$ allows a correct plotting of all observations (including the first and last) on probability paper (equiprobability test). For the return periods, the ranks $m$ and $m-1$ correspond to the observed exceedance and recurrence intervals. Generally the corrected return periods pass for increasing values of the variate from the exceedance to the recurrence intervals, provided the variate is unlimited and possesses a single mode. The asymptotic standard error of the partition values may be used to construct confidence bands for the ogive, the equiprobability test, and the return periods. This control for the fit between theory and observation may be applied to all observations which are not extreme. (Received July 30, 1943.)

### 239. Henry Scheffé: *On a measure-theoretic problem arising in the theory of non-parametric tests.*

Let $\mu$ be any measure on the real line, such that the measure of the whole line is unity, and form the "power" measure $\mu^k$ in Euclidean $k$-space—that is, the product measure obtained by using $\mu$ on each axis. The problem is, for a given class $C$ of measures $\mu$, what can be said about the existence and structure of sets of points in the $k$-space which have the property that their "power" measure is the same for all measures $\mu$ in $C$? Necessary conditions and sufficient conditions for sets to have the above property are found for certain large classes $C$ of statistical interest. (Received July 24, 1943.)

## TOPOLOGY

### 240. H. D. Huskey: *On polyhedra and polyhedral path surfaces.*

In the study of surfaces in parametric form there are several definitions of polyhedra or of polyhedral path surfaces in common use. The equivalence of these definitions is discussed in this paper. A single polyhedral path surface may have several representations; the paper goes on to show directly that the area of a polyhedral path surface is independent of the representation (this result follows indirectly from results in the literature, see T. Radó: *On the semicontinuity of double integrals in parametric form*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 336–361). Finally it is shown that the Lebesgue area of a surface is independent of the type of polyhedra used in the definition. (Received July 16, 1943.)

### 241. A. D. Wallace: *A structural property of transformations.*

Let $M$ and $N$ be compact connected Hausdorff spaces and let $f(M)=N$ be continuous. Suppose further that $f$ satisfies the condition: If $M=A+B$ where $A$ and $B$ are closed and $A \cdot B$ is a point, then for any continuum $K$ in $M$ it is true that $f(K \cdot A) = f(K) \cdot f(A)$. The following results have been obtained: (1) If $E$ is a prime-chain in $N$ there exists a unique prime-chain $F$ in $M$ such that $E$ is contained in $f(F)$ and (2) If $X$ is a chain in $M$ then $f(X)$ is a chain in $N$. For $f$ non-alternating and metric $M$ and $N$ the above results are due respectively to G. T. Whyburn and G. E. Schweigert. If $f$ is non-alternating then the above condition on $f$ holds, but not conversely. Indeed $f$ may be completely alternating. (Received July 28, 1943.)

### 242. Paul White: *On r-regular convergence.*

In this paper the notion of regular convergence as introduced by G. T. Whyburn (Fund. Math. vol. 25 (1935) is further studied. It is shown that a sequence of irreducible membranes of homologies or of irreducible carriers of $r$-dimensional cycles has as its limit under $s$-regular convergence ($s \leqq r$) a set of the same type. Finally it is shown that under this type of convergence a sequence of $r$-dimensional closed Cantorian manifolds converges to another $r$-dimensional closed Cantorian manifold. (Received July 2, 1943.)

# NEW PUBLICATIONS

BRENKE, W. C. See CELL, J. W.

CASTELNUOVO, G. Lecciones de geometría analítica. Translated by A. LEVIALDI and M. SADOSKY from 7th Italian edition. La Plata, Editorial Mundo Científico, 1943. 7+657 pp.

CELL, J. W., BRENKE, W. C., MOORE, G. E., and TORRANCE, C. C. Engineering problems illustrating mathematics. New York and London, McGraw-Hill, 1943. 11+172 pp. $1.75.

EMDE, F. See JAHNKE, E.

FOLLEY, K. W. See NELSON, A. L.

HART, W. L., WILSON, W. A., and TRACEY, J. I. First year college mathematics. Boston, Heath, 1943. 7+279+2+124 pp. $4.00.

JACOBSON, N. The theory of rings. (Mathematical Surveys, no. 2.) New York, American Mathematical Society, 1943, 6+150 pp. $2.00.

JAHNKE, E., and EMDE, F. Tables of functions with formulae and curves. Revised edition. New York, Dover, 1943. 14+303+76 pp.

LEVIALDI, A. See CASTELNUOVO, G.

LOVE, C. E. Differential and integral calculus. 4th edition. New York, Macmillan, 1943. 15+483 pp. $3.25.

MOORE, G. E. See CELL, J. W.

MORGAN, F. M. College algebra. New York, American, 1943. 6+368 pp. $2.00.

NELSON, A. L., and FOLLEY, K. W. Plane and spherical trigonometry. Mathematical tables. Revised edition. New York, Harper, 1943. 14+247+135 pp. $2.40.

SADOSKY, N. See CASTELNUOVO, G.

SHOHAT, J. A., and TAMARKIN, J. D. The problem of moments. (Mathematical Surveys, no. 1.) New York, American Mathematical Society, 1943. 14+140 pp. $2.50.

TAMARKIN, J. D. See SHOHAT, J. A.

TORRANCE, C. C. See CELL, J. W.

TRACEY, J. I. See HART, W. L.

WILSON, W. A. See HART, W. L.

# RADICAL EXTENSIONS AND CROSSED CHARACTERS

REINHOLD BAER

E. Witt[1] has given a theory of abelian extensions of fields containing sufficiently many roots of unity which consists essentially, as has been remarked before,[2] in applying the theory of characters of finite abelian groups. There exists now a sufficiently developed theory of crossed characters,[3] and it is the object of this note to show that a fairly complete and simple theory of radical extensions may be obtained if one follows Witt's treatment[4] of abelian extensions, only substituting for the classical theory of characters the theory of crossed characters.[5]

Suppose that the commutative field[6] $K$ is a finite, normal, and separable extension of the field $F$, that the characteristic of the field $K$ is either 0 or prime to the given integer $m$, and that $E$ is the group of the $m$th roots of unity contained[7] in $K$. The Galois group $G$ of the extension $K$ of $F$ consists of all the $F$-automorphisms of the field $K$ (automorphisms of the field $K$ which leave the elements in $F$ invariant). Every automorphism $g$ in $G$ induces in $E$ an automorphism which we also denote by $g$, and the correspondence mapping the automorphism $g$ in $G$ upon the automorphism $g$ of $E$ shall be denoted by $C$.

A $C$-character of the group $G$ is a single-valued $G$ to $E$ function $f(g)$, satisfying the functional equation $f(u)^v f(v) = f(uv)$.

LEMMA 1. *The function $v^{1-g}$ of the element $g$ in $G$, for $v$ an element in $K$, is a $C$-character of $G$ if, and only if, $v^m$ is an element, not* 0, *in $F$.*

PROOF. If $v^{1-g}$ is a $C$-character of $G$, then $v^{1-g}$ is for every $g$ in $G$ an element in $E$, so that $(v^m)^{1-g} = 1$ for every $g$ in $G$. Thus $v^m$ is a fixed element of the Galois group of $K$ over $F$, proving that $v^m$ is an element, not 0, in $F$.

Assume conversely that $v^m \neq 0$ is in $F$. Then $(v^{1-g})^m = (v^m)^{1-g} = 1$ for

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[1] Witt [1], [2]; the references refer to the bibliography at the end of the paper.

[2] Baer [1].

[3] Baer [2].

[4] Or the treatment as suggested in Baer [1].

[5] The importance of the theory of crossed characters for the theory of radical extensions has recently been stressed by MacLane-Schilling [1].

[6] As all the fields will be commutative, we shall omit the word "commutative" in the future.

[7] $K$ need not contain $m$ distinct $m$th roots of unity; cp. Theorem 3 below.

every $g$ in $G$, so that $v^{1-g}$ is a single-valued $G$ to $E$ function. If $s$, $t$ are elements in $G$, then $v^{1-st}=v^{1-t}v^{t-st}=(v^{1-s})^t v^{1-t}$, and we have shown that $v^{1-g}$ is a $C$-character of $G$.

We denote, as customary, by $F^*$ the multiplicative group of the elements, not 0, in $F$, and it will be convenient to denote by $K_m$ the multiplicative group of those elements in $K$ whose $m$th power is in $F^*$. Clearly $F^*$ is a subgroup of $K_m$.

THEOREM 1. *An isomorphism of the group $K_m/F^*$ upon the group of all the $C$-characters of $G$ is effected by mapping the element $v$ in $K_m$ upon the function $v^{1-g}$ of the elements in $G$.*

PROOF. It is a consequence of Lemma 1 that $v^{1-g}$ is a $C$-character of $G$ whenever $v$ is an element in $K_m$, and it is readily seen that a homomorphism of $K_m$ into the group of all the $C$-characters of $G$ is effected by mapping $v$ upon the function $v^{1-g}$ of the elements in $G$. Furthermore $v^{1-g}=1$ for every $g$ in $G$ if, and only if, $v$ is an element in $F^*$, as follows from the fundamental theorem of Galois theory, and this shows that exactly the elements in $F^*$ are mapped upon the $C$-character 1. To complete the proof of our theorem we need show only that[8] every $C$-character of $G$ may be represented in the form $v^{1-g}$ for some $v$ in $K_m$. Suppose therefore that $f(g)$ is a $C$-character of $G$. Assume that there did not exist an element $w$ in $K$ such that $\sum_{g \text{ in } G} w^g f(g)\neq 0$. Then $\sum_{g \text{ in } G} (w^h)^g f(g)=0$ for every $h$ in $G$ and every $w$ in $K$. Since the elements $f(g)$ are different from 0 as elements in $E$, this implies that the determinant $D(w)=|w^{hg}|=0$ for every $w$ in $K$. But this is impossible, since the normal basis theorem[9] assures the existence of elements $w$ in $K$ such that $D(w)\neq 0$. Consequently there exists an element $w$ in $K$ such that $v=\sum_{g \text{ in } G} w^g f(g)\neq 0$. If $h$ is some element in $G$, then

$$v^h f(h) = \sum_{g \text{ in } G} w^{gh} f(g)^h f(h) = \sum_{g \text{ in } G} w^{gh} f(gh) = v,$$

since $f$ is a $C$-character and $gh$ ranges over $G$ with $g$. Since $v\neq 0$, we find that $v^{1-h}=f(h)$ for every $h$ in $G$, and it is a consequence of Lemma 1 that $v$ belongs to $K_m$, as was to be shown.

The following restatement of Theorem 1 will prove helpful: If $V$ is a coset of $K_m/F^*$, then $V^{1-g}$ is a $C$-character of $G$, and every $C$-character of $G$ may be represented in one and only one way in the form $V^{1-g}$.

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[8] The following arguments are essentially a restatement of a proof of Speiser [1] p. 3.

[9] Cp. Deuring [1].

We recall that a group $B$ of $C$-characters of $G$ is termed[10] *complete* if $B(g)=1$ implies $g=1$.

THEOREM 2. *If $M$ is a multiplicative group between $F^*$ and $K_m$, then the following two properties of $M$ are equivalent:*

(i) $K=F(M)$.

(ii) *The group of the C-characters $V^{1-\sigma}$ for $V$ in $M/F^*$ is complete.*

PROOF. The statement (ii) is obviously equivalent to the fact that 1 is the only $F(M)$-automorphism of $K$, and that this fact is equivalent to (i) is readily deduced from the Galois theory.

The group $G$ has been termed[11] *C-complete* if 1 is the only element in $G$ which is mapped upon 1 by every $C$-character of $G$. The following statement is an immediate consequence of Theorem 2.

COROLLARY 1. *The group $G$ is C-complete if, and only if, $K$ may be obtained by adjoining to $F$ $m$th roots of elements in $F$.*

The $C$-characters of $G$ which have the form $e^{1-\sigma}$ for suitable $e$ *in the group $E$* have been termed *principal* $C$-characters of $G$, and the group of principal $C$-characters is called the *principal genus*. A complete group of $C$-characters of $G$ is said to be *strictly complete* if it contains the principal genus.[12]

COROLLARY 2. *If $M$ is a multiplicative group between $F^*$ and $K_m$, then the following two conditions are necessary and sufficient for the group of C-characters of the form $V^{1-\sigma}$ with $V$ in $M/F^*$ to be strictly complete:*

(a) $K=F(M)$;

(b) *$M$ is the set of all the elements in $K$ whose $m$th powers are in the subgroup $M^m$ of $F^*$.*

PROOF. It is readily seen that condition (b) is equivalent to the inequality $E\leq M$, and hence (b) is true if, and only if, the group of $C$-characters of the form $V^{1-\sigma}$ for $V$ in $M/F^*$ contains the principal genus. Thus Corollary 2 is an immediate consequence of Theorem 2.

The importance of Corollary 2 stems from the comparative rarity of strictly complete groups of $C$-characters which are different from the group of all the $C$-characters.[13]

If $M$ is a group between $F^*$ and $K_m$, then $M^m$ is a group between $F^{*m}$ and $K_m^m\subseteq F^*$. However, the groups $M/F^*$ and $M^m/F^{*m}$ need not be isomorphic, since the homomorphism $x\rightarrow x^m$ maps upon 1 all the $m$th roots of unity in $M$, and these need not be in $F^*$.

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[10] Baer [2] introduction to chapter IV.

[11] Baer [2] I.3.

[12] Baer [2] introduction to chapter IV.

[13] Baer [2] Corollary IV.4.2.

In an investigation of extensions by $m$th roots it is clearly no loss of generality to assume that $m$ has been chosen as small as possible. Then the positive integer $m$ has been determined in such a way that $K=F(K_m)$, though $F(K_{m'})<K$ for every proper divisor $m'$ of $m$, and this signifies in group-theoretical language that $m$ is the maximum order of the elements in the multiplicative abelian group $K_m/F^*$. If we use the notation $m=\prod_p p^{m(p)}$, where the product ranges over all the prime numbers $p$ and where almost all the exponents $m(p)$ vanish, then the maximum order of the elements in $K_m/F^*$ is $m$ if, and only if, $K_m/F^*$ contains elements of order $p^{m(p)}$ for every $p$, since the orders of the elements in $K_m/F^*$ are clearly divisors of $m$.

THEOREM 3. *The following conditions are necessary and sufficient for* $m$ *to be the maximum order of the elements in* $K_m/F^*$:

(a) $K$ *contains* $m$ *distinct* $m$*th roots of unity, so that* $m$ *is the order of* $E$.

(b) *If* $0<m(p)$, *and if a* $p$*th root of unity different from* 1 *is contained in* $F$, *then* $K_m^m$ *is not part of* $F^{*p}$.

PROOF. If condition (a) were not satisfied by $K$, then the order $m'$ of $E$ would be a proper divisor of $m$; and if $w$ is an element in $K_m$, $g$ an $F$-automorphism of $K$, then $(w^{1-g})^{m'}=1$, since $w^{1-g}$ belongs to $E$, proving that $w^{m'}=(w^{m'})^g$ for every $F$-automorphism $g$ of $K$, and that therefore every $w^{m'}$, for $w$ in $K_m$, is in $F^*$. Hence the maximum order of the elements in $K_m/F^*$ is a divisor of the proper divisor $m'$ of $m$, proving the necessity of (a).

To prove the necessity of (b) we assume that (b) does not hold for some particular prime $p$. Then $0<m(p)$, $F$ contains $p$ distinct $p$th roots of unity (so that every $p$th root of unity is in $F$) and $K_m^m\subseteq F^{*p}$; and we are going to show that $K_m/F^*$ does not contain elements of order $p^{m(p)}$. Suppose namely that $b$ is an element in $K$ such that $b^{p^{m(p)}}$ is in $F^*$. Then there exists an integer $r$, prime to $p$, such that $rmp^{-m(p)}\equiv 1$ modulo $p^{m(p)}$, since $mp^{-m(p)}$ is itself prime to $p$. Hence $b\equiv b^{rmp^{-m(p)}}$ modulo $F^*$, and there exists therefore an element $f$ in $F^*$ such that $b=b^{rmp^{-m(p)}}f$. Since $b^{rm}=b^{p^{m(p)}}f^{-p^{m(p)}}$ is in $F^*$, it is in $K_m^m$ and therefore in $F^{*p}$. Thus there exists an element $c$ in $F^*$ such that $c^p=b^{rm}$. If we put $d=cf^{p^{m(p)-1}}$, then $d$ is an element in $F^*$ and $(b^{p^{m(p)-1}}d^{-1})^p=b^{p^{m(p)}}f^{-p^{m(p)}}c^{-p}=1$. Hence $e=b^{p^{m(p)-1}}d^{-1}$ is a $p$th root of unity and as such is contained in $F^*$, proving finally that $b^{p^{m(p)-1}}=ed$ is an element in $F^*$. This completes the proof of the necessity of condition (b).

Suppose conversely that the conditions (a) and (b) are satisfied by $K$. If $p$ is any prime number such that $m(p)=0$, then it is obvious that $K_m/F^*$ contains elements of order $p^{m(p)}$ $(=1)$. If $0<m(p)$, and if $F^*$ does not contain any $p$th roots of unity different from 1, then we infer from (a) that $K$ contains a primitive $p^{m(p)}$th root of unity $e$; and it is obvious that the order of $e$ modulo $F^*$ is exactly $p^{m(p)}$, proving again the existence of an element of order $p^{m(p)}$ in $K_m/F^*$. If finally $0<m(p)$, though $F^*$ contains $p$th roots of unity different from 1, then we infer from condition (b) the existence of an element $b$ in $K_m^m$ which is not contained in $F^{*p}$. It is clear that $b$ is an element in $F^*$ and that there exists an element $c$ in $K$ satisfying $b=c^{p^{m(p)}}=(c^{p^{m(p)-1}})^p$. Since $b$ is not in $F^{*p}$, it is impossible that $c^{p^{m(p)-1}}$ belongs to $F^*$, though $c^{p^{m(p)}}$ is in $F^*$, and this shows that $c$ is an element in $K$ whose order modulo $F^*$ is exactly $p^{m(p)}$. Thus we have shown that $K_m/F^*$ contains elements of order $p^{m(p)}$ for every $p$, and that consequently the maximum order of the elements in $K_m/F^*$ is exactly $m$, as was to be proved.

REMARKS. 1. If $K=F(K_m)$, then condition (a) may be seen readily to be equivalent to the following condition:

(a′) *$K$ is obtained by adjoining to $F$ all the $m$th roots of the elements in the subgroup $K_m^m$ of $F^*$.*

On the basis of Theorem 3 we may restrict ourselves without loss of generality to the consideration of extensions $K$ of $F$ which meet the requirement (a′).

2. Since the second root of unity $-1$ is always contained in $F$, it follows from $0<m(2)$ that the characteristic of $F$ is different from 2, and that therefore condition (b) implies $K_m^m \not\subseteq F^{*2}$.

3. Suppose that $0<m(p)$. It is well known that an automorphism of a cyclic group of order $p^{m(p)}$ possesses fixed elements different from 1 if, and only if, the order of the automorphism under consideration is a power of $p$. We note furthermore that a $p$th root of unity different from 1 is contained in $F$ if, and only if, it is contained in $K$ and is left invariant by every $F$-automorphism of $K$. Thus it follows that the condition (b) is equivalent to the following condition (b′), provided (a) is satisfied by $K$.

(b′) *If $0<m(p)$, and if the group of automorphisms induced by $F$-automorphisms of $K$ in the group of the $p^{m(p)}$th roots of unity in $K$ is of order a power of $p$, then $K_m^m \not\subseteq F^{*p}$.*

For a further analysis it will be necessary to translate the fundamental concepts of the theory of crossed characters into concepts from the theory of extensions of fields. We denote by $E(p)$ the group of the $p^{m(p)}$th roots of unity in $K$ so that $E$ is the direct product of its cyclic subgroups $E(p)$; we note that condition (a) of Theorem 3 is equivalent to the assertion that $E(p)$ is of order $p^{m(p)}$ for every $p$.

As we denoted by $C$ the homomorphism mapping the $F$-automorphism $g$ of $K$ upon the automorphism $g$ of $E$ which it induces in $E$, so we denote by $Cp$ the homomorphism of $G$ which maps $g$ in $G$ upon the automorphism $g$ which it induces in $E(p)$. The groups $G_C$ and $G_{Cp}$ consist of all those elements in $G$ which induce in $E$ and $E(p)$ respectively the identity automorphism, and we find that $G_C$ is the group of $F(E)$-automorphisms of $K$, and that $G_{Cp}$ is the group of $F(E(p))$-automorphisms of $K$. The subgroup $H(p)$ of $G$ has been defined[14] as the group of all those elements in $G$ which are mapped upon 1 by every $Cp$-character of $G$, and it is an immediate consequence of Theorem 1 that $H(p)$ is the group of all the $F(K_{p^{m(p)}})$-automorphisms of $K$.

Theorem 4. *There exists one and only one multiplicative group $S$ between $F^{*m}$ and $F^*$ such that $K$ may be obtained by adjoining all the mth roots of elements in $S$ to $F$ if, and only if, the following conditions are satisfied by $K$:*

(i) *$K$ contains $m$ distinct mth roots of unity, and $K = F(K_m)$.*

(ii) *If the element $g$ in $G_{Cp}$ is of order $p$ modulo $H(p)$, then it is contained in $H(p)G_C$.*

(iii) *If $1 < m(2)$, and if there exists an $F$-automorphism of $K$ which maps every element in $E(2)$ upon its inverse, then $G/(H(2)G_{C2}^2)$ is not commutative.*

Proof. It is readily verified that condition (i) is necessary and sufficient for the existence of at least one group $S$ between $F^{*m}$ and $F^*$ such that $K$ may be obtained by adjoining all the $m$th roots of elements in $S$ to $F$ (take $S = K_m^m$). If (i) is satisfied, then there exists one and only one group $S$ with the required property if, and only if, $K_m$ is the only group $T$ between $F^*$ and $K_m$ such that $K = F(T)$ and $E \subseteq T$. It is an immediate consequence of Theorem 2 and of Corollary 2 to Theorem 2 that $K_m$ is the only group $T$ between $F^*E$ and $K_m$ satisfying $K = F(T)$ if, and only if, the group of all the $C$-characters of $G$ is the only strictly complete group of $C$-characters of $G$; and that this later property is equivalent to conditions (ii) and (iii) is an immediate consequence of Baer [2], Corollary IV.4.2.

On account of the considerations centered around Theorem 3 we define: *the field $K$ is an $m$-extension of its subfield $F$ if* (1) *$K$ is finite, normal, separable over $F$*; (2) *$K$ contains $m$ distinct mth roots of unity*; (3) $K = F(K_m)$. We note that (2) implies in particular that the characteristic of $K$ is either 0 or prime to $m$.

If $K$ is an $m$-extension of $F$, then $K$ may be obtained by adjoining to $F$ all the $m$th roots of elements in the multiplicative group $K_m^m$ between $F^{*m}$ and $F^*$; and it is obvious that $K$ is uniquely determined, up to equivalence, by $m$ and $K_m^m$.

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[14] Baer [2] I.3; for a more detailed analysis of $H(p)$, cp. Baer [2] III 2.

THEOREM 5. *If $S$ is a multiplicative group between $F^{*m}$ and $F^*$, then the following conditions are necessary and sufficient for obtaining an $m$-extension of $F$ by adjoining to $F$ all*[13] *the $m$th roots of elements in $S$.*

(i) *The characteristic of $F$ is either 0 or prime to $m$.*

(ii) *$S/F^{*m}$ is a finite group.*

PROOF. The necessity of (i) has been pointed out before. To show the necessity of (ii), we consider an $m$-extension $K$ of $F$ which may be generated by adjoining the $m$th roots of elements in $S$ to $F$. Then $K=F(K_m)$ and $F^{*m}\subseteq S\subseteq K_m^m\subseteq F^*$. It is a consequence of the finiteness of $K$ over $F$ and of Theorem 1 that $K_m/F^*$ is a finite group. A homomorphism of $K_m/F^*$ upon $K_m^m/F^{*m}$ is effected by mapping the element $x$ in $K_m$ upon the element $x^m$ in $K_m^m$, proving the finiteness of $K_m^m/F^{*m}$, and this implies the necessity of condition (ii).

The sufficiency of the conditions (i) and (ii) is readily verified.

Our theory would be complete if we could prove that different groups $S$ between $F^{*m}$ and $F^*$, meeting the above requirements (i), (ii), lead to essentially different $m$-extensions. This, however, cannot be expected, as may be seen from Theorem 4.

We are now going to impose two conditions upon the Galois group of the equation $x^m-1=0$ in the field $F$. This equation is supposed to be separable, a property that may be expressed in two equivalent ways by saying either that the characteristic of $F$, if not 0, should be prime to $m$, or that the equation has $m$ distinct roots in a suitable extension of $F$. Properties of the Galois group of $x^m-1=0$ are best described by using some finite, normal, and separable extension of $F$ which contains $m$ distinct $m$th roots of unity, as the Galois group of the equation is independent of the particular choice of this extension. Thus we denote by $E$ the group of the $m$ distinct $m$th roots of unity contained in some finite, normal, separable extension $H$ of $F$ (for example, $H=F(E)$), and by $E(p)$ the subgroup of $E$ consisting of the $p^{m(p)}$th roots of unity. Now we may state our two requirements, as follows:

(A) *If $1<m(2)$, then none of the automorphisms of the group $E$, contained in the Galois group of $x^m-1=0$ over $F$, maps every element in $E(2)$ upon its inverse.*

(B) *If the automorphisms of the Galois group of $x^m-1=0$ over $F$ induce in $E(p)$ a group of order a power of $p$, and if $E(p)\neq 1$, then*

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[13] This is best understood by restricting one's attention to subfields of some fixed algebraically closed extension of the field $F$.

*the order of every automorphism in the Galois group of* $x^m-1=0$ *over* $F$ *which leaves the elements in* $E(p)$ *invariant is prime to* $p$.

It should be noted that condition (A) concerns only the Galois group of $x^{2^{m(2)}}-1=0$ over $F$, whereas (B) may be restated without reference to the groups $E$, $E(p)$ as follows:

(B′) *If the Galois group of* $x^{p^{m(p)}}-1=0$ *over* $F$ *is of order a power of* $p$, *and if* $0<m(p)$, *then the modulo* $x^{p^{m(p)}}-1=0$ *reduced Galois group of* $x^m-1=0$ *over* $F$ *is of an order prime to* $p$.

Witt in his theory of abelian extensions[16] had to require that $m$ distinct $m$th roots of unity are contained in $F$. If we substitute for this condition the above properties (A), (B), then we obtain an extension of Witt's theory which comprises certain classes of $m$-extensions, as may be seen from the next theorem together with our remarks in connection with Theorem 5.

THEOREM 6. *If conditions* (A) *and* (B) *are satisfied by the field* $F$ *and the integer* $m$ (*which is prime to the characteristic of* $F$ *in case the characteristic of* $F$ *should be different from* 0), *if* $S$ *and* $T$ *are multiplicative groups between* $F^{*m}$ *and* $F^*$, *and if* $S/F^{*m}$ *and* $T/F^{*m}$ *are both finite groups, then the following two properties imply each other*:

(i) $S=T$.

(ii) *The extensions of* $F$ *which are obtained by adjoining to* $F$ *all the* $m$*th roots of elements in* $S$ *and in* $T$ *respectively are equivalent.*

PROOF. It has been pointed out before that (ii) is a consequence of (i). If conversely (ii) is satisfied by $S$ and $T$, then there exists a finite, normal, separable extension $K$ of $F$ such that $K=F(S')=F(T')$ where $X'$ (for $X=S$ or $T$) is the group of all the elements in $K$ whose $m$th powers are in $X$ and $X'$ consists of all the $m$th roots of elements in $X$. Clearly $X$ is a group between $F^*$ and $K_m$ which contains $m$ distinct $m$th roots of unity. It is a consequence of Theorem 2 and of Corollary 2 that $X'/F^*$ is (essentially) a strictly complete group of $C$-characters of $G$. It is a consequence of conditions (A), (B) and of Baer [**2**], Theorem IV.1.1, that the only strictly complete group of $C$-characters of $G$ is the group of all the $C$-characters of $G$. But it follows from Theorem 1 that this latter group is essentially the same as $K_m/F^*$, proving that $X'=K_m$. Thus we have shown that $S'=T'=K_m$, and this implies clearly $S=T$.

REMARK. If $m$ is in particular an odd prime power, then conditions (A) and (B) are satisfied by $F$ and $m$; if $m$ is a power of 2, then (B) is satisfied by $F$ and $m$, and (A) would be satisfied, for example, if $F$

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[16] Witt [**1**].

contained all the fourth roots of unity. In these cases the above theorem may be applied without reservation.

In case condition (B) is not satisfied by $F$ and $m$, we have to analyze the situation still further in order to obtain an extension of Theorem 6.

If $K$ is an $m$-extension of $F$, then $K=F(K_m)$, and the multiplicative group $K_m$ is the product of the multiplicative groups $K_{p^{m(p)}}$ consisting of all the elements in $K$ whose $p^{m(p)}$th power is in $F^*$. Clearly $K$ is the composite of its uniquely determined subfields $F(K_{p^{m(p)}})=K^{(p)}$, and it is readily verified that $K^{(p)}$ is a $p^{m(p)}$-extension of $F$. It has been remarked just now that condition (B) is satisfied by $F$ and $p^{m(p)}$; and thus Theorem 6 may be applied upon $K^{(p)}$, provided (A) is satisfied by $F$ and $2^{m(2)}$. Furthermore it should be remarked that a composite of $p^{m(p)}$-extensions of $F$ is an $m$-extension of $F$. Combining these remarks with Theorem 5 we obtain the following result.

THEOREM 7. *Suppose that the characteristic of the field $F$ is either* 0 *or prime to the integer $m$.*

(a) *The field $K$ is an $m$-extension of the field $F$ if, and only if, there exists, for every prime $p$, a multiplicative group $S_p$ between $F^{*p^{m(p)}}$ and $F^*$ such that $S_p/F^{*p^{m(p)}}$ is finite, and such that $K$ is obtained by adjoining to $F$ all the $p^{m(p)}$th roots of the elements in $S_p$.*

(b) *If condition* (A) *is satisfied by $F$ and $m$, if $S_p$ and $T_p$ are, for every $p$, multiplicative groups between $F^{*p^{m(p)}}$ and $F^*$ such that both $S_p/F^{*p^{m(p)}}$ and $T_p/F^{*p^{m(p)}}$ are finite, then the following two properties are equivalent*:

(b, i) $S_p=T_p$ *for every prime $p$.*

(b, ii) *The $m$-extensions of $F$ which are determined by the $S_p$ and by the $T_p$ respectively through statement* (a) *are equivalent extensions of $F$.*

If, however, condition (A) fails to hold, then we have to fall back upon Theorem 4.

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# A NOTE ON DIFFERENTIAL POLYNOMIALS

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The following theorem indicates to what extent the expression of a differential polynomial[1] $G$ as an element of the differential ideal determined by $F$ is unique.

THEOREM I. *Let $F \neq 0$, $C_0, C_1, \cdots, C_s$ be differential polynomials in the unknowns $y_1, \cdots, y_n$ with coefficients in an abstract differential field $\mathcal{J}$. Let $F^{(i)}$ be the ith derivative of $F$ and let*

$$C_0F + C_1F' + \cdots + C_sF^{(s)} \tag{1}$$

*be identically zero. Then each $C_i$ is in the perfect ideal generated by $F$.*[2]

We need merely show that any solution $y_j = \bar{y}_j$ $(j = 1, \cdots, n)$, in any extension $\mathcal{J}_1$ of $\mathcal{J}$, of the form $F$ is a solution of each $C_i$.[3] Since this is true if $F$ has no solutions, we may assume that $F$ effectively involves the unknowns. Make the substitution $y_j = z_j + \bar{y}_j$ in (1). Let $A$ consist of the terms of $F$ of lowest degree in the $z_j$ and their derivatives. Collecting terms of the same degree, we see that

$$C_0(\bar{y})A + \cdots + C_s(\bar{y})A^{(s)} = 0, \tag{2}$$

where $C_i(\bar{y})$ is the element of $\mathcal{J}_1$ obtained by substituting $y_j = \bar{y}_j$ $(j = 1, \cdots, n)$ in $C_i$. Let $A$ be of order $p \geqq 0$ in some $z_k$ which it effectively involves, let $z_{k,m}$ be the $m$th derivative of $z_k$, and let $S$ be the partial derivative of $A$ with respect to $z_{k,p}$. For $i > 0$, $A^{(i)}$ can be written as $Sz_{k,p+i} + B_i$, where $B_i$ is some form of order less than $p+i$ in $z_k$. Now (2) becomes

$$C_s(\bar{y})Sz_{k,p+s} + D = 0 \tag{3}$$

where $D$ has order less than $p+s$ in $z_k$. Hence $C_s(\bar{y}) = 0$. In turn $C_{s-1}, \cdots, C_0$ must vanish for $y_j = \bar{y}_j$ as desired.

Using the ideas of the above proof together with a uniqueness result of J. F. Ritt,[4] one can very easily prove the following generalization.

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[1] For definitions of differential fields, polynomials, and ideals, see H. W. Raudenbush, Ann. of Math. (2) vol. 34 (1933) pp. 509–517.

[2] For a result analogous to Theorem I for ordinary polynomials, see Satz 1 of E. Lasker, *Zur Theorie der Moduln und Ideale*, Math. Ann. vol. 60 (1905) pp. 20–116.

[3] H. W. Raudenbush, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 361–368.

[4] *On singular solutions* · · · , Ann. of Math. vol. 37 (1936) pp. 552–617, §§1–3.

THEOREM II. *Let $C_1P_1 + \cdots + C_sP_s$ be identically zero, where the $P_i$ are distinct power products each of degree $d>0$ in a nonzero $F$ and its derivatives. Then each $C_i$ is in the perfect ideal generated by $F$.*

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# ON THE NON-EXISTENCE OF ODD PERFECT NUMBERS OF FORM $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$

ALFRED BRAUER[1]

One of the oldest unsolved mathematical problems is the following one: Are there odd perfect numbers?[2] If such a number $n$ exists, it must have the form

$$n = p^{\alpha} q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_t^{2\beta_t}$$

where $p, q_1, q_2, \cdots, q_t$ are primes and $p \equiv \alpha \equiv 1 \pmod 4$. This has been proved by Euler.[3] Sylvester[4] obtained estimates for $t$, in particular $t \geqq 4$, and $t \geqq 7$ if $n \not\equiv 0 \pmod 3$. Recently, it was shown by R. Steuerwald[5] that the case $\beta_1 = \beta_2 = \cdots = \beta_t = 1$ is impossible, and by H. J. Kanold[6] that the same is true for $\beta_1 = \beta_2 = \cdots = \beta_t = 2$. Moreover Kanold proved that $n$ is not perfect if the greatest common divisor $d$ of $2\beta_1+1, 2\beta_2+1, \cdots, 2\beta_t+1$ is divisible by 9, 15, 21, or 33, and some similar results. All these results deal with the case $d>1$.

In the following, it will be proved that no odd perfect number $n$ of form $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$ exists. Here we have $d=1$. For the proof I use

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[1] The relation of the results of this paper to another paper by H. J. Kanold, *Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, J. Reine Angew. Math. vol. 184 (1942) pp. 116–124, will be considered in an addendum to be published in the December Bulletin.

[2] For the history of the problem see Dickson, *History of the theory of numbers*, vol. 1, 1919, pp. 1–33.

[3] *Commentationes arithmeticae collectae*, vol. 2, *Tractatus de numerorum doctrina* 1849, p. 514; *Opera postuma*, vol. 1, 1862, pp. 14–15.

[4] *Sur l'impossibilité de l'existence d'un nombre parfait impair qui ne contient pas au moins 5 diviseurs premiers distincts*, C. R. Acad. Sci. Paris vol. 106 (1888) pp. 522–526; *Collected mathematical papers*, vol. 4, 1912, pp. 611–614. *Sur une classe spéciale des diviseurs de la somme d'une série géométrique*, C. R. Acad. Sci. Paris vol. 106 (1888) pp. 446–450; *Collected mathematical papers*, vol. 4, 1912, pp. 607–610.

[5] *Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, Sitzungsberichte der mathematisch-naturwissenschaftlichen Abteilung der Bayerischen Akademie der Wissenschaften zu München, 1937, pp. 68–72.

[6] *Untersuchungen über ungerade vollkommene Zahlen*, J. Reine Angew. Math. vol. 183 (1941) pp. 98–109.

theorems of T. Nagell on Diophantine equations. With the same method similar results may be obtained.

LEMMA 1. *Let $q$ be a positive prime. The Diophantine equation*

$$q^2 + q + 1 = y^m$$

*has no solution for $m > 1$.*

PROOF. T. Nagell[7] has proved the following theorem: *If $m > 1$ is not a power of 3, then the Diophantine equation $x^2+x+1=y^m$ has no solutions in integers $x$, $y$ with $y \neq \pm 1$.* In order to obtain all the solutions of

$$x^2 + x + 1 = y^3, \tag{1}$$

it is sufficient to solve the cubic Diophantine equation

$$a^3 - 3ab^2 + b^3 = 1 \tag{2}$$

and to set

$$x = a^3 - 3a^2b + b^3 - 1 \quad \text{and} \quad x = -a^3 + 3a^2b - b^3. \tag{3}$$

It follows from the Theorem of Thue-Siegel that (2) has only a finite number of solutions. Nagell gives the solutions

$$a=1,\ b=0;\quad a=0,\ b=1;\quad a=b=-1;\quad a=2,\ b=-1;\quad a=1,\ b=3. \tag{4}$$

But it is unknown whether there are other solutions in integers $a$, $b$. Since the discriminant of (2) is positive, we can not apply the general theorems of Delaunay and Nagell for cubic Diophantine equations. It follows from (3) and (4) that (1) has at least the following solutions: $x=0$, $y=1$; $x=-1$, $y=1$; $x=18$, $y=7$; $x=-19$, $y=7$.

By Nagell's theorem we have only to consider the case $m=3^k$ for the proof of our lemma. But we do not need the complete solution of (1); it is sufficient to prove that this equation has no solution where $x=q$ is a positive prime.

For $q=3$ we have $q^2+q+1=13$. This is no cube. If $q\equiv 1 \pmod 3$, then

$$q^2 + q + 1 \equiv 0 \pmod 3, \tag{5}$$

but

$$q^2 + q + 1 \not\equiv 0 \pmod{3^2} \tag{6}$$

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[7] *Des équations indéterminées $x^2+x+1=y^n$ et $x^2+x+1=3y^n$*, Norsk Matematisk Forenings Skrifter (1) no. 2 (1921). Cf. *L'analyse indéterminée de degré supérieure*, Mémorial des Sciences Mathématiques, vol. 39 (1929), p. 58.

since it is well known that the prime divisors of the $p$th cyclotomic polynomial $f_p(x)$, where $p$ is a prime, are the primes of form $ph+1$ and $p$ itself; but $f_p(x)$ is not divisible by $p^2$ for any integer $x$. It follows from (5) and (6) that $q^2+q+1$ is not a cube for $q\equiv 1 \pmod 3$.

Now, let $q$ be a prime of form $3h+2$. Since $y^2+y+1<y^3$ for $y\geqq 2$, it follows from $q^2+q+1=y^3$ that $q>y$. Moreover we have $q(q+1) =(y-1)(y^2+y+1)$. Since $q$ is a prime and greater than $y$, it follows that $y-1$ is relatively prime to $q$. Hence $q$ is a divisor of $y^2+y+1$. This gives a contradiction because $y^2+y+1$ has no prime divisor of form $3h+2$.

LEMMA 2. *Let $r$ and $s$ be different positive integers and $p$ be a prime. The system of simultaneous Diophantine equations $x^2+x+1=3p^r$, $y^2+y+1=3p^s$, has no solutions in positive integers $x$, $y$.*

PROOF. Nagell[8] has proved that the Diophantine equation $x^2+x+1 =3z^k$ $(k>2)$ has no solution with $z>1$. Hence we have only to consider the case $r=1$, $s=2$. Then we have

$$x^2 + x + 1 = 3p, \qquad y^2 + y + 1 = 3p^2. \tag{7}$$

If these equations have a solution in positive integers, then it follows from (7) that

$$0 < x < p < y < 2p \tag{8}$$

and, on the other hand, that

$$(2x + 1)^2 \equiv - 3 \equiv (2y + 1)^2 \pmod p,$$
$$2x + 1 \equiv \pm (2y + 1) \pmod p$$

since $p$ is a prime, hence either

$$x \equiv y \pmod p \tag{9}$$

or

$$x \equiv - y - 1 \pmod p. \tag{10}$$

In the first case, it follows from (8) and (9) that

$$y = p + x \tag{11}$$

and in the second case, from (8) and (10) that

$$y = 2p - x - 1. \tag{12}$$

On the other hand, it follows from (7) that $p\neq 3$ and that $x\equiv y\equiv 1$

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[8] Loc. cit.

(mod 3). This contradicts (11) and (12).

Now we are able to prove our theorem.

THEOREM. *An odd number of form* $n=p^{\alpha}q_1^2q_2^2\cdots q_{l-1}^2q_l^4$ *is not perfect.*

PROOF. We change the notation and write $n$ in the following form

$$n = p^{\alpha}q_1^2q_2^2\cdots q_k^2r_1^2r_2^2\cdots r_l^2s^4 \qquad (k \geqq 0,\ l \geqq 0)$$

where the primes $q_\kappa$ are congruent to 1 (mod 3) and where the primes $r_\lambda$ are incongruent to 1 (mod 3). Let us assume that $n$ is perfect, then we have

$$2n = \sigma(n) = \sigma(p^{\alpha})\sigma(q_1^2)\sigma(q_2^2)\cdots\sigma(q_k^2)\sigma(r_1^2)\sigma(r_2^2)\cdots\sigma(r_l^2)\sigma(s^4),$$

where $\sigma(n)$ denotes the sum of the divisors of $n$. It follows that

$$\begin{aligned}2n &= 2p^{\alpha}q_1^2q_2^2\cdots q_k^2r_1^2r_2^2\cdots r_l^2s^4\\ &= \sigma(p^{\alpha})\prod_{\kappa=1}^{k}(1+q_\kappa+q_\kappa^2)\prod_{\lambda=1}^{l}\{(1+r_\lambda+r_\lambda^2)\}(1+s+s^2+s^3+s^4).\end{aligned} \tag{13}$$

Each factor $1+q_\kappa+q_\kappa^2$ is divisible by 3, but not by 9; each factor $1+r_\lambda+r_\lambda^2$ is not divisible by 3. All the other prime divisors of

$$\prod_{\kappa=1}^{k}(1+q_\kappa+q_\kappa^2)\prod_{\lambda=1}^{l}(1+r_\lambda+r_\lambda^2)$$

have the form $3h+1$. We have now to distinguish between some cases.

I. $n\not\equiv 0$ (mod 3). Here we have $k=0$. Since $n$ is not divisible by 3, it follows from Sylvester's theorem mentioned above that $n$ must contain at least 8 different primes; hence $l\geqq 6$. Moreover we obtain from (13) that $\prod_{\lambda=1}^{l}(1+r_\lambda+r_\lambda^2)$ is a divisor of $p^{\alpha}s^4$. It may happen that one of the $l$ factors of this product equals $p$, but each of the $l-1$ remaining factors cannot be a power of $p$ by Lemma 1. Hence each of these $l-1$ factors must be divisible by $s$, and their product must be divisible by $s^5$. This gives a contradiction.

II. $n\equiv 0$ (mod 3), $n\not\equiv 0$ (mod 27). Here $k\leqq 2$. One of the primes $r_\lambda$, say $r_1$, equals 3 since $p=3$ is not possible. It follows that $n$ is divisible exactly by $3^2$, and hence by $\sigma(3^2)=13$, because of (13). Therefore we have either $13=p$, $13=q_1$, or $13=s$.

IIa. $p=13$. Since $\alpha+1$ is even, $\sigma(p^{\alpha})=(p^{\alpha+1}-1)/(p-1)$ is divisible by $p+1$. Hence $n$ must be divisible by 7, and it follows that either one of the primes $q_\kappa$, say $q_1=7$, or $s=7$.

IIa$\alpha$. $q_1=7$. $n$ must be divisible by $\sigma(q_1^2)=57$, therefore by 19, and we have either $q_2=19$ or $s=19$.

IIa$\alpha$1. $q_2=19$. $n$ is divisible by $\sigma(q_2^2)=381$, hence by 127. Since $127\equiv 1 \pmod 3$ and $k\leq 2$, we have $127=s$. Thus

$$n = 13^{\alpha}\cdot 3^2\cdot 7^2\cdot 19^2\cdot 127^4 r_2^2 r_3^2 \cdots r_l^2,$$

$$\begin{aligned} 1 &= \frac{\sigma(n)}{2n} = \frac{\sigma(13^{\alpha})\cdot 13\cdot 57\cdot 381\cdot \sigma(127^4)\sigma(r_2^2)\cdots\sigma(r_l^2)}{2\cdot 13^{\alpha}\cdot 9\cdot 49\cdot 361\cdot 127^4\cdot r_2^2\cdots r_l^2} \\ &= \frac{\{(1/14)\cdot\sigma(13^{\alpha})\}\sigma(127^4)\sigma(r_2^2)\cdots\sigma(r_l^2)}{13^{\alpha-1}\cdot 7\cdot 19\cdot 127^3\cdot r_2^2\cdots r_l^2}. \end{aligned} \tag{14}$$

Since the prime divisors of $\sigma(127^4)$ have the form $5h+1$, they are different from 7, 13, 19, and 127. It follows now from (14) that they are of form $3h+2$, hence of form $15h+11$. Since

$$\sigma(127^4) \equiv 1+7+4+13+1 \equiv 11 \pmod{15},$$

the number of prime divisors of $\sigma(127^4)$ must be odd if $n$ should be perfect.

Let us first assume that $\sigma(127^4)$ is a prime. Then we have $\sigma(127^4)=r_\lambda$,

$$\begin{aligned} \sigma(r_\lambda^2) = 1 + r_\lambda + r_\lambda^2 = 1 + 1 &+ 127 + 127^2 + 127^3 + 127^4 \\ &+ (1 + 127 + 127^2 + 127^3 + 127^4)^2. \end{aligned} \tag{15}$$

Setting $\sigma(r_\lambda^2)=A$ it follows from (14) that $A$ can have only the prime divisors 7, 13, 19, and 127. But, by (15),

$$A \not\equiv 0 \pmod 7, \qquad A \not\equiv 0 \pmod{19}, \qquad A \not\equiv 0 \pmod{127}.$$

Since $A>13$, it follows that $A$ is a power of 13. This contradicts Lemma 1.

Let us assume now that $\sigma(127^4)$ is composite. Since the number of its prime divisors would be odd, there would be at least 3 factors $r_\lambda$, and at least one of them, say $r_2$, must be less than $\{\sigma(127^4)\}^{1/3}$. But

$$\{\sigma(127^4)\}^{1/3} < \{127^3(127 + 1 + .01)\}^{1/3} < 641,$$

therefore $r_2$ would be one of the following primes of form $15h+11$: 11, 41, 71, 101, 131, 191, 251, 281, 311, 401, 431, 461, 491, 521. Since $\sigma(r_2^2)\neq 7$, 13, 19, 127, it must be divisible by at least two of these primes, by Lemma 1. But $f(x)=1+x+x^2\equiv 0 \pmod 7$ for $x\equiv 2, 4$; $f(x)\equiv 0 \pmod{13}$ for $x\equiv 3, 9$; $f(x)\equiv 0 \pmod{19}$ for $x\equiv 7, 11$. Hence it is easy to see that $\sigma(r_2^2)$ is relatively prime to $7\cdot 13\cdot 19$ for $r_2=41$, 71, 101, 131, 251, 281, 461, 491, 521. For $r_2=401$ and 431 we have $(\sigma(r_2^2), 13\cdot 19)=1$ and $\sigma(r_2^2)\not\equiv 0 \pmod{127}$. For the remaining primes 11 and 191 we have $\sigma(127^4)\not\equiv 0 \pmod{11}$ and $\sigma(127^4)\not\equiv 0 \pmod{191}$. This is impossible since $r_2$ was a divisor of $\sigma(127^4)$.

IIa$\alpha$2. $s=19$. Since $\sigma(19^4)$ is divisible by 151, we have $q_2=151$, and since $\sigma(151^2)=22953=3\cdot 7\cdot 1093$, we have $q_3=1093$. This contradicts $k\leqq 2$.

IIa$\beta$. $s=7$. Since $\sigma(7^4)=2801$ is a prime and since $\sigma(2801^2)=37\cdot 43\cdot 4933$, it follows that $q_1=37$, $q_2=43$, $q_3=4933$. This is impossible.

IIb. $q_1=13$. Since $\sigma(q_1^2)=3\cdot 61$, we have either $p$, $q_2$ or $s=61$.

IIb$\alpha$. $p=61$. Here $n$ is divisible by $(p+1)/2$, hence either $q_2=31$ or $s=31$.

IIb$\alpha$1. $q_2=31$. Since $\sigma(q_2^2)=3\cdot 331$ and $k\leqq 2$, we have $s=331$. Therefore $n$ is divisible by $\sigma(331^4)$, hence by 5, and $n$ has the form

$$n = 61^{\alpha}\cdot 3^2\cdot 13^2\cdot 31^2\cdot 5^2\cdot 331^4\cdot M^2. \tag{16}$$

Since $\sigma(p^{\alpha})/p^{\alpha}\geqq (p^{\alpha}+p^{\alpha-1})/p^{\alpha}=(p+1)/p$, it follows from (16) that

$$\sigma(n)/2n > (62\cdot 13\cdot 183\cdot 993\cdot 31)/(2\cdot 61\cdot 9\cdot 13^2\cdot 31^2\cdot 25) = 331/325 > 1.$$

This is impossible.

IIb$\alpha$2. $s=31$. $\sigma(31^4)\equiv 0 \pmod{11}$ and $\sigma(11^2)=7\cdot 19$. Hence $q_2=7$ and $q_3=19$. This contradicts $k\leqq 2$.

IIb$\beta$. $q_2=61$. Since $\sigma(61^2)=3\cdot 13\cdot 97$, we have either $p=97$ or $s=97$. If $p=97$, then $n$ is divisible by $(p+1)/2=7^2$, therefore $s=7$ since $q_1=13$, $q_2=61$, and $k\leqq 2$. But it was proved in IIa$\beta$ that $s=7$ is impossible. For $s=97$ the proof is the same as in IIb$\alpha$2 since $97\equiv 31 \pmod{11}$.

IIb$\gamma$. $s=61$. $\sigma(61^4)$ is divisible by 5 and by 131. Therefore $r_2=131$, and either $p=5$ or $r_3=5$. If $p=5$, then $\sigma(p^{\alpha})$ contains the factor $p+1=6$. Thus $\sigma(n)$ can contain only one other factor 3, hence $k=1$. Since $\sigma(131^2)\not\equiv 0 \pmod{13}$ and $\sigma(131^2)\not\equiv 0 \pmod{61}$, it is necessary that $\sigma(131^2)$ and $n$ contain another prime factor of form $3h+1$. This is impossible. If $r_3=5$, then $n$ is divisible by $\sigma(5^2)=31$, hence $q_2=31$, $q_3=331$. This contradicts $k\leqq 2$.

IIc. $s=13$. Since $\sigma(13^4)=30941$ is a prime, we have either $p=30941$ or $r_2=30941$. But $p=30941$ is impossible because $n$ must be divisible by $(p+1)/2$ and $30941\equiv -1 \pmod{27}$. Hence $n\equiv 0 \pmod{27}$; this contradicts $k\leqq 2$. Therefore $r_2=30941$, and $n$ is divisible by $\sigma(30941^2)$.

Let us first assume that $\sigma(30941^2)$ is a prime, then $\sigma(30941^2)=q_1$ since $\sigma(30941^2)\equiv 3 \pmod{4}$. Now $\sigma(q_1^2)\equiv 0 \pmod{151}$, therefore

$q_2=151$, and $n$ is divisible by $\sigma(151^2)$, hence by 7. This contradicts $k\leqq 2$.

Let us now assume that $\sigma(30941^2)$ is composite. It is easy to see that all the prime factors of $\sigma(30941^2)$ are greater than 151. Since these prime factors are congruent to 1 (mod 3), it follows that either $q_1>150$ and $q_2>150$, or $q_1>150$ and $p>150$.

In the first case we have $k=2$ and hence $p\not\equiv 2 \pmod 3$, therefore $p\equiv 1 \pmod{12}$. Since $(p+1)/2$ is a divisor of $n$ and since 19 cannot be a divisor of $n$, the case $p=37$ is impossible. It follows that $p>60$.

In the latter case we have $q_2>60$ if $k=2$, because otherwise $\sigma(q_2^2)$ is divisible by one of the primes 19, 127, 331, 7, or 631; this is impossible since these primes are not divisors of $\sigma(30941^2)$. Hence, if $k=2$, two of the primes $q_1$, $q_2$, $p$ are greater than 150 and the third is greater than 60. The product

$$P=\prod_{\kappa=1}^{2}\sigma(q_\kappa^2)\prod_{\lambda=1}^{l}\sigma(r_\lambda^2)$$

must be a divisor of $9p^\alpha q_1^2q_2^2s^4$. Each factor $\sigma(q_\kappa^2)$ is divisible by 3. But only one of them may have the form $3p^t$ by Lemma 2. Therefore at least one of the 8 prime factors of $q_1^2q_2^2s^4$ is a divisor of $\sigma(q_1^2)\sigma(q_2^2)$; hence only 7 of these factors may be divisors of the product $\sigma(r_1^2)\sigma(r_2^2)\cdots\sigma(r_l^2)$. It may happen that one of these $l$ factors equals $p$; each of the others is not a power of $p$, by Lemma 1, and contains, therefore, at least one of the remaining 7 prime factors of $q_1^2q_2^2s^4$. It follows that $l\leqq 8$.

We have $r_1=3$; $r_2=30941$. The primes $r_\lambda$ are different from 5, 11, 23, 47, 53, 83, since $q_1>60$ and $q_2>60$. Therefore $r_3\geqq 17$, $r_4\geqq 29$, $r_5\geqq 41$, $r_6\geqq 59$, $r_7\geqq 71$, $r_8\geqq 89$ and

$$\frac{\sigma(n)}{2n}<\frac{\sigma(151^2)\cdot 151\cdot 61\cdot 13\cdot 30941\cdot\sigma(17^2)\cdot\sigma(29^2)\cdot 41\cdot 59\cdot 71\ \ 89\cdot 30941}{2\cdot 151^2\cdot 150\cdot 60\cdot 9\cdot 13^4\cdot 17^2\cdot 29^2\cdot 40\cdot 58\cdot 70\cdot 88\cdot 30940}<1.$$

This is not possible. For $k=1$, it follows similarly that $l\leqq 7$, and the proof is the same.

III. $s=3$. Here we have $k\leqq 4$. Since $n$ must be divisible by $\sigma(3^4)=121$, it follows successively that $n$ is divisible by 7, 19, 127, 5419, and 31. These five primes are congruent to 1 (mod 3) and congruent to 3 (mod 4). Hence $k\geqq 5$; this is impossible. Herewith our theorem is proved for each case.

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# TWO NOTES ON MEASURE THEORY

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I. In a recent paper [1],[1] Saks has indicated a construction whereby a Carathéodory outer measure can be produced on any compact metric space $M$, provided that a certain linear functional $\Phi$ is defined on the set $\mathfrak{C}$ of all continuous real-valued functions whose domain is $M$. The functional $\Phi$ is required to be non-negative for non-negative functions, and to have the property that if the sequence $\{f_n\}$ has the uniform limit 0, then the sequence $\Phi(f_n)$ is a null-sequence. (The measure itself can be defined without this last property.) The purpose of this note is to show that such a linear functional always exists, in a non-trivial form, specifically, so that $\Phi(1)=1$.

We consider the set $\mathfrak{C}$ as a linear space, and together with $\mathfrak{C}$ the linear space $\mathfrak{K}\subset\mathfrak{C}$, where $\mathfrak{K}$ consists of all constant functions. On the entire space $\mathfrak{C}$, we define a functional $p(f)=\sup_{x\in M} f(x)$. This least upper bound always exists, since $M$, being a compact metric space, is a bicompact space, on which every continuous real-valued function is bounded. It is easy to verify that $p(f+g)\leqq p(f)+p(g)$, for all $f$, $g\in\mathfrak{C}$, and that $p(tf)=tp(f)$ whenever $t$ is a non-negative real number. We define a linear functional $\Phi$ on the subspace $\mathfrak{K}$ as follows: $\Phi(f)=f(x)$ for an arbitrary $x\in M$. It is clear that $\Phi(f)=p(f)$ for $f\in\mathfrak{K}$ and that $\Phi$ is linear on $\mathfrak{K}$. By virtue of the celebrated theorem of Hahn-Banach, it appears that $\Phi$ can be extended linearly to all of $\mathfrak{C}$ in such a fashion that $\Phi(f)\leqq p(f)$ for all $f\in\mathfrak{C}$. We further observe that $\Phi$ may be taken non-negative for non-negative functions. For, if $\Phi$ has been defined by the Hahn-Banach construction for all $f\in\mathfrak{B}$, where $\mathfrak{K}\subset\mathfrak{B}\subset\mathfrak{C}$, $\mathfrak{B}\neq\mathfrak{C}$, and if $g\in\mathfrak{C}-\mathfrak{B}$ and $g\geqq 0$, then the number $a=\inf_{f\in\mathfrak{B}}(p(f+g)-\Phi(f))$ is an upper bound to possible values for $\Phi(g)$. $a$, however, is plainly non-negative, so that $\Phi(g)$ may always be taken non-negative. Suppose now that the sequence of functions $\{f_n\}$ has the uniform limit 0. The function $\epsilon-f_n$ is non-negative for all $n>N(\epsilon)$, $N(\epsilon)$ being some natural number dependent upon the arbitrary positive number $\epsilon$. Accordingly, $\Phi(\epsilon-f_n)=\Phi(\epsilon)-\Phi(f_n)=\epsilon\Phi(1)-\Phi(f_n)=\epsilon-\Phi(f_n)\geqq 0$. Likewise, it is easy to show that $\epsilon+\Phi(f_n)\geqq 0$ for all sufficiently large $n$. It follows at once that $\lim_{n\to\infty}\Phi(f_n)=0$. It is proved in Saks [1] that the functional $\Phi$ can

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[1] Numbers in brackets refer to correspondingly numbered articles in the bibliography at the end of the paper.

be used to define a Carathéodory outer measure under which every Borel set is measurable.

II. The present note has as its object the proof of the following result.

THEOREM. *If $E$ is any infinite set, there exists a non-negative real-valued function $\Gamma$ defined on all subsets of $E$ such that:*

(1) $\Gamma(A\cup B)\leqq\Gamma(A)+\Gamma(B)$;
(2) $\Gamma(A)\leqq\Gamma(B)$ *if* $A\subset B$;
(3) $\Gamma(\sum_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\Gamma(A_n)$ *if* $A_n\cap A_m=0$ *for* $m\neq n$;
(4) $\Gamma(E)=1$;
(5) $\Gamma(0)=0$;
(6) *the function* $\Gamma$ *assumes an infinite number of different values;*
(7) $\Gamma(\{p\})=0$ *for all points* $p\in E$ *except for those in a countable subset of* $E$.

Since Ulam has proved that a function enjoying properties (4) and (3) cannot vanish for all subsets containing exactly one point (where $E$ has any of a wide class of cardinal numbers), it appears that the present theorem is the strongest result possible.

The proof of this theorem depends upon a consideration of the family $\mathfrak{B}$ of all bounded real-valued functions defined on the set $E$. As in the preceding note, it is easy to prove the existence of a linear functional $\Phi$ defined on the family $\mathfrak{B}$ considered as a linear space. The construction, for our present purposes, will be considered in more detail. Let $E$ be partitioned into $\aleph_0$ disjoint sets $E_1, E_2, E_3, \cdots, E_n, \cdots$, each having cardinal number equal to the cardinal number of $E$. Let $\omega_n$ be the characteristic function of the set $E_n$. It is obvious that $\omega_n\in\mathfrak{B}$ for every $n$. We shall first define the linear functional $\Phi$ on the linear spaces $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \cdots, \mathfrak{P}_n, \cdots$ obtained from $\mathfrak{K}$, the space of constant functions, by adjoining $\omega_1, \omega_2, \omega_3, \cdots, \omega_n, \cdots$ in succession and forming all possible linear combinations. As in the preceding note, we define $p(f)$ as $\sup_{x\in E} f(x)$, and $\Phi(f)$ as $f(x)$ for $f\in\mathfrak{K}$, $x$ being any point of $E$. By the Hahn-Banach construction, if $\Phi(f)$ is to be bounded by $p(f)$, we must have, when we calculate $\Phi(\omega_1)$, $a_1\leqq\Phi(\omega_1)\leqq b_1$, where $a_1=\sup_{f\in\mathfrak{K}}\ (-p(-f-\omega_1)-\Phi(f))$ and $b_1=\inf_{f\in\mathfrak{K}}\ (p(f+\omega_1)-\Phi(f))$. It is easy to show that $b_1=1$ and that $a_1=0$. We may, then, in accordance with the Hahn-Banach construction, take $\Phi(\omega_1)$ as $1/2$.

The numbers $a_2=\sup_{f\in\mathfrak{P}_1}\ (-p(-f-\omega_2)-\Phi(f))$ and $b_2=\inf_{f\in\mathfrak{P}_1}$ $(p(f+\omega_2)-\Phi(f))$ are lower and upper bounds, respectively, for $\Phi(\omega_2)$. $a_2$ may be computed as 0, and $b_2$, as it is easy to see, is equal to 1/2. We may thus put $\Phi(\omega_2)=1/4$. This process may be continued by finite induction; it is found that the function $\omega_n$ may be assigned the value $1/2^n$ under the functional $\Phi$. $\Phi$ having been defined for the linear space generated by $\mathfrak{R}$ and the sequence $\{\omega_n\}$, the Hahn-Banach construction is carried out for the rest of $\mathfrak{B}$ in any fashion consonant with the restrictions of that theorem, provided that $\Phi(f)\geqq 0$ for non-negative functions $f$. We thus have a linear non-negative functional defined on all of the space $\mathfrak{B}$.

The measure $\Gamma(A)$ for every subset $A$ of $E$ can now be defined: $\Gamma(A)=\Phi(\omega_A)$, $\omega_A$ being the characteristic function of the set $A$. Properties (1)–(7) can now be established. It is plain that $\omega_{A\cup B}\leqq\omega_A+\omega_B$, whence $\Phi(\omega_A+\omega_B-\omega_{A\cup B})\geqq 0$, and consequently $\Phi(\omega_{A\cup B})\leqq\Phi(\omega_A)+\Phi(\omega_B)$, which inequality establishes (1). It is also obvious that $A\subset B$ implies that $\omega_A\leqq\omega_B$. From this, we infer property (2).

We examine (3) in some detail. If $A$ and $B$ are disjoint sets, it follows that $\omega_{A\cup B}=\omega_A+\omega_B$, and consequently $\Phi(\omega_{A\cup B})=\Phi(\omega_A)+\Phi(\omega_B)$, that is, the measure is additive for all subsets of $E$. We may thus state that all subsets of $E$ are measurable in the sense of Carathéodory. It is easy to prove from this fact that if $\{A_n\}$ is any sequence of pairwise disjoint sets, then $\Gamma(\sum_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\Gamma(A_n)$. The proof may be carried over word for word from a similar proof in Saks [2, chap. 2, §4, p. 44, Theorem 4.1].

Statements (4), (5), and (6) are immediate consequences of the definitions of $\Phi$ and $\Gamma$.

To prove that $\Gamma$ vanishes for all points except those in a countable subset, we assume the contrary. If an uncountable set $T$ of points exist such that $\Gamma(\{p\})>0$ for every $p\in T$, then there is some $\epsilon>0$ with $\Gamma(\{p_n\})>\epsilon$, where $p_n\in T$, and $n=1, 2, 3, \cdots$. On account of property (3), we have $\Gamma(\sum_{n=1}^{\infty}\{p_n\})=\sum_{n=1}^{\infty}\Gamma(\{p_n\})=\infty$. Since $1=\Gamma(E)\geqq\Gamma(\sum_{n=1}^{\infty}\{p_n\})$, a contradiction is apparent.

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# ON CERTAIN PAIRS OF SURFACES IN ORDINARY SPACE

BUCHIN SU

1. **Introduction.** In a recent paper[1] Jesse Douglas has proposed and solved the following problem: To determine the form of the linear element of a surface in ordinary space upon which exists a family of $\infty^2$ curves possessing two properties: (1) The angular excess of any triangle $ABC$ formed by curves of the family $\mathcal{J}$ is proportional to the area of the triangle:

$$\mathcal{E} = A + B + C - \pi = k\mathcal{A}, \tag{1}$$

where $k$ denotes a constant; (2) The curves of $\mathcal{J}$ are a *linear* system; that is, a point transformation exists which converts them into the straight lines of a plane. It is natural to inquire what class of surfaces we shall obtain if, instead of using property (2), we make the less specific demand that a point transformation exists which converts the curves of $\mathcal{J}$ into the *geodesics* of another surface. Here we have found certain pairs of surfaces $S$ and $S_1$ which furnish the complete solution of our generalized problem. According to whether the constant $k$ is zero or not, the linear elements of $S$ and $S_1$ take different types, whose derivation constitutes the purpose of the present paper.

2. **Conditions for the property $\mathcal{E} = k\mathcal{A}$.** As was shown by Douglas,[2] the necessary and sufficient conditions that every curve of a family $\mathcal{J}$ upon a surface $S$ should have the property $\mathcal{E} = k\mathcal{A}$ can be expressed by the relation

$$ds/\rho = Pdu + Qdv, \tag{2}$$

where $1/\rho$ is the geodesic curvature of the curve and $P$, $Q$ obey the condition

$$Q_u - P_v = (k - K)W. \tag{3}$$

For the subsequent discussion it is convenient to consider both surfaces $S$ and $S_1$, wherein the curves of $\mathcal{J}$ upon $S$ correspond to the geodesics of $S_1$. Let $(u, v)$ be general coordinates of the corresponding points on these surfaces, so that the first fundamental form of $S$ is

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[1] J. Douglas, *A new special form of the linear element of a surface*, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 101–116.

[2] Douglas, loc. cit., p. 108.

$$ds^2 = E du^2 + 2F du dv + G dv^2 \tag{4}$$

and that of $S_1$ is

$$ds_1^2 = E_1 du^2 + 2F_1 du dv + G_1 dv^2. \tag{5}$$

According to the classical theorem of Tissot,[3] there exists upon $S$ one and, in general, only one orthogonal system of curves which corresponds to an orthogonal system upon $S_1$. Suppose that these surfaces $S$ and $S_1$ are referred to the orthogonal curves which correspond to each other; then we have

$$ds^2 = E du^2 + G dv^2, \tag{6}$$

$$ds_1^2 = E_1 du^2 + G_1 dv^2. \tag{7}$$

In orthogonal coordinates $(u, v)$, the geodesic curvature $1/\rho$ of any curve $v = v(u)$ upon the surface $S$ is given by[4]

$$\begin{aligned} ds/\rho = {} & (EG)^{-1/2}(E + Gv'^2)^{-1}\{EGv'' - (1/2)EE_v \\ & + (EG_u - (1/2)GE_u)v' + ((1/2)EG_v - GE_v)v'^2 + (1/2)GG_u v'^3\} du. \end{aligned} \tag{8}$$

Therefore, for a family $\mathcal{J}$, we have by (2):

$$v'' = A + Bv' + Cv'^2 + Dv'^3, \tag{9}$$

where

$$\begin{cases} A = (1/2)E_v/G + (E/G)^{1/2}P, \\ B = (1/2)E_u/E - G_u/G + (E/G)^{1/2}Q, \\ C = E_v/E - (1/2)G_v/G + (G/E)^{1/2}P, \\ D = -(1/2)G_u/E + (G/E)^{1/2}Q, \end{cases} \tag{10}$$

and

$$Q_u - P_v = (k - K)(EG)^{1/2}. \tag{11}$$

That is: *the form* (9) *with additional condition* (11) *is characteristic of curves having the property* $\mathcal{E} = k\mathcal{A}$ *upon the surface* $S$.

**3. Geodesic representation of the family $\mathcal{J}$.** We now have to impose the further property on the family $\mathcal{J}$.

Since the parametric curves on the surface $S_1$ form an orthogonal system, the differential equation of geodesics of $S_1$ is found to be[5]

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[3] Cf. G. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 3, 1894, p. 47.

[4] Cf. W. Blaschke, *Vorlesungen über Differentialgeometrie*, 3d edition, 1930, p. 175. Write $F=0$, $u'=1$, $u''=0$.

[5] Darboux, loc. cit., p. 49.

$$
\begin{aligned}
v'' = \frac{1}{2}\frac{\partial E_1}{\partial v}\Big/G_1 &+ \left(\frac{1}{2}\frac{\partial E_1}{\partial u}\Big/E_1 - \frac{\partial G_1}{\partial u}\Big/G_1\right)v' \\
&+ \left(\frac{\partial E_1}{\partial v}\Big/E_1 - \frac{1}{2}\frac{\partial G_1}{\partial v}\Big/G_1\right)v'^2 - \frac{1}{2}\left(\frac{\partial G_1}{\partial u}\Big/E_1\right)v'^3.
\end{aligned}
\tag{12}
$$

In order that they should correspond to the curves of the family $\mathcal{J}$ upon $S$, it is necessary and sufficient that the differential equations (9) and (12) be coincident with each other. This gives

$$
\left\{
\begin{aligned}
&\frac{1}{2}\frac{\partial E}{\partial v}\Big/G + (E/G)^{1/2}P = \frac{1}{2}\frac{\partial E_1}{\partial v}\Big/G_1, \\
&\frac{1}{2}\frac{\partial E}{\partial u}\Big/E - \frac{\partial G}{\partial u}\Big/G + (E/G)^{1/2}Q = \frac{1}{2}\frac{\partial E_1}{\partial u}\Big/E_1 - \frac{\partial G_1}{\partial u}\Big/G_1, \\
&\frac{\partial E}{\partial v}\Big/E - \frac{1}{2}\frac{\partial G}{\partial v}\Big/G + (G/E)^{1/2}P = \frac{\partial E_1}{\partial v}\Big/E_1 - \frac{1}{2}\frac{\partial G_1}{\partial v}\Big/G_1, \\
&\frac{1}{2}\frac{\partial G}{\partial u}\Big/E - (G/E)^{1/2}Q = \frac{1}{2}\frac{\partial G_1}{\partial u}\Big/E_1.
\end{aligned}
\right.
\tag{13}
$$

From the first and the last of these equations we have the expressions for $P$ and $Q$:

$$
\left\{
\begin{aligned}
P &= \frac{1}{2}(G/E)^{1/2}\left\{\frac{\partial E_1}{\partial v}\Big/G_1 - \frac{\partial E}{\partial v}\Big/G\right\}, \\
Q &= \frac{1}{2}(E/G)^{1/2}\left\{\frac{\partial G}{\partial u}\Big/E - \frac{\partial G_1}{\partial u}\Big/E_1\right\}.
\end{aligned}
\right.
\tag{14}
$$

Substitution of these expressions in the remaining equations of (13) shows that the fundamental quantities $E$, $G$; $E_1$, $G_1$ are related by

$$
\left\{
\begin{aligned}
\frac{\partial}{\partial v}\log\frac{EG_1}{GE_1} &= \left(1 - \frac{GE_1}{EG_1}\right)\frac{\partial \log E_1}{\partial v}, \\
\frac{\partial}{\partial u}\log\frac{EG_1}{GE_1} &= \left(\frac{EG_1}{GE_1} - 1\right)\frac{\partial \log G_1}{\partial u}.
\end{aligned}
\right.
\tag{15}
$$

In interpreting these conditions, it is important to distinguish the case $(E/G):(E_1/G_1) = 1$ from $(E/G):(E_1/G_1) \neq 1$. If $(E/G):(E_1/G_1) = 1$, then we can put

$$
E_1 = \rho E, \qquad G_1 = \rho G,
\tag{16}
$$

where $\rho \neq 0$, so that the surfaces $S$ and $S_1$ are conformal. In this case the equations (15) are satisfied identically, while the equations (14) give

$$(17)\qquad \begin{cases} P = \dfrac{1}{2}(E/G)^{1/2}\dfrac{\partial \log \rho}{\partial v}, \\[2ex] Q = -\dfrac{1}{2}(G/E)^{1/2}\dfrac{\partial \log \rho}{\partial u}. \end{cases}$$

Substituting these expressions in (11) we have

$$(18)\qquad \frac{\partial}{\partial v}\left\{(E/G)^{1/2}\,\frac{\partial \log \rho}{\partial v}\right\} + \frac{\partial}{\partial u}\left\{(G/E)^{1/2}\,\frac{\partial \log \rho}{\partial u}\right\} + 2(k - K)(EG)^{1/2} = 0.$$

This condition can be interpreted geometrically as follows: by means of the formula of G. Frobenius for Gaussian curvature[6] we obtain for the surface $S$ with linear element (6)

$$(19)\qquad K = -\frac{1}{2W}\left\{\frac{\partial}{\partial v}(E_v/W) + \frac{\partial}{\partial u}(G_u/W)\right\},$$

where $W = (EG)^{1/2}$, and similarly, for the surface $S_1$ with linear element (7)

$$(20)\qquad K_1 = -\frac{1}{2W_1}\left\{\frac{\partial}{\partial v}\left(\frac{\partial E_1}{\partial v}\Big/ W_1\right) + \frac{\partial}{\partial u}\left(\frac{\partial G_1}{\partial u}\Big/ W_1\right)\right\},$$

where $W_1 = (E_1G_1)^{1/2}$.

If the expressions of $E_1$ and $G_1$ given by (16) are substituted in the right-hand member of (20), then

$$K_1 = -\frac{1}{2\rho W}\left\{\frac{\partial}{\partial v}(E_v/W) + \frac{\partial}{\partial u}(G_u/W)\right\} - \frac{1}{2\rho W}\left\{\frac{\partial}{\partial v}\left((E/G)^{1/2}\,\frac{\partial \log \rho}{\partial v}\right) + \frac{\partial}{\partial u}\left((G/E)^{1/2}\,\frac{\partial \log \rho}{\partial u}\right)\right\}.$$

A reference to (18) and (19) gives immediately the relation

$$(21)\qquad \rho K_1 = k,$$

which is, of course, equivalent to (18).

If $k \neq 0$, then we have

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[6] Blaschke, loc. cit., p. 117.

$$(22)\qquad \begin{cases} ds^2 = (K_1/k)(E_1du^2 + G_1dv^2), \\ ds_1^2 = E_1du^2 + G_1dv^2. \end{cases}$$

Therefore *the surface* $S_1$ *may be arbitrarily selected, the only restriction being that it is non-developable, and any surface* $S$, *necessarily conformal to* $S_1$, *with the linear element*

$$(23)\qquad ds^2 = (K_1/k)ds_1^2$$

*possesses the two stated properties*, where $K_1$ is the Gaussian curvature of $S_1$ and $k$ a constant different from zero. In particular when a surface of constant Gaussian curvature $k$ is taken for $S_1$, the corresponding surface $S$ is applicable to $S_1$.

This result not only proves the existence but also furnishes a remarkable class of the surfaces under consideration.

On the contrary, if $k=0$ in the relation (21), then $K_1$ must necessarily vanish, because $\rho$ is by no means zero, so that *the surface* $S_1$ *is developable.* In this case, the surface $S$ is arbitrary and $\mathcal{J}$ must be a conformal image of the $\infty^2$ straight lines of a plane. That no other family of curves upon a generic surface $S$ can be linear and such that the sum of the angles in every triangle of curves in the family is two right angles has been proved analytically by E. Kasner[7] and synthetically by J. Douglas.[8]

We now consider the case where

$$(24)\qquad (E/G):(E_1/G_1) \neq 1.$$

The partial differential equations (15) are easily integrated, and the result may be written in the form

$$(25)\qquad (EG_1)/(GE_1) = 1 - E_1U^2, \quad (GE_1)/(EG_1) = 1 + G_1V^2,$$

where $U$ denotes any function of $u$ alone, and $V$ any function of $v$ alone. The assumption (24) shows that neither $U$ nor $V$ is zero.

From (25) follows the relation

$$(26)\qquad E_1U^2(G_1V^2 + 2) = G_1V^2(2 - E_1U^2).$$

It may happen that both members of (26) are zero. Since $E_1$, $G_1$, $U$, $V$ are different from zero, we have in this case

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[7] E. Kasner, *A characteristic property of isothermal systems of curves*, Math. Ann. vol. 59 (1904) pp. 352–354.

[8] J. Douglas, *A criterion for the conformal equivalence of a Riemann space to a Euclidean space*, Trans. Amer. Math. Soc. vol. 27 (1925) pp. 299–306.

$$E_1 = 2U^{-2}, \qquad G_1 = -2V^{-2}, \tag{27}$$

so that

$$E/G = V^2/U^2. \tag{28}$$

The expressions (14) now become

$$P = -\frac{1}{2}\frac{V}{U}\frac{\partial \log E}{\partial v}, \qquad Q = \frac{1}{2}\frac{U}{V}\frac{\partial \log G}{\partial u}. \tag{29}$$

Substituting them in (18) and remembering the formula (19) in addition to the relation (28), we obtain

$$k = 0. \tag{30}$$

That is, *the sum of the angles in every triangle formed by three curves of the family J under consideration must be two right angles*, and the linear elements of the surfaces $S$ and $S_1$ are, after a suitable transformation of the type

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v), \tag{31}$$

reducible to the form

$$\begin{cases} ds^2 = \lambda(u, v)(du^2 + dv^2), \\ ds_1^2 = du^2 - dv^2. \end{cases} \tag{32}$$

It remains for us to consider the case

$$(2 - E_1U^2)(G_1V^2 + 2) \neq 0. \tag{33}$$

Setting (26) in the form

$$(E_1U^2)/(2 - E_1U^2) = (G_1V^2)(G_1V^2 + 2) = \tau, \tag{34}$$

we obtain

$$E_1 = 2\tau(1 + \tau)^{-1}U^{-2}, \qquad G_1 = 2\tau(1 - \tau)^{-1}V^{-2} \tag{35}$$

and

$$(E/G)^{1/2} = (1 - \tau)(1 + \tau)^{-1}VU^{-1}. \tag{36}$$

The expressions (14) for $P$ and $Q$ may be written in the form

$$\begin{cases} P = \dfrac{1}{2}\left\{\dfrac{V}{U}\dfrac{\partial \log E_1}{\partial v} - \dfrac{1}{W}\dfrac{\partial E}{\partial v}\right\}, \\ Q = -\dfrac{1}{2}\left\{\dfrac{U}{V}\dfrac{\partial \log G_1}{\partial u} - \dfrac{1}{W}\dfrac{\partial G}{\partial u}\right\}. \end{cases} \tag{37}$$

Consequently, we have

$$Q_u - P_v = -\frac{1}{2V}\frac{\partial}{\partial u}\left(U\frac{\partial}{\partial u}\log G_1\right) - \frac{1}{2U}\frac{\partial}{\partial v}\left(V\frac{\partial}{\partial v}\log E_1\right) + \frac{1}{2}\left\{\frac{\partial}{\partial u}(G_u/W) + \frac{\partial}{\partial v}(E_v/W)\right\},$$

namely,

$$\begin{aligned} Q_u - P_v = & -\frac{1}{2V}\frac{\partial}{\partial u}\left(U\frac{\partial}{\partial u}\log G_1\right) \\ & - \frac{1}{2U}\frac{\partial}{\partial v}\left(V\frac{\partial}{\partial v}\log E_1\right) - WK, \end{aligned} \tag{38}$$

as may easily be seen on account of (19). By comparison of (38) with (11) it follows that

$$2k(EG)^{1/2} = -\frac{1}{V}\frac{\partial}{\partial u}\left(U\frac{\partial}{\partial u}\log G_1\right) - \frac{1}{U}\frac{\partial}{\partial v}\left(V\frac{\partial}{\partial v}\log E_1\right). \tag{39}$$

In virtue of (35) we reach the relation:

$$2k(EG)^{1/2} = -\frac{1}{UV}\left[\left(U\frac{\partial}{\partial u}\right)^2 \log\frac{\tau}{1-\tau} + \left(V\frac{\partial}{\partial v}\right)^2 \log\frac{\tau}{1+\tau}\right]. \tag{40}$$

It is important to distinguish the case $k=0$ from $k\neq 0$. The latter case is of more interest; we find by means of (40) an additional relation between $E$ and $G$, namely,

$$(EG)^{1/2} = -(1/2kUV)\left[\left(U\frac{\partial}{\partial u}\right)^2 \log\frac{\tau}{1-\tau} + \left(V\frac{\partial}{\partial v}\right)^2 \log\frac{\tau}{1+\tau}\right]. \tag{41}$$

Therefore the linear elements of $S$ and $S_1$ are, from (35), (36) and (41), given by

$$\left\{\begin{aligned} ds^2 = & -\frac{1}{2k}\left[\left(U\frac{\partial}{\partial u}\right)^2 \log\frac{\tau}{1-\tau} + \left(V\frac{\partial}{\partial v}\right)^2 \log\frac{\tau}{1+\tau}\right] \\ & \cdot\left[\frac{1-\tau}{1+\tau}U^{-2}du^2 + \frac{1+\tau}{1-\tau}V^{-2}dv^2\right], \\ ds_1^2 = & \frac{2\tau}{1+\tau}U^{-2}du^2 + \frac{2\tau}{1-\tau}V^{-2}dv^2, \end{aligned}\right. \tag{42}$$

where $\tau$ denotes an arbitrary function of $u$, $v$.

If the parameters $u$, $v$ are subjected to a suitable transformation of type (31), then the pair of surfaces $S$ and $S_1$ is characterized by the linear elements of a special form:

$$(43)\quad \begin{cases} ds^2 = -\dfrac{1}{2k}\left\{\dfrac{\partial^2}{\partial u^2}\log\dfrac{\tau}{1-\tau}+\dfrac{\partial^2}{\partial v^2}\log\dfrac{\tau}{1+\tau}\right\}\left\{\dfrac{1-\tau}{1+\tau}du^2+\dfrac{1+\tau}{1-\tau}dv^2\right\}, \\ ds_1^2 = \dfrac{2\tau}{1+\tau}du^2+\dfrac{2\tau}{1-\tau}dv^2, \end{cases}$$

involving an arbitrary function $\tau$ of $u$ and $v$.

If $k=0$, then (39) becomes, by applying a transformation of type (31),

$$(44)\qquad \frac{\partial^2}{\partial v^2}\log E_1+\frac{\partial^2}{\partial u^2}\log G_1 = 0.$$

Therefore we obtain

$$(45)\qquad \begin{cases} ds_1^2 = (E^{-1/2}-G^{-1/2})(E^{1/2}du^2+G^{1/2}dv^2), \\ ds^2 = Edu^2+Gdv^2, \end{cases}$$

where the quantities $E$ and $G$ are related by

$$(46)\qquad \frac{\partial^2}{\partial v^2}\log\left(1-(E/G)^{1/2}\right)+\frac{\partial^2}{\partial u^2}\log\left((G/E)^{1/2}-1\right) = 0.$$

Thus the problem of determining the form of linear elements of a pair of surfaces $S$ and $S_1$ with the stated properties is completely solved.

It should be observed that the analogous problem for higher dimensional spaces may be of some interest. We hope to consider it in a future paper.

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# SUBSERIES OF A CONVERGENT SERIES

HARRY POLLARD

In a recent paper J. D. Hill[1] has discussed the mean-value of the subseries of any absolutely convergent series $s=\sum u_n$. Simplifying his method by use of the Rademacher functions, we obtain a mapping of the subseries into the interval $0\leqq x\leqq 1$ by defining[2]

$$\phi(x)=\sum_{n=1}^{\infty}\frac{1+R_n(x)}{2}u_n. \tag{1}$$

Hill's result states that if $\sum|u_n|$ converges, then the mean-value is given by

$$\int_0^1\phi(x)dx=s/2. \tag{2}$$

In the theorem below we point out the weakest condition on the series $\sum u_n$ for which this result persists.

LEMMA. *If* (1) *converges on a set of positive measure it converges almost everywhere.*

Let $D$ be the set of points on which (1) converges. Let $x=a_1a_2a_3\cdots$ (in binary notation) be a point of $D$. If a finite number of the $a_i$ are changed then the new point still belongs to $D$, for by the definition of the Rademacher functions this operation changes only a finite number of the terms of the series (1). Then $D$ is a "homogeneous" set not of measure 0; hence it must be of measure 1.[3]

THEOREM. *A necessary and sufficient condition that the series* (1) *converge on a set of positive measure is that the two series $\sum u_n$ and $\sum u_n^2$ converge. Then* (1) *converges almost everywhere and* (2) *is valid.*

(i) Suppose that (1) converges on a set of positive measure. Then it must, by the lemma, converge almost everywhere. Then there exist

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[1] Bull. Amer. Math. Soc. vol. 48 (1942) p. 103.

[2] The mapping is not 1-1 at the points $x=k/2^n$, but this does not affect the results. For the properties of the Rademacher functions used in this paper see Kacmarcz and Steinhaus, *Le système orthogonale de M. Rademacher*, Studia Mathematica vol. 2 (1939) p. 231.

[3] C. Visser, *The law of nought-or-one in the theory of probability*, Studia Mathematica vol. 7 (1938) pp. 146–147.

two points $x_0$ and $1-x_0$, symmetric in $x=1/2$, at which (1) converges. Inasmuch as $R_n(x_0)+R_n(1-x_0)=0$ we have from (1) that $\phi(x_0)+\phi(1-x_0)=u_n/2$, so that $\sum u_n$ converges. Hence $\sum u_nR_n(x)$ converges a.e., so that $\sum u_n^2$ converges.[4]

(ii) Suppose now that $s=\sum u_n$ and $\sum u_n^2$ converge. Then the series of (1) converges a.e. to a function $\phi(x)$. The $R_n(x)$ are orthonormal, so that by the Riesz-Fischer theorem the series $\sum u_nR_n(x)/2$ converges in the mean to a function of $L^2$; this function must coincide a.e. with $\phi(x)-s/2$.

To establish (2) note that by the Schwarz inequality

$$\begin{aligned}\int_0^1 \left| \phi(x) - s/2 - \sum_1^N (u_n/2)R_n(x) \right| dx \\ \leq \left( \int_0^1 \left| \phi(x) - s/2 - \sum_1^N (u_n/2)R_n(x) \right|^2 dx \right)^{1/2} \\ = o(1) \qquad (N \to \infty).\end{aligned}$$

Since $\int_0^1 R_n(x)dx=0$, (2) is an immediate consequence.

We conclude with some remarks.

(i) Hill points out that if in our theorem we take $\sum u_n$ to be conditionally convergent then $D$ is of the first category[1] though of measure 1.[5]

(ii) Ulam notes that if $\sum |u_n|$ converges then the set of values taken on by $\phi(x)$ is a perfect set and asks what perfect sets can be obtained this way.[5]

(iii) We can obtain part of the above theorem from the laws of 0 or 1 of probability.[6] For (1) may be regarded as a series of independent random variables of mean-value and standard deviation $u_n/2$.

(iv) The above method furnishes a simple proof of a theorem of Steinhaus: the series (1) is $(C, 1)$ summable a.e. if and only if $\sum u_n$ is $(C, 1)$ summable and $\sum u_n^2$ converges.[7]

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[4] Kacmarcz and Steinhaus, op. cit. p. 234.

[5] Written communication to the author.

[6] See, for example, Khintchine and Kolmogoroff, *Über Konvergenz von Reihen, deren Glieder durch den Zufall bestimmt werden*, Rec. Math. (Mat. Sbornik) N. S. vol. 32 (1925) p. 668.

[7] Incorporated into a paper of Paley and Zygmund, *On some series of functions* (2), Proc. Cambridge Philos. Soc. vol. 26 (1930) p. 473.

# RANDOM SUMMABILITY AND FOURIER SERIES

RICHARD BELLMAN

Fejér's theorem that the first arithmetic mean of the partial sums of the Fourier series of a function converges to the function p.p. is equivalent to saying the average values of the partial sums, taken in order, are close to the function. It is interesting to ask whether, if the partial sums are chosen *at random*, and then averaged, the new averages will be close to the function. We shall show that this is true *most of the time.*

Let us define *random summability* in terms of the existence of the limit

$$\lim_{n\to\infty} \frac{\sum_1^n s_k(x)(1 + r_k(t))}{\sum_1^n (1 + r_k(t))}$$

where $r_k(t)$ are the Rademacher functions, $r_k(t) = \text{sign} \sin 2^{k+1}\pi t$, and $s_k(x)$ are the partial sums of $\sigma(f) = a_0 + \sum_1^\infty (a_k \cos kx + b_k \sin kx)$.

The only fact we require concerning the Rademacher functions is [1][1]: If $\sum_1^\infty a_k^2 < \infty$, $\sum_1^\infty a_k r_k(t)$ converges p.p. in $t$.

Now we have

$$\frac{\sum_1^n s_k(x)(1 + r_k(t))}{\sum_1^n (1 + r_k(t))} = \frac{\sum_1^n s_k(x)}{n} \cdot \frac{n}{\sum_1^n (1 + r_k(t))} + \frac{\sum_1^n s_k(x) r_k(t)}{n} \cdot \frac{n}{\sum_1^n (1 + r_k(t))}.$$

Also $\sum_1^\infty r_k(t)/k^{1/2} \log (k+1)$ converges p.p. in $t$, by the above, since $\sum_1^\infty 1/k \log (k+1)^2 < \infty$, so that $\sum_1^n r_k(t) = o(n^{1/2} \log n)$, by Kronecker's Lemma (much more is known, but this is more than needed). Hence, certainly $\sum_1^n r_k(t) = o(n)$, and $\sum_1^n (1 + r_k(t)) \sim n$.

By Fejér's theorem $n^{-1}\sum_1^n s_k(x) \to f(x)$ p.p. in $x$, so that it remains to prove $n^{-1}\sum_1^n s_k(x) r_k(t) \to 0$ p.p. in $x$ and $t$.

But $\sum_1^\infty (s_k(x)/k)^2 < \infty$ p.p. in $x$, since $s_k(x) = o(\log k)$ p.p. in $x$ [2]. Therefore $\sum_1^\infty s_k(x) r_k(t)/k$ converges p.p. in $x$ and $t$, and using Kronecker's Lemma again $\sum_1^n s_k(x) r_k(t) = o(n)$.

Thus, we have proved: If $f(x)$ belongs to $L$, the Fourier series of $f(x)$ is *random summable* p.p. in $x$, *most of the time.*

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[1] Numbers in brackets indicate references at end of paper.

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1. Zygmund, *Trigonometrical series*, chap. 5, p. 123.
2. Ibid., chap. 2, p. 32.

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# ON FIBRE SPACES. II

RALPH H. FOX

This paper is primarily concerned with fibre mappings[1] into an absolute neighborhood retract. Theorem[2] 3 is a converse of the covering homotopy theorem; it characterizes fibre mappings (into a compact ANR) as mappings for which the covering homotopy theorem holds. Theorem 4 is Borsuk's fibre theorem;[3] the proof[4] which I present here is new. It seems to me that this theorem is a promising tool in function-space theory. Also I think that it furnishes conclusive justification for the generality of the Hurewicz-Steenrod definition of a fibre space. In fact, a fibre space of the type constructed by Borsuk's theorem almost never has a compact base space and almost never has its fibres of the same topological type.

The common denominator of the proofs of Theorems 3 and 4 is a property which I call *local equiconnectivity*. Local equiconnectivity is a strengthened form of local contractibility and a weakened form of the absolute neighborhood retract property (Theorems 1 and 2). Definitions and notations are those of FS. I.[5]

Let $\Delta$ be the diagonal subset $\sum_{b\in B}(b, b)$ of $B\times B$. I shall call the space $B$ *locally equiconnected* (or, to be specific, $(U, V)$-equiconnected) if there are neighborhoods $U$ and $V$ of $\Delta$ and a homotopy $\lambda$ in $B$ between the two projections of $U$ which does not move the points of $\Delta$ and which is uniform[6] with respect to $V$. Precisely:

(1) $\lambda_t(b_0, b_1)$ is defined for all $(b_0, b_1)\in U$,

(2) $\lambda_0(b_0, b_1)=b_0$,

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[1] W. Hurewicz and N. Steenrod, Proc. Nat. Acad. Sci. U. S. A. vol. 27 (1941) p. 61.

[2] This theorem was announced in Hurewicz-Steenrod, op. cit. footnote 3.

[3] K. Borsuk, Fund. Math. vol. 28 (1937) p. 99.

[4] This proof was announced in the author's paper *On the deformation retraction of some function spaces* · · · , Ann. of Math. vol. 44 (1943) p. 52.

[5] $\bar{\pi}(x, b)=(\pi(x), b)$ as in R. H. Fox, *On fibre spaces*. I, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 555–557.

(3) $\lambda_1(b_0, b_1) = b_1$,

(4) $\lambda_t(b, b) = b$ for every $(b, b) \in \Delta$, $0 \leqq t \leqq 1$,

(5) there is a $\delta > 0$ such that $|t - t'| < \delta$ implies that $\sum_{(b_0, b_1) \in U}(\lambda_t(b_0, b_1), \lambda_{t'}(b_0, b_1)) \subset V$.

Roughly speaking, $B$ is locally equiconnected if there are paths between sufficiently nearby points such that the paths depend continuously on the end points.

THEOREM 1. *A locally equiconnected space is locally contractible.*

Let $N$ be a neighborhood of some point $b_1$ of $B$ and let $M$ denote the set of points $b_0$ such that $\sum_{0 \leqq t \leqq 1} \lambda_t(b_0, b_1) \subset N$. By (4), $b_1 \in M$; a simple continuity argument shows that $M$ is a neighborhood of $b_1$. Since $M$ is contractible to $b_1$ in $N$ the theorem is proved.

THEOREM 2. *A compact ANR-set is locally equiconnected.*

Let $B$ be a neighborhood retract of the Hilbert parallelotope $Q$ and let $r$ be a retraction of an open neighborhood $N$ of $B$ onto $B$. Since $Q - N$ and $B$ are disjoint compact sets $\epsilon = d(B, Q - N)/2 > 0$. Let $U_\epsilon$ be the closed neighborhood of $\Delta$ determined by the covering of $B$ by $\epsilon$-spheres and let $\lambda_t(b_0, b_1) = r((1-t)b_0 + tb_1)$ for $(b_0, b_1) \in U_\epsilon$, $0 \leqq t \leqq 1$. Conditions (1), (2), (3), and (4) are obviously satisfied. Condition (5) follows, for any $V$, from the compactness of $U_\epsilon$.

From Theorems 1 and 2 it follows,[6] for finite dimensional compacta, that local contractibility, local equiconnectivity and the ANR property are equivalent. For infinite dimensional spaces no more is known than is implied above.

THEOREM 3 (CONVERSE OF THE COVERING HOMOTOPY THEOREM). *Let $B$ be a $(U, V)$-equiconnected space and let $\pi \in B^X$. Suppose that for every mapping $g \in X^Y$ and homotopy $h$ in $B$ which is uniform with respect to $V$ and has initial value*[5] *$\pi g$ there exists a covering homotopy $h^*$ in $X$ with initial value $g$. Then $\pi$ is a fibre mapping relative to $U$.*

Let $h_t(x, b) = \lambda_t(\pi(x), b)$. Since $h$ is uniform with respect to $V$ there is a covering homotopy $h^*$ such that $h_0^*(x, b) = x$. Let $\phi(x, b) = h_1^*(x, b)$. Then[5] $\phi$ maps $\pi^{-1}(U)$ continuously into $X$ and $\pi\phi(x, b) = b$. Since $h_{[0,1]}(x, \pi(x)) = \pi(x)$ it follows that $\phi(x, \pi(x)) = h_1^*(x, \pi(x)) = h_0^*(x, \pi(x)) = x$. Thus $\phi$ is a slicing function.

Let $A$ be a closed subset of $X$ and let $\pi$ denote the sectioning operation $\pi(f) = f|A$, $f \in Y^X$.

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[6] K. Borsuk, Fund. Math. vol. 19 (1932) p. 240, Theorem 32.

THEOREM 4 (BORSUK'S FIBRE THEOREM). *If $A$ is closed in $X$ and $Y$ is a compact ANR-set then $\pi$ is a fibre mapping.*

By Theorem 2, $Y$ is locally equiconnected and, if it is suitably metrized, there is a positive number $\epsilon$ such that $\lambda_t(y_0, y_1)$ is defined whenever $d(y_0, y_1) < \epsilon$. Let $\Gamma_0$ denote the graph of $\pi$ and let $\Gamma_\epsilon$ denote the subset of $Y^X \times Y^A$ defined by the rule $(f, g) \in \Gamma_\epsilon$ when $d(\pi(f), g) < \epsilon$. Because $Y$ is compact $\Gamma_\epsilon$ is a neighborhood of $\Gamma_0$. Define

$$\psi_t(f, g, x) = \begin{cases} \lambda_t(f(x), g(x)) & \text{for } (f, g, x) \in \Gamma_\epsilon \times A, \\ f(x) & \text{for } (f, g, x) \in \Gamma_0 \times X. \end{cases}$$

Thus $\psi$ is a homotopy in $Y$; each $\psi_t$ is defined on the closed subset $C = \Gamma_\epsilon \times A + \Gamma_0 \times X$ of $\Gamma_\epsilon \times A$. But $\psi_0(f, g, x) = f(x)$ for every $(f, g, x) \in C$, and this map has the extension $\psi_0^*(f, g, x) = f(x)$ defined for every $(f, g, x) \in \Gamma_\epsilon \times X$. It follows[7] that $\psi_1$ can be extended to $\Gamma_\epsilon \times X$. Let $\psi_1^*$ denote an extension of $\psi_1$ and set $\phi(f, g)(x) = \psi_1^*(f, g, x)$ for $(f, g) \in \Gamma_\epsilon$ and $x \in X$, so that $\phi(f, g) \in Y^X$ for every fixed $(f, g) \in \Gamma_\epsilon$. Then $\phi$ maps $\Gamma_\epsilon$ into $Y^X$, $\pi\phi(f, g) = g$, $\phi(f, \pi(f)) = f$. Thus $\phi$ is a slicing function for $\pi$.

Since the image set of a fibre mapping is necessarily open and closed in the base space, an example[8] "$\mathcal{E}$" shows that Theorem 4 is false for non-compact ANR-sets $Y$. However if neither $X$ nor $Y$ are compact (as in "$\mathcal{E}$") the topology of $Y^X$ (and also of $Y^A$) depends on the metrization of $Y$. Thus it may be possible (as it is in "$\mathcal{E}$") to remetrize an ANR-set $Y$ so as to make the sectioning operations fibre mappings. It should be observed that Borsuk has shown that Theorem 4 is false (with or without remetrization) if $Y$ is not locally contractible.[4]

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[7] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941, p. 86.

[8] R. H. Fox, Bull. Amer. Math. Soc. vol. 48 (1942) p. 271 footnote 3.

# EXTENSIONS OF CERTAIN DYNAMICAL THEOREMS OF HALPHEN AND KASNER

JOHN DeCICCO

**1. Introduction.** We wish to present here some extensions of certain theorems of Halphen and Kasner concerning the dynamical trajectories of positional fields of force in space.

Kasner has developed the differential geometry of the dynamical trajectories of general positional fields of force in the plane and in space in his Princeton Colloquium Lectures.[1] Recently Kasner and the author have introduced the concept of *generalized* fields of force which depend not only upon the position of the point but also upon the direction through the point.[2] In a generalized field of force in space, there are $\infty^5$ dynamical trajectories just as in the ordinary positional case.

The theorem of Halphen which we wish to extend to generalized fields of force is that all positional fields of force whose $\infty^5$ dynamical trajectories are plane curves are of the central or parallel type. That is, the lines of force are all straight lines through a fixed point $O$ (which may be finite or at infinity). Moreover the $\infty^5$ trajectories consist of the $\infty^2$ systems of $\infty^3$ ordinary plane dynamical trajectories of the central or parallel type, each such system lying in a plane through the point $O$. There are $\infty^{3+f(3)}$ such systems of $\infty^5$ dynamical trajectories,[3] and the number of central fields of force is $\infty^{4+f(3)}$.

We find that all generalized fields of force whose $\infty^5$ dynamical trajectories are plane curves form a more extensive class. The $\infty^5$ trajectories consist, in general, of $\infty^2$ systems of $\infty^3$ generalized plane dynamical trajectories, each such system lying in a plane tangent to a base surface $\Sigma$. The various degenerate situations are four in number and may be described as follows: (i) The base planes are tangent to a given curve $C$, (ii) the base planes pass through a given finite point $O$, (iii) the base planes are all parallel to the tangent planes of a given

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[1] Kasner, *Differential geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 3, 1913, 1934. Also see Trans. Amer. Math. Soc. vols. 7–11 (1906–1910).

[2] See (1) *A generalized theory of dynamical trajectories*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 23–38; and (2) *Generalized dynamical trajectories in space* abstract 49-3-120.

[3] The symbol $\infty^{kf(n)}$ denotes the content of a geometric manifold which depends on $k$ functions of $n$ variables only. See Kasner, *A notation for infinite manifolds*, Amer. Math. Monthly vol. 49 (1942) pp. 243–244.

cone $\Gamma$, and (iv) the base planes are all parallel to a fixed line $L$. The number of such planar systems of generalized dynamical trajectories is $\infty^{f(2)+2f(5)}$, and the number of associated generalized fields of force is $\infty^{f(2)+f(4)+2f(5)}$.

The theorem of Kasner which we wish to generalize is concerned with arbitrary positional fields of force and may be stated in the following manner. If a particle starts at rest from any given point $O$ the resulting rest trajectory which it describes will be initially tangent to, and also will possess the same osculating plane as, the line of force at $O$; moreover, the ratio of the curvature of the rest trajectory to that of the line of force is 1:3.

In a generalized field of force this theorem is no longer valid. It is true that the rest trajectory and the line of force will be initially tangent, but, in general, they will have different osculating planes at $O$. We determine all those fields of force such that, at any point $O$ of space, the rest trajectory and the line of force will have the same osculating plane. In the final part of our paper, we obtain a converse of Kasner's theorem.

**2. The differential equations of the $\infty^5$ generalized dynamical trajectories.** We now consider any generalized field of force in space. Observe that an ordinary positional field of force could be called *isotropic* whereas a generalized field of force could be termed *anisotropic*. (A state of stress in a solid medium is an example of a generalized field of force, although it is not of the most general character.)

Let $F(\phi, \psi, \chi)$ be the rectangular components of the generalized force vector which corresponds to the lineal element $E(x, y, z, y', z')$. (Note that the primes denote differentiation with respect to $x$). We assume that the direction of our force vector $F(\phi, \psi, \chi)$ does not identically agree with that of the corresponding lineal element $E$. The equations of motion of a particle of unit mass are

$$(1)\qquad \begin{gathered} d^2x/dt^2 = \phi(x, y, z, y', z'), \qquad d^2y/dt^2 = \psi(x, y, z, y', z'), \\ d^2z/dt^2 = \chi(x, y, z, y', z'), \end{gathered}$$

where, of course, $t$ is the time.

Upon eliminating the time $t$ from these equations, it is found that the $\infty^5$ dynamical trajectories of a generalized field of force are given as the integral curves of either of the two Monge differential equations of third order

$$(2.1)\qquad \begin{aligned} (\psi - y'\phi)y''' = {} & [(\psi_x + y'\psi_y + z'\psi_z) - y'(\phi_x + y'\phi_y + z'\phi_z)]y'' \\ & + [\psi_{y'} - y'\phi_{y'} - 3\phi]y''^2 + (\psi_{z'} - y'\phi_{z'})y''z'', \end{aligned}$$

or

$$(2.2)\quad \begin{aligned}(\chi - z'\phi)z''' = {} & [(\chi_x + y'\chi_y + z'\chi_z) - z'(\phi_x + y'\phi_y + z'\phi_z)]z'' \\ & + [\chi_{z'} - z'\phi_{z'} - 3\phi]z''^2 + (\chi_{y'} - z'\phi_{y'})y''z'',\end{aligned}$$

solved together with the Monge differential equation of second order

$$(3)\quad z'' = [(\chi - z'\phi)/(\psi - y'\phi)]y''.$$

Therefore the generalized dynamical trajectories are of *the three-dimensional type* (G), namely

$$(\mathrm{G})\quad \begin{aligned} z'' &= K(x, y, z, y', z')y'', \\ y''' &= G(x, y, z, y', z')y'' + H(x, y, z, y', z')y''^2. \end{aligned}$$

Conversely any system of $\infty^5$ curves of the three-dimensional type (G) represents the dynamical trajectories of $\infty^{f(4)}$ generalized fields of force.

Kasner has shown that the three-dimensional type (G) may be characterized by the two geometric properties listed below. For each of the $\infty^1$ trajectories passing through a given lineal element $E$, construct the osculating plane and the osculating sphere at $E$. The two properties are:

*Property* I. The $\infty^1$ trajectories possess the same osculating plane at $E$.

*Property* II. The locus of the centers of the osculating spheres at $E$ is a straight line.

3. **Extension of Halphen's theorem.** Now we shall discuss our extension of the Halphen problem.

THEOREM 1. *A system of* $\infty^5$ *generalized dynamical trajectories consists wholly of plane curves if and only if it is represented by a system of differential equations of the three-dimensional type* (G) *where the function* $K$ *is defined as a solution of an equation of the form*

$$(4)\quad F[z' - y'K, K, z - x(z' - y'K) - yK] = 0,$$

*where* $F$ *is an arbitrary function of three variables.*

There are thus $\infty^{f(2)+2f(3)}$ systems of generalized dynamical trajectories which consist wholly of plane curves.

We proceed with the proof of Theorem 1. Upon substituting the three-dimensional type (G) into the condition $y''z''' - y'''z'' = 0$ for plane curves and simplifying, we obtain the equation

$$(5)\quad y''^3(K_{y'} + KK_{z'}) + y''^2(K_x + y'K_y + z'K_z) = 0.$$

By this equation we see that, upon excluding the $\infty^4$ straight lines which are trajectories of any three-dimensional type (G), there are *at most* $\infty^4$ trajectories in any system of $\infty^5$ dynamical trajectories, which are plane curves.

Now if our three-dimensional type (G) is to contain more than $\infty^4$ plane trajectories, we observe that (5) must be an identity in $y''$. Therefore the function $K$ must satisfy the two partial differential equations of the first order

$$K_{y'} + KK_{z'} = 0, \qquad K_x + y'K_y + z'K_z = 0. \tag{6}$$

First let us suppose that $K$ is independent of $y'$ or $z'$. From these equations we see that $K$ must be constant. This solution for $K$ is given by the equation (4) where $F$ is a function of the second variable $K$ alone. Therefore, we can henceforth suppose that $K$ contains $y'$ or $z'$ explicitly.

By the first of equations (6) and by the fact that $K$ contains $y'$ or $z'$, we find that $K$ must be a solution of a finite equation of the form

$$z' - y'K = \lambda(K, x, y, z), \tag{7}$$

where $\lambda$ is an arbitrary function of four variables.[4] Obtaining the partial derivatives of $K$ with respect to $x$, $y$, and $z$ from this equation and substituting into the second equation of (6), we find

$$\lambda_x + y'\lambda_y + z'\lambda_z = 0. \tag{8}$$

Eliminating $z'$ from this equation and equation (7), we find

$$y'(\lambda_y + K\lambda_z) + (\lambda_x + \lambda\lambda_z) = 0. \tag{9}$$

Now this must be an identity in $y'$. Assume the contrary. We then can solve this equation for $y'$ and differentiate the result with respect to $z'$. Since, by (7), $K_{z'} \neq 0$ we find

$$\frac{\partial}{\partial K}\left(\frac{\lambda_x + \lambda\lambda_z}{\lambda_y + K\lambda_z}\right) = 0. \tag{10}$$

This will mean that the equation (9) solved for $y'$ is independent of $y'$. This is absurd. Hence (9) is an identity in $y'$.

By the preceding argument, the function $\lambda$ of $(K, x, y, z)$ must satisfy the two partial differential equations of the first order

$$\lambda_y + K\lambda_z = 0, \qquad \lambda_x + \lambda\lambda_z = 0. \tag{11}$$

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[4] This equation (7) arises in the characterization of ordinary positional fields of force in space. See Kasner, *Dynamical trajectories: The motion of a particle in an arbitrary field of force*, Trans. Amer. Math. Soc. vol. 8 (1907) pp. 135–158.

Suppose now that $\lambda_z=0$. These equations show that $\lambda$ is independent of $(x, y, z)$. Therefore $\lambda$ is a function of $K$ only. Substituting this result into (7), we discover that the function $K$ is given by the equation (4) where $F$ is independent of the third variable. Henceforth we may suppose that $\lambda_z\neq 0$.

Since $\lambda_z\neq 0$, we find from the second of equations (11) that $\lambda$ must satisfy an equation of the form

$$z - x\lambda = \mu(\lambda, K, y), \tag{12}$$

where $\mu$ is an arbitrary function of three variables. Obtaining the partial derivatives of $\lambda$ with respect to $y$ and $z$ from this equation, and substituting into the first of equations (11), we find

$$\mu_y = K. \tag{13}$$

The integration of this equation yields the following solution for $\mu$,

$$\mu = yK + f(\lambda, K), \tag{14}$$

where $f$ is an arbitrary function of two variables. Substituting this into (12), we find

$$z = x\lambda + yK + f(\lambda, K). \tag{15}$$

Finally, eliminating $\lambda$ from this equation and equation (7), we discover that

$$z = x(z' - y'K) + yK + f(z' - y'K, K). \tag{16}$$

This may be obtained as a solution of the equation (4) for the third variable. Thus the proof of Theorem 1 is complete.

To obtain Halphen's theorem as a consequence of Theorem 1, we first observe that, for the ordinary positional case, equation (3) demonstrates that equation (4) must be a function of a linear integral function in $(z'-Ky')$ and $K$ with coefficients functions of $(x, y, z)$ only. If the third variable is missing in (4), then the coefficients must be constants and the field of force is of the parallel type. On the other hand, if the third variable is present in (4), the equation may be written in the form (16), where $f$ is integral linear in the two variables with constant coefficients. Solving the resulting equation for $K$, we find that the field of force is of the central type. This completes the proof of Halphen's theorem.

THEOREM 2. *The $\infty^5$ generalized dynamical trajectories of Theorem 1 consist of $\infty^2$ systems of $\infty^3$ generalized plane dynamical trajectories, each such system lying in a plane which is, in general, tangent to a base surface $\Sigma$.*

The degenerate situations which arise are four in number and may be described as follows: (i) The base planes are tangent to a given curve $C$, (ii) the base planes pass through a given finite point $O$, (iii) the base planes are all parallel to the tangent planes of a given cone $\Gamma$, and (iv) the base planes are all parallel to a fixed line $L$.

To prove Theorem 2, we first observe that the differential equations of the dynamical trajectories of Theorem 1 may be written as

$$(17)\qquad \begin{gathered} F[z' - y'z''/y'', z''/y'', z - x(z' - y'z''/y'') - yz''/y''] = 0, \\ y''' = Gy'' + Hy''^2. \end{gathered}$$

To find out what curves of this system (17) lie in the plane $z=ax+by+c$, we eliminate $z$ between the first of equations (17) and the equation of the plane. The result of this elimination yields

$$(18)\qquad F(a, b, c) = 0.$$

Therefore we conclude that for a plane to contain at least one curve of (17), its parameters $(a, b, c)$ must satisfy the preceding equation. Moreover any such plane will contain $\infty^3$ spatial generalized dynamical trajectories, and these form a generalized plane dynamical system of arbitrary character. The conclusion of our theorem is obtained from the observation that the $\infty^2$ planes whose parameters $(a, b, c)$ satisfy (18) are, in general, tangent to a surface $\Sigma$. The degenerate situations also can be obtained by a discussion of the various forms that the function $F$ may assume.

It must be noted here that any plane system of $\infty^3$ generalized dynamical trajectories of Theorem 2 is of perfectly general character, whereas in Halphen's theorem any such plane system must be of the central or parallel type.

**4. The lines of force and the rest trajectories.** In the remainder of this paper, we shall suppose that

$$(19.1)\qquad (\psi_{y'}-y'\phi_{y'}-\phi)(\chi_{z'}-z'\phi_{z'}-\phi)-(\psi_{z'}-y'\phi_{z'})(\chi_{y'}-z'\phi_{y'})\neq 0$$

and

$$(19.2)\qquad (\psi_{y'}-y'\phi_{y'}-3\phi)(\chi_{z'}-z'\phi_{z'}-3\phi)-(\psi_{z'}-y'\phi_{z'})(\chi_{y'}-z'\phi_{y'})\neq 0.$$

Let us first of all consider the lines of force in a generalized field of force. At any point $O$ there is, in general, one (and at most $\infty^1$) lineal element $E_0$ whose direction coincides with that of the corresponding force vector $F_0$. The lines of force are the integral curves of the differential equations defined by all the lineal elements $E_0$ of a certain region of space. This system of differential equations is

$$\psi - y'\phi = 0, \qquad \chi - z'\phi = 0. \tag{20}$$

By (19.1), it follows that there are, in general, $\infty^2$ lines of force. Otherwise if the left-hand side of (19.1) is identically zero, there are $\infty^{f(1)}$, or $\infty^1$, or a finite number of lines of force.

Consider next the rest trajectories in a generalized field of force. A rest trajectory is defined as the trajectory described by a particle of unit mass which starts at rest from any point $O$. By (19.2) there are, in general, $\infty^3$ rest trajectories in a generalized field of force. Otherwise if the left-hand member of (19.2) is identically zero, there are at most $\infty^{f(1)}$ rest trajectories.

Thus it is obvious that under the assumptions (19.1) and (19.2) there are, in general, $\infty^2$ lines of force and $\infty^3$ rest trajectories.

## 5. The extension of Kasner's theorem concerning one-third of the curvatures.

In this section we shall determine all generalized fields of force such that, at any point $O$, the rest trajectory and the line of force will have the same osculating plane.

In the first place, we can easily show that the rest trajectory through the point $O$ is tangent to the line of force at $O$. This is an immediate consequence of the equations

$$\begin{aligned} dy/dt &= y'dx/dt, & d^2y/dt^2 &= y'd^2x/dt^2 + y''(dx/dt)^2, \\ dz/dt &= z'dx/dt, & d^2z/dt^2 &= z'd^2x/dt^2 + z''(dx/dt)^2, \end{aligned} \tag{21}$$

together with the initial conditions $(dx/dt)_0 = (dy/dt)_0 = (dz/dt)_0 = 0$ and the equations (20) defining the lines of force.

We discuss our extension of Kasner's theorem.

THEOREM 3. *The rectangular components $F(\phi, \psi, \chi)$ of any generalized field of force for which, at any point $O$ of space, the rest trajectory and the line of force possess the same osculating plane at $O$ must be such that the single eliminant with respect to $y'$ and $z'$ of the three equations*

$$\psi - y'\phi = 0; \quad \chi - z'\phi = 0, \quad (AC + BF)/(DC + EF) = C/F, \tag{22}$$

*where $A$, $B$, $C$, $D$, $E$, $F$ are defined by the expressions*

$$\begin{aligned} A &= \psi_{y'} - y'\phi_{y'}, & B &= \psi_{z'} - y'\phi_{z'}, \\ C &= (\psi_x + y'\psi_y + z'\psi_z) - y'(\phi_x + y'\phi_y + z'\phi_z), \\ D &= \chi_{y'} - z'\phi_{y'}, & E &= \chi_{z'} - z'\phi_{z'}, \\ F &= (\chi_x + y'\chi_y + z'\chi_z) - z'(\phi_x + y'\phi_y + z'\phi_z), \end{aligned} \tag{23}$$

*is identically zero.*

In the first place, the first derivatives $(y', z')$ of the line of force and the rest trajectory through the point $O$ are given as solutions of the equations (20).

By obtaining the total derivatives of equations (20) with respect to $x$ and using the formulas (23), we find that the equations defining the second derivatives $(Y'', Z'')$ of the line of force through the point $O$ are

$$(24) \qquad (A - \phi)Y'' + BZ'' + C = 0, \qquad DY'' + (E - \phi)Z'' + F = 0.$$

By equations (2.1) and (2.2), we see that the equations defining the second derivatives $(y'', z'')$ of the rest trajectory through the point $O$ are

$$(25) \qquad (A - 3\phi)y'' + Bz'' + C = 0, \qquad Dy'' + (E - 3\phi)z'' + F = 0.$$

It is obvious that, in general, the osculating planes can not be identical. Let us impose the condition that the two osculating planes be identical. Then there must exist a number $\rho$ such that

$$(26) \qquad y'' = \rho Y'', \qquad z'' = \rho Z''.$$

Eliminating $Y''$ and $Z''$ from equations (24), (25), and (26), we find

$$(27) \qquad \begin{aligned} y'' &= (1 - \rho)C/2\phi, \qquad z'' = (1 - \rho)F/2\phi, \\ \rho &= (AC + BF - \phi C)/(AC + BF - 3\phi C) \\ &= (DC + EF - \phi F)/(DC + EF - 3\phi F). \end{aligned}$$

This last equation establishes the validity of Theorem 3.

For the generalized fields of force of Theorem 3, let $\lambda$ be defined by the two equal expressions

$$(28) \qquad \lambda = (AC + BF)/\phi C = (DC + EF)/\phi F.$$

THEOREM 4. *For the generalized fields of force of Theorem* 3, *the ratio* $\rho$ *of the curvature of the rest trajectory to that of the line of force is given by the formula*

$$(29) \qquad \rho = (1 - \lambda)/(3 - \lambda).$$

This result may be considered to be an extension of Kasner's one-third theorem.

**6. A converse of Kasner's theorem concerning one-third the curvatures.** If, in addition to the possession of the same osculating plane, we impose the condition that the ratio $\rho = 1/3$, we obtain the following converse of Kasner's theorem.

THEOREM 5. *The rectangular components $F(\phi, \psi, \chi)$ of any generalized field of force for which, at any point $O$ of space, the rest trajectory and the line of force possess the same osculating plane and the ratio $\rho$ of the curvature of the rest trajectory to that of the line of force is $1/3$ must be such that the two eliminants with respect to $y'$ and $z'$ of the four equations*

$$\psi - y'\phi = 0, \qquad \chi - z'\phi = 0, \qquad AC + BF = 0, \qquad DC + EF = 0, \tag{30}$$

*where $(A, B, C, D, E, F)$ are defined by equations (23), are identically zero.*

Thus it is seen that, in the generalized fields of force in space, there are no general geometric relations between the lines of force and the rest trajectories except for the trivial one concerning the possession of the common tangent line. This is entirely different from the two-dimensional case (see footnote 2) where there is a theory—the general theorem being quite similar to our Theorem 4.

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# UNIFORM CONVEXITY. III

MAHLON M. DAY

It is the purpose of this note to fill out certain results given in two recent papers on uniform convexity of normed vector spaces.[1] A normed vector space[2] $B$ is called ***uniformly convex*** with ***modulus of convexity*** $\delta$ if for each $\epsilon>0$ there exists a $\delta(\epsilon)>0$ such that for every two points $b$ and $b'$ of $B$ satisfying the conditions $\|b\|=\|b'\|=1$ and $\|b-b'\|\geqq\epsilon$ the quantity $\|b+b'\|\leqq 2(1-\delta(\epsilon))$. If $\|b_0\|=1$, $B$ is said to be ***locally uniformly convex near*** $b_0$ if there is a sphere about $b_0$ in which the condition for uniform convexity holds. Theorem 1 shows that all properties of normed-vector spaces which are invariant under isomorphism are the same for uniformly convex and locally uniformly convex spaces. Theorem 2 gives a necessary condition for isomorphism with a uniformly convex space. The condition is in terms of isomorphisms of finite dimensional subspaces and is suggested by examples given in [I]; it is not known whether the condition is sufficient. Theorem 3 is somewhat more general than Theorem 3 of [II]; it uses uniformly convex function spaces instead of the $l_p$ spaces of [II].

A *cone* $C$ in $B$ is a set which contains all of every half line from the origin through each point of $C$.

LEMMA 1. *A normed vector space $B$ is locally uniformly convex near $b_0$ if and only if there exists a convex cone $C$, with $b_0$ in its interior, such that for every $\epsilon$ there is a $\delta_1(\epsilon)>0$ such that the conditions $\|b\|\leqq 1$, $\|b'\|\leqq 1$, and $\|b-b'\|\geqq\epsilon$ imply $\|b+b'\|\leqq 2(1-\delta_1(\epsilon))$ for every pair of points $b$ and $b'$ in $C$.*

If this condition is satisfied there is obviously a sphere about $b_0$ inside $C$, so that in that sphere $\delta(\epsilon)$ can be taken equal to $\delta_1(\epsilon)$. On the other hand, if there is a sphere of radius $2k$ about $b_0$ in which $\delta$ can be defined, it can be shown that it suffices to let $C$ be the cone through points of the sphere of radius $k$ about $b_0$ and to let $\delta_1(\epsilon)=\inf\,[\epsilon/10,\ \delta(4\epsilon/5)/2]$.

LEMMA 2. *If the cone $C$ of Lemma 1 contains a sphere about $b_0$ of radius $k$, if $\|b\|\leqq 1$ and if $\|b-b_0\|\geqq k$, then $\|b+b_0\|<2-\delta_1(k)$.*

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[1] These papers are [I] *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 313–317, and [II] *Some more uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 504–507.

[2] See Banach, *Théorie des opérations linéaires*, Warsaw, 1932, for general definitions.

This is obvious if $\|b-b_0\|=k$. If $\|b\|\leqq 1$ and $\|b-b_0\|>k$ there exists a point $b_1=\lambda b+(1-\lambda)b_0$, $0<\lambda<1$, on the line segment from $b_0$ to $b$ such that $\|b_0-b_1\|=k$ while $\|b_1\|\leqq 1$; hence $\|b_1+b_0\|\leqq 2(1-\delta_1(k))$. Let $f$ be a linear functional such that $f(b')\leqq\|b'\|$ for all $b'$ in $B$ and such that the line $\{b'|f(b')=1\}$ in the plane of 0, $b_0$ and $b$ touches the unit sphere in $B$ at the point of intersection of that sphere with the half line from 0 through $b+b_0$, so that $f(b+b_0)=\|b+b_0\|$. Then $f(b_0+b_1)\leqq\|b_0+b_1\|\leqq 2(1-\delta_1(k))$. Two cases can now be distinguished: If $f(b_0)\geqq f(b)$, $2-2\delta_1(k)\geqq f(b_1+b_0)\geqq f(b+b_0)=\|b+b_0\|$. If $f(b_0)<f(b)$, $2(1-\delta_1(k))\geqq f(b_1+b_0)=f(b_1)+f(b_0)>2f(b_0)$, so $\|b+b_0\|=f(b+b_0)=f(b_0)+f(b)<1-\delta_1(k)+1$.

THEOREM 1. *If $B$ is locally uniformly convex near some point $b_0$, then $B$ is isomorphic to a uniformly convex space. If $k$ is the radius of the sphere which exists by Lemma 1 about $b_0$, a suitable modulus of convexity for the new space is given in terms of the old by $\delta_1'(\epsilon)=1-1/[1+\delta_1(\delta_1(k)\epsilon/4)/(k+\delta_1(k)/4)]$.*

Suppose the cone $C$ of Lemma 1 contains a sphere $\{b|\|b-b_0\|\leqq k\}$ about $b_0$; let $\alpha=1-\delta_1(k)/4$ and consider the two spheres $E_1=\{b|\|b-\alpha b_0\|\leqq 1\}$ and $E_2=\{b|\|b+\alpha b_0\|\leqq 1\}$. If $S$ is the intersection of $E_1$ and $E_2$, it is clear that $S$ is convex and symmetric about the origin, and that $S$ contains the sphere $\{b|\|b\|\leqq\delta_1(k)/4\}$.

To show $\|b\|\leqq k+\delta_1(k)/4$ for each $b$ in $S$, it suffices to show that $b\in S$ implies that $b+\alpha b_0$ is within $k$ of $b_0$. If this is false, that is, if $\|b+\alpha b_0-b_0\|>k$, then, by Lemma 2, $\|b+\alpha b_0+b_0\|<2-\delta_1(k)$. However $\|b+\alpha b_0+b_0\|=\|b-\alpha b_0+(1+2\alpha)b_0\|\geqq(1+2\alpha)\|b_0\|-\|b-\alpha b_0\|\geqq 1+2\alpha-1=2\alpha=2-\delta_1(k)/2>2-\delta_1(k)$.

Let $|b|$ be the smallest non-negative value of $t$ for which the point $b/t$ is in $S$. Then $|\cdots|$ defines a new norm in $B$ and it is clear from the inequalities thus far derived that $[\delta_1(k)/4]|b|\leqq\|b\|\leqq[k+\delta_1(k)/4]|b|$, so this new norm defines a space isomorphic to the original and all that need be proved is that $|\cdots|$ is uniformly convex. If $b_1$, $b_2\in S$, and $|b_1-b_2|>\epsilon$, then $\|(b_1+\alpha b_0)-(b_2+\alpha b_0)\|=\|b_1-b_2\|\geqq\delta_1(k)\epsilon/4$. Also $\|b_i+\alpha b_0-b_0\|\leqq k$ by the preceding paragraph so, by the original hypotheses near $b_0$, $\|b_1+b_2+2\alpha b_0\|\leqq 2[1-\delta_1(\delta_1(k)\epsilon/4)]=2(1-\mu(\epsilon))$; that is, $(b_1+b_2)/2=E_1'=\{b|\|b+\alpha b_0\|\leqq 1-\mu(\epsilon)\}$. The same argument with $-\alpha b_0$ and $-b_0$ shows that $(b_1+b_2)/2\in E_2'=\{b|\|b-\alpha b_0\|\leqq 1-\mu(\epsilon)\}$.

It will now suffice to show that there is a $\delta_1'(\epsilon)>0$ such that $|b|<1-\delta_1'(\epsilon)$ if $b\in E_1'\cdot E_2'$. $E_i'\subset E_i$, $i=1, 2$, so for any $b$ in $E_1'\cdot E_2'$ there is a number $t\geqq 1$ such that $|tb|=1$; hence, either $\|tb+\alpha b_0\|=1$ or $\|tb-\alpha b_0\|=1$. These cases are interchanged by replacing $b$ by

$-b$ so it suffices to consider the first; then $1-\mu(\epsilon)\geqq\|\alpha b_0-b\| = \|\alpha b_0-tb+tb-b\| \geqq \|\alpha b_0-tb\|-\|tb-b\| = 1-(t-1)\|b\|$. Therefore $(t-1)\|b\|\geqq\mu(\epsilon)$ or $t\geqq 1+\mu(\epsilon)/\|b\|\geqq 1+\delta_1(\delta_1(k)\epsilon/4)/[k+\delta_1(k)/4]$. Letting $1-\delta_1'(\epsilon)$ be the reciprocal of the last term in the preceding inequality gives $|b| = 1/t\leqq 1-\delta_1'(\epsilon)$ if $b\in E_1'\cdot E_2'$.

We turn now to a necessary condition for isomorphism of $B$ with a uniformly convex space. The effect of uniform convexity on the finite dimensional subspaces of an isomorphic space was used implicitly in [I]; it is given explicit formulation here. Let $B_0$ and $B$ be two normed vector spaces; then there exist linear operations of norm $\leqq 1$ defined on $B_0$ with values in $B$. For each such operator $U$ there is a largest number $k_U$, $0\leqq k_U\leqq 1$, such that $\|b_0\|\geqq\|U(b_0)\| \geqq k_U\|b_0\|$ for each $b_0$ in $B_0$, and this number $k_U$ can be taken as a measure of the distortion of $B_0$ under the mapping $U$ into $B$. Define $k(B_0, B)$ to be the least upper bound of $k_U$ as $U$ runs over the linear operators from $B_0$ to $B$ of norm not greater than 1; explicitly, $k(B_0, B)=\sup_{\|U\|\leqq 1}\inf_{\|b_0\|=1}\|U(b_0)\|$. $k(B_0, B)$ is then a measure of how nearly $B_0$ approaches isometry with a subspace of $B$; if $k(B_0, B)=1$, there are operations which come arbitrarily near preserving distances; $k(B_0, B)>0$ if and only if $B_0$ is isomorphic to a subspace of $B$. For the present it suffices to choose certain finite dimensional spaces for $B_0$. Let $M_n$ and $L_n$ be the $n$-dimensional spaces of sequences $t=(t_1, \cdots, t_n)$ of $n$ real numbers, where $\|t\|_{M_n}=\|(t_1, \cdots, t_n)\|_{M_n}=\max_{1\leqq i\leqq n}|t_i|$ and $\|t\|_{L_n}=\|(t_1, \cdots, t_n)\|_{L_n} =\sum_{1\leqq i\leqq n}|t_i|$. Then $k(M_n, B)=k(L_n, B)=0$ if and only if the dimension of $B$ is less than $n$; also $k(M_n, B)$ and $k(L_n, B)$ are nonincreasing functions of $n$ for each $B$.

LEMMA 3. *If $U$ is a one-to-one linear operator from $B_1$ onto $B_2$ such that for some $a\geqq 0$, $\|b_1\|\geqq\|Ub_1\|\geqq a\|b_1\|$ for each $b_1$ in $B_1$, then for any normed vector space $T$, $k(T, B_1)\geqq ak(T, B_2)\geqq a^2k(T, B_1)$.*

If $a=0$, this is obvious. If $a>0$ and $F$ is any linear operator from $T$ into $B$, with $\|F\|\leqq 1$, let $UF$ be defined by $UF(t)=U(F(t))$ for every $t$ in $T$. Then $\|UF\|\leqq 1$ and $\|UF(t)\|\geqq a\|F(t)\|$ for every $t$. Hence $\inf_{\|t\|=1}\|UF(t)\|\geqq a\inf_{\|t\|=1}\|F(t)\|$ so $k(T, B_2)\geqq ak(T, B_1)$. If $F'$ is any linear operator of norm $\leqq 1$ from $T$ into $B_2$, the same argument, using the operator $aU^{-1}F'$, shows that $k(T, B_1)\geqq ak(T, B_2)$.

Note that if $U$ maps $B_1$ on only part of $B_2$ or is not 1-1 but is of norm $\leqq 1$, the first half of the proof still holds (although $a=0$ in the second case); it follows that if $B_1$ is a subspace of $B_2$, then $k(T, B_1) \leqq k(T, B_2)$.

Sobczyk[3] has defined a special embedding of $l_1$ into $m$ which can easily be modified to define an isometry of $L_{n+1}$ and a subspace of $M_{2^n}$ so $k(L_{n+1}, B) \geqq k(M_{2^n}, B)$ for every integer $n$. In particular, $L_2$ and $M_2$ are isometric so $k(L_2, B) = k(M_2, B)$.

LEMMA 4. *If $\delta_1$ satisfies Lemma* 1 *in the whole unit sphere of $B$ and is continuous on the left, then*

(1) $k(M_n, B) \leqq [1-\delta_1(2k(M_n, B))]^{n-1}$,

(2) $k(L_{2^n}, B) \leqq [1-\delta_1(2k(L_{2^n}, B))]^n$.

If $F$ is an operation from $M_n$ into $B$ such that $\|t\| \geqq \|F(t)\| \geqq k\|t\|$ for all $t$, where $k>0$, let $\epsilon_i = \pm 1$ for $i=1, \cdots, n$; then the points $F(\epsilon_1, \cdots, \epsilon_n)$ lie in the unit sphere of $B$ since $\|F(\epsilon_1, \cdots, \epsilon_n)\| \leqq \|\epsilon_1, \cdots, \epsilon_n\| = 1$. If $\epsilon_1, \cdots, \epsilon_n$ and $\epsilon_1', \cdots, \epsilon_n'$ are different, $\|F(\epsilon_1, \cdots, \epsilon_n) - F(\epsilon_1', \cdots, \epsilon_n')\| \geqq k\|(\epsilon_1, \cdots, \epsilon_n) - (\epsilon_1', \cdots, \epsilon_n)\| = 2k$; hence $\|F(\epsilon_1, \cdots, \epsilon_{n-1}, 0)\| = \|F(\epsilon_1, \cdots, \epsilon_{n-1}, 1) - F(\epsilon_1, \cdots, \epsilon_{n-1}, -1)\|/2 \leqq 1-\delta_1(2k)$; that is, $\|F(\epsilon_1, \cdots, \epsilon_{n-1}, 0)/[1-\delta_1(2k)]\| \leqq 1$ for all $\epsilon_1, \cdots, \epsilon_{n-1}$. These points are at least $2k$ apart for different $\epsilon_i$, so this process can be applied $n-1$ times to show that $\|F(1, 0, 0, \cdots, 0)/[1-\delta_1(2k)]^{n-1}\| \leqq 1$. Hence $k = k\|1, 0, 0, \cdots, 0\| \leqq \|F(1, 0, 0, \cdots, 0)\| \leqq [1-\delta_1(2k)]^{n-1}$. Taking $k = k(M_n, B)$ or, if that is impossible, taking the limit as $k$ increases toward $k(M_n, B)$ gives (1).

If $F$ maps $L_{2^n}$ into $B$ so that $\|t\| \geqq \|F(t)\| \geqq k\|t\|$, $k>0$, for all $t$, the same sort of argument can be carried through using the points of $L_{2^n}$ which have one coordinate equal to one, the others all zero. It leads to the inequality $k = k\|(2^{-n}, \cdots, 2^{-n})\| \leqq \|F(2^{-n}, \cdots, 2^{-n})\| \leqq [1-\delta_1(2k)]^n$ which gives (2).

THEOREM 2. *If $B$ is isomorphic to a space which is locally uniformly convex near any point, then* $\lim_n k(M_n, B) = \lim_n k(L_n, B) = 0$.

By Theorem 1, $B$ is isomorphic to a uniformly convex space $B'$. By Lemma 4, $k(L_{2^n}, B') < [1-\delta_1(2k(L_{2^n}, B'))]^n$ for all $n$. If $k(L_{2^n}, B') > k > 0$ for all $n$, then $0 < k \leqq k(L_{2^n}, B') \leqq (1-\delta_1(2k))^n \to 0$ as $n \to \infty$; this contradiction and the monotony of $k(L_n, B')$ show that $k(L_n, B') \to 0$. Lemma 3 shows that $k(L_n, B) \to 0$ also. A similar proof holds for $k(M_n, B)$; this can also be proved by using the remark before Lemma 4 and the fact that $k(L_n, B) \to 0$.

This theorem has as a corollary the result of [I]: *If $B = \mathfrak{P}^p(B_i)$, where $B_i = l^{p_i}$ or $L^{p_i}$, and if the numbers $p_i$ are not bounded away from*

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[3] A. Sobczyk, *Projection of the space $m$ on its subspace $c_0$*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 938–947; the construction is given in the proof of Theorem 3.

1 *and* $\infty$, *then* $B$ *is not isomorphic to a uniformly convex space.*

It is not difficult to give a direct proof of Theorem 2 not using Theorem 1. I have also shown that if $B^*$ is uniformly convex, then $k(L_n, B)\to 0$ (as does $k(M_n, B)$); whether this condition is sufficient as well as necessary for isomorphism of $B$ or $B^*$ with a uniformly convex space is a question which I am, so far, unable to answer.

Some remarks may be made about the minimum values, $k(L_n)$ and $k(M_n)$, of $k(L_n, B)$ and $k(M_n, B)$ taken for $n$ fixed and $B$ varying over the spaces of dimension at least $n$. $k(M_n, l^p)=n^{-1/p}$ if $2\leqq p<\infty$ and $k(L_n, l^p)=n^{-1/p'}$ if $1\leqq p\leqq 2$ where $1/p'+1/p=1$. Hence $k(L_n)\leqq k(L_n, l^2)=n^{-1/2}$ and $k(M_n)\leqq k(M_n, l^2)=n^{-1/2}$ for all $n$. The plane with a regular hexagon for unit sphere is an example showing that $k(L_2)=k(M_2)\leqq 2/3$ $(<2^{-1/2})$. A tedious computation has shown that $2/3$ is precise; that is, that $k(L_2)=k(M_2)=2/3$. So far all my attempts to show $k(L_n)$ and $k(M_n)\geqq 1/n$ have failed for $n>2$.

The rest of this paper is devoted to extending the results of [II]. A normed vector space $T$ of real-valued functions $t=\{t_s\}$ on some set of indices $S$ will be called a *proper function space* if for every function $t=\{t_s\}$ in $T$ with $0\leqq t_s$ for all $s$ (a) for every real-valued function $\{t_s'\}$ with $0\leqq t_s'\leqq t_s$ for all $s$, the function $\{t_s'\}\in T$ and (b) $0\leqq\|\{t_s'\}\|\leqq\|\{t_s\}\|$. If $T$ is a proper function space and $B_s$, $s\in S$, are normed vector spaces, let $\mathfrak{P}_T\{B_s\}$ be the space of functions $b=\{b_s\}$ where $b_s\in B_s$ and the function $\{\|b_s\|\}\in T$; in $\mathfrak{P}_T\{B_s\}$, $\|b\|=\|\{b_s\}\|=\|\{\|b_s\|\}\|$. (In [II] $S$ was countable and only the special product spaces $\mathfrak{P}^p\{B_s\}=\mathfrak{P}_{l^p}\{B_s\}$ were used.)

THEOREM 3. *If* $T$ *is a proper function space, then* $\mathfrak{P}_T\{B_s\}$ *is uniformly convex if and only if* $T$ *is uniformly convex and the spaces* $B_s$ *have a common modulus of convexity.*

As the proof follows the lines of the proof of Theorem 3 of [II] except at one point it suffices to give the first half of the sufficiency proof; that is, the special case in which $\|b\|=\|b'\|=1$, $\|b-b'\|\geqq\epsilon$ and $\|b_s\|=\|b_s'\|$ for every $s$. Let $\beta_s=\|b_s\|$ and $\gamma_s=\|b_s-b_s'\|$; then for each $s$, $\|b_s+b_s'\|\leqq 2(1-\delta(\gamma_s/\beta_s))\beta_s$ where $\delta$ is a common modulus of convexity for all $B_s$. Hence

$$\|b+b'\|=\|\{\|b_s+b_s'\|\}\|_T\leqq 2\|\{1-\delta(\gamma_s/\beta_s)\beta_s\}\|_T. \tag{1}$$

Clearly $\gamma_s\leqq 2\beta_s$ for all $s$; let $E$ be the set of all $s$ for which $\gamma_s/\beta_s>\epsilon/4$; then in $F$, the complement of $E$, $\beta_s\geqq 4\gamma_s/\epsilon$. If $\{t_s\}$ is any element of $T$, let $t_{sE}=t_s$ if $s\in E$, $t_{sE}=0$ if $s\notin E$; then

$$1\geqq\|\{\beta_s\}\|_T\geqq\|\{\beta_{sF}\}\|\geqq\|\{4\gamma_{sF}/\epsilon\}\|=(4/\epsilon)\|\{\gamma_{sF}\}\|.$$

Hence $\|\{\gamma_{s,F}\}\| \leqq \epsilon/4$ and

$$\|\{\gamma_{s,E}\}\| = \|\{\gamma_s\} - \{\gamma_{s,F}\}\| \geqq \|\{\gamma_s\}\| - \|\{\gamma_{s,F}\}\| \geqq 3\epsilon/4.$$

Hence $\|\{\beta_{s,E}\}\| \geqq \|\{\gamma_{s,E}\}\|/2 \geqq 3\epsilon/8$.

Now let $t = \{\beta_{s,F}\}$, $t' = \{\beta_{s,E}\}$ and $t'' = (1 - 2\delta(\epsilon/4))t'$; then $\|t + t''\| \leqq \|t + t'\| = 1$ and $\|t + t' - (t + t'')\| = \|t' - t''\| = 2\delta(\epsilon/4)\|t'\| \geqq 3\delta(\epsilon/4)\epsilon/4$. Call this last quantity $\alpha(\epsilon)$; then

$$(2) \quad \|(1 - \delta(\epsilon/4))t' + t\| = (1/2)\|t + t' + t + t''\| \leqq 1 - \delta_1(\alpha(\epsilon))$$

where $\delta_1$ is the function which exists in $T$ by Lemma 1. By (1) and (2)

$$\|b + b'\| \leqq \|\{(1 - \delta(\gamma_s/\beta_s))\beta_{s,E}\} + \{\beta_{s,F}\}\| \leqq \|(1 - \delta(\epsilon/4))t' + t\|$$
$$\leqq 1 - \delta_1(\alpha(\epsilon)) = 1 - \delta_2(\epsilon).$$

The remainder of the proof is exactly that given in [II] (beginning with line 4 on p. 506); it shows that a suitable value of $\delta_2$ in $\Phi_T\{B_s\}$ is given if $\delta_2(\epsilon) = \delta_1(\eta)$ where $\eta$ is so chosen that $\eta/2 + \delta_1(\eta) < \delta_2(\epsilon)$. Since $\delta_2$ depends only on the moduli of convexity in $T$ and all $B_s$, we have the following result, more general than Corollary 1 of [II].

COROLLARY. *If $\{T\}$ is a collection of proper function spaces, if $\{B\}$ is a collection of normed vector spaces, and if all these spaces have a common modulus of convexity, then all the spaces $\Phi_T\{B_s\}$ with $T$ in $\{T\}$ and all $B_s$ in $\{B\}$ have a common modulus of convexity.*

Some extensions of Theorem 3 may be made; for instance, it is clear that the condition (a) on a proper function space is imposed to make sure that such functions as $\{\|b_s + b_s'\|\}$ are in $T$. For example, if $S$ is a space in which a measure is defined and all $B_s$ are the same space $B_0$, it suffices to take $T = L^p_S$, $1 < p < \infty$ and to consider only Bochner-measurable functions[4] $\{b_s\}$ for which $\{\|b_s\|\} \in T$. In this case all the functions constructed are again in $T$ so the proof can be carried through showing directly that $L^p(B_0)$ is uniformly convex if $1 < p < \infty$ and $B_0$ is uniformly convex. In fact, if the norm in $T$ satisfies (b) and if it is assumed only that every measurable real-valued function dominated by a function in $T$ is again in $T$, the proof can be carried through for the space of Bochner measurable functions from $S$ into $B$ for which $\{\|b_s\|\} \in T$.

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[4] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fund. Math. vol. 20 (1933) pp. 262–276.

# THE RADICAL OF A GROUP WITH OPERATORS

GARRETT BIRKHOFF

In a recent note (*The radical of a non-associative algebra*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 893–897) A. A. Albert has defined the radical of a general linear algebra, and deduced some of its properties from his theory of isotopy. The purpose of the present note is to extend this concept to groups with operators, and to derive similar properties from lattice theory.

Let $G$ be a group with a class $\Omega$ of endomorphisms including all inner automorphisms.[1] By an *ideal*, we mean a subgroup $S$ of $G$ such that $s \in S$ and $w \in \Omega$ imply $sw \in S$. Clearly $G$ and the group identity 0 are ideals; any other ideal is called a *proper ideal*. We recall that the ideals of $G$ form a *modular lattice*;[2] we shall assume below that this has *finite length*.

We define $G$ to be *prime* if and only if it has no proper ideals; it is well known (and easy to prove) that $G/S$ is prime if and only if $S$ is maximal. Now let $\mathfrak{Z}$ be *any* class of groups with operators (specializing to zero algebras) which is invariant under isomorphisms,[3] and let us define a *simple* group (with operators) to be any prime group not in the exceptional class $\mathfrak{Z}$. We define a direct sum of prime groups to be *semiprime*, and (following Albert and others) a direct sum of simple groups to be *semisimple*.

We further define the *$\phi$-ideal* $\Phi$ of $G$ as the intersection of all maximal ideals (by analogy with the $\phi$-subgroup of a group), the *radical* $R$ of $G$ as the intersection of all maximal ideals $S$ such that $G/S$ is simple, and denote by $Z$ the intersection of all maximal ideals $T$ such that $(G/T) \in \mathfrak{Z}$.

THEOREM 1. *The quotient group $G/S$ is semiprime if and only if $S$ contains $\Phi$.*

PROOF. If $G/S$ is semiprime, then it can be written as a direct sum $G/S = (S_1/S) + \cdots + (S_r/S)$, where the $(S_i/S)$ are prime. The $B_i = S_1 \cup \cdots \cup S_{i-1} \cup S_{i+1} \cup \cdots \cup S_r$ are maximal ideals; and

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[1] If $G$ is commutative, $\Omega$ may be void. The case of a linear algebra is included by requiring $\Omega$ to consist of all scalar multiplications $x \rightarrow \lambda x$, all left multiplications $x \rightarrow ax$, and all right multiplications $x \rightarrow xa$. It follows that ideals and direct sums correspond.

[2] In the sense of the author's *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1940. This will be referred to below as [LT].

[3] It is to be observed that a zero algebra is *not* a "zero" group with operators: $\lambda x \neq 0$ for scalar operators $\lambda$.

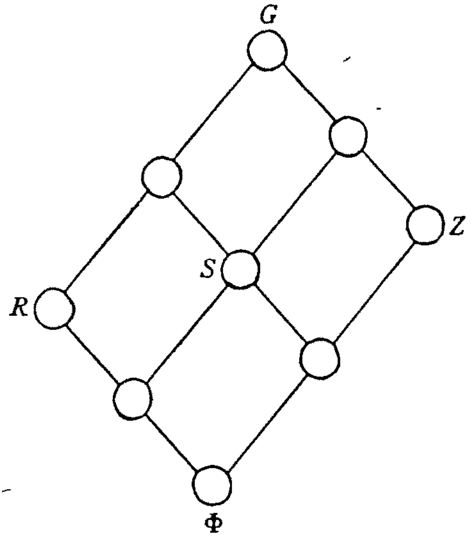
$$S = \bigwedge_{i=1}^{r} B_i \geqq \Phi.$$

To prove the converse, note that $\Phi$ is the meet of dual points (in the sense of lattice theory), and hence (by [LT, Theorem 4.1]) the modular lattice of ideals $X$ satisfying $\Phi \leqq X \leqq G$ is *complemented*. It follows (ibid, §65) that we can find ideals $D_1, \cdots, D_r$ prime over $X$ such that

$$(D_1 \cup \cdots \cup D_{k-1}) \cap D_k = X \text{ for all } k,$$

whence $G/X = (D_1/X) + \cdots + (D_r/X)$, in the sense of being a direct sum.

THEOREM 2. *The quotient group $G/S$ is semisimple if and only if $S$ contains $R$*



PROOF. We consider the complemented modular lattice just described. "Projective" quotients (in the sense of lattice theory) are operator-isomorphic [LT, Theorem 3.16]; hence no $S$ for which $G/S$ is simple is projective with an $S$ for which $G/S$ is in $\mathfrak{Z}$. It follows [LT, Theorem 4.6 and Corollary] that $R/\Phi$ and $Z/\Phi$ are "neutral" complements in $G/\Phi$. Consequently any $S$ such that $G/S$ is semiprime (let alone semisimple) fits into the lattice just described according to the scheme indicated graphically above. We have, by well known theorems, the isomorphisms

$$G/S \cong (R \cup S/S) + (Z \cup S/S),$$
$$S/\Phi \cong (R \cap S/\Phi) + (Z \cap S/\Phi),$$

where $R\cup S/S\cong R/R\cap S\cong G/Z\cup S$ is a direct sum of groups in $\mathfrak{Z}$ and $Z\cup S/S\cong Z/Z\cap S\cong G/R\cup S$ is semisimple. We omit the details, which are identical with those of the second part of the proof of Theorem 1 as regards representation as direct sums of prime groups, and involve [LT, Theorem 4.5], for the basic isomorphisms. Theorem 2 appears as an obvious corollary of these facts and Theorem 1.

The remarkable fact about the radical of a linear associative algebra from this point of view is that it is ***nilpotent***. Also, the radical of a complex Lie algebra $A$ is ***solvable***, and $[A, A]$ is ***nilpotent***. The radical of an alternative algebra is also nilpotent. The generalization of these properties to more general classes of algebras would be extremely important.

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# MINIMAL $A$-SETS, INFINITE ORBITS, AND FIXED ELEMENTS

G. E. SCHWEIGERT

Throughout this note $S$ denotes a semi-locally-connected continuum[1] and $T(S)=S$ an onto-homeomorphism. If $E$ is a set of $S$ such that $T^k(E)=E$ for some positive integer $k$, then $E$ has a *finite period*, otherwise the period for $E$ is *infinite*. The set of all images of $E$ under $T$ and its inverse $T^{-1}$ is said to be the *orbit* of $E$. If $E$ is of period $k=1$, then $E$ is *invariant* under $T$. We investigate the least invariant $A$-set which contains the orbit of a cyclic element $E$, when the period and orbit for $E$ are infinite. This work occupies an intermediate position between a previous paper,[2] wherein certain general results are directed toward the study of finite orbits, and the problem of the action of $T(S)=S$ in general. We follow here the spirit of the work of Ayres begun in his paper, *On transformations having periodic properties*, Fund. Math. vol. 33 (1939) pp. 95–103. With Ayres we denote the unique cyclic chain between two cyclic elements $E$ and $D$ by $C(E, D)$.

THEOREM[3] A. *If $E$ is any cyclic element of $S$ with an infinite period and $B$ is the least (invariant[4]) $A$-set containing the orbit of $E$ then one of the following cases must occur:*

(a) *$B$ contains exactly one fixed element $F$. In this case if $B$ is cyclic then $E$ is a single point which is a cut point of $S$ lying in $B$. The set $S-B$ then has infinitely many distinct components bounded by images of $E$. (Of course it may also have other components not bounded by these images.) If $B$ is not cyclic then for any element $E'$ of the orbit of $E$ the set $B$ is the closure of the orbit of the cyclic chain $C(E', F)$.*

(b) *$B$ contains exactly two fixed elements $X$ and $Y$. In this case $X$*

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[1] See G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications vol. 28, 1942, pp. 64–98. It is assumed that the reader is familiar with the cyclic element theory and the general terminology of this publication. We refer to this book hereafter as ATW with the numbers for theorems in parentheses.

[2] *Fixed elements and periodic types for homeomorphism on s.l.c. continua*, Proc. Nat. Acad. Sci. U. S. A. vol. 29 (1942) p. 52. To appear in full in the Amer. J. Math. under the same title. This paper will be referred to as F; the numbers in parentheses refer to theorems of the complete paper.

[3] Compare F(2.4). Each theorem of this paper has as a corollary the special case in which $S$ is a dendrite. The form used in stating Theorem A and the general nature of the proof are at the suggestion of the referee.

[4] The least $A$-set containing an invariant set is invariant.

*and $Y$ are end points of $B$ and all elements of $B$ except $X$ and $Y$ have infinite periods. Furthermore:*

(1) *if $B=C(X, Y)$ then the images $T^n(E)$ are linearly ordered on $C(X, Y)$ as they occur under $T$ for $n$ in the range of all integers.*

(2) *if $B\neq C(X, Y)$ then $C(X, Y)$ contains an element $E^*$ satisfying (1) and such that $C(X, Y)$ is the minimal invariant $A$-set in $S$ containing the orbit of $E^*$. Moreover, if $E_n$ and $E_n^*$ denote the nth images of $E$ and $E^*$, respectively, under $T$, then the cyclic chain $C(E_n, E_n^*)$ in $B$ contains the component of $B-E_n^*$ which has $E_n$ as a subset of its closure.*

PROOF. By a theorem of Ayres[5] $B$ must contain at least one invariant cyclic element. If it contains exactly one such invariant element $F$ then (a) holds and the results in this case are a very easy exercise for the reader. Otherwise, $B$ must contain at least two invariant cyclic elements and we denote any two such elements by $X$ and $Y$. Thus the cyclic chain $C(X, Y)$ is invariant under $T$ and every element which is not fixed has an infinite orbit. Now if $C(X, Y)$ contains an element of the orbit of $E$ then it must contain the entire orbit and we have $B=C(X, Y)$. It follows at once that $C(X, Y)$ can contain no other invariant element by the fact that $B$ is minimal and $X$ and $Y$ were arbitrary. From the minimal character of $B$ we see that $X$ and $Y$ are end points of $B$, and (1) holds. In the event that no element in the orbit of $E$ is in $B$ there is again no third invariant element in $B$. Such an element $Z$ would imply that $Z\subset\bar{K}$, where $K$ is some component of $B-C(X, Y)$. From the minimal property of $B$ it follows that $\bar{K}$ must contain an element of the orbit of $E$. Thus the entire orbit of $E$ must lie in both of the $S$ cyclic chains $C(Z, X)$ and $C(Z, Y)$. This easily gives an $A$-set which contains the orbit of $E$ and is a proper subset of $B$, contrary to the definition of this set. Thus $B$ contains exactly two invariant cyclic elements $X$ and $Y$. Let $K(E)$ denote the component of $B-C(X, Y)$ having $E$ as a subset of its closure, and $p$ the boundary point of this component. Then if $E^*$ is the minimal cyclic element of $C(X, Y)$ containing $p$ we see that $E^*$ has an infinite orbit. This gives us (2) without difficulty and completes the proof of Theorem A.

The property mentioned below is one of five previously shown[6] to be equivalent.

COROLLARY. *Let $E$ be any cyclic element in $S$ such that $E$ and $E_1=T(E)$ are distinct. In order to insure that $C(E, E_1)$ has the third property of Ayres—contains exactly one invariant element $Z$—it is suffi-*

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[5] ATW (2.51) p. 242.

[6] F (2.6).

*cient (and necessary) to assume that an invariant element in $C(E, E_1)$ exists provided $E$ has an infinite period. There is no assumption when $E$ has finite period.*

PROOF. If the period for $E$ is infinite this follows from Theorem A. If $E$ has finite period, the least $A$-set $M$ containing the orbit of $E$ also contains a fixed cyclic element $Z$ by virtue of the theorem of Ayres. It is readily shown that: (1) $M$ is the finite orbit of $C(E, E_1)$, (2) $Z \subset C(E, E_1)$, (3) $Z$ is unique.

If $M$ is a sum of cyclic elements of $S$ and each element of $M$ has a finite period then $T$ is said to be ***elementwise periodic*** on $M$. The case $M=S-L$, where $L$ is the set of all end points of $S$, has received considerable attention;[7] however, the question treated below awaited a study of infinite orbits.

THEOREM B. *If $T(S)=S$ is elementwise periodic on $S-L$ and $p \in L$ has an infinite orbit, then the least $A$-set $M$ containing the orbit of $p$ has a unique invariant element ("hub") $E$, and the cyclic element hyperspace of $M$ is structurally similar to an $n$-adic tree.*[8]

PROOF. The set $M$ is invariant and contains the closure of the orbit of $p$. In view of the density of the elements of finite period and Theorem A it is evident that $M$ contains a unique invariant element $E$. Let $k$ be the least positive value of the period function on $M-E$ and $I(k)$ the set of all cyclic elements of $M$ fixed under $T^k$. It follows[9] that $I(k)$ is an $A$-set. Furthermore the minimal property for $M$ insures that $M-E$ is a component-orbit, the finite orbit of some component $K$ of $M-E$. We select one particular $K$ and consider $\overline{K}=K+q$, with $q$ a cut point of $M$ on $E$. (Note: $T(q) \neq q$ although $T^k(q)=q$ is possible.) Let $q \in C$, where $C$ is a true cyclic element of $K$, then $q \in I(k)$ implies $C \subset I(k)$. If $C$ does not exist, $q$ is an end point of $K$, and hence, by a previous theorem,[10] we again get $I(k) \cdot K \neq 0$. Thus, in any case, there is a node $X$ of $I(k)$ in $\overline{K}$ by virtue of well known results.[11] Finally $C(X, E) \subset I(k)$ also follows from a previous theorem.[12] We now let $N=I(k) \cdot \overline{K}$ and investigate the nature of this $A$-set and its complement. If $R$ is a component of $\overline{K}-N$, $F(R)=x$, and $E_x$ the cyclic element of $N$ which contains $x$, then $T^k(R)\cdot R=0$ and $T^k(x)\in E_x$; for otherwise $T^k(R)=R$ and $I(k)\cdot R\neq 0$ by the argument used to establish $X$ above. Thus $\bar{K}-N$ is a component-orbit under $T^k$ and each component in this finite orbit is bounded by a point of $E_x$. If $x\notin X$ then it is possible to construct an $A$-set $M^*$ which is a proper subset of $M$ and contains the orbit of $p$. This would be contrary to the definition of $M$. A similar set $M^*$ is possible if there is a node of $I(k)$, other than $X$, in $N$. Hence $N+E=C(X, E)=C(X, q)+E$ is the only possibility. Moreover each component of $\bar{K}-N$ has as its boundary a point of $X$; that is, $X=E_x$. This suffices to give the structural character of $M$ "near" the "center of rotation" $E$ in $M$; namely $C(X, q)$ and its $k-1$ images extending radially as "spokes" from the "hub" $E$, or a variant of this (when the period of $q$ is less than $k$) in which several spokes meet in the same cut point on the orbit of $q$. More specifically, $I(k)=\sum_{n=1}^{k}T^n(C(X, q))+E$ and the interior of this $A$-set contains $I(k)-\sum T^n(X)$.

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[7] F (4.2) and ATW (4.6) p. 248.

[8] Tree = dendrite of ATW. For the dyadic tree see *A note on the limit of orbits*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 968. Also see Leo Zippin, *Transformation groups*, in *Lectures in topology*, University of Michigan Press, 1941, p. 196.

[9] ATW (4.4) p. 248 adapted in F (2.6).

[10] F (2.1).

[11] ATW (8.1) and (8.2) p. 77.

[12] ATW (4.3) p. 248 adapted in F (2.6).

Since $I(k)$ contains no point of the orbit of $p$ we may iterate the discussion above using $\bar{K}$ and $T^k$ in the roles $M$ and $T$. If $l>k$ is the least positive value of the period function on $K-N$ and $I(l)$ is the set of all elements in $M$ fixed under $T^l$, then $l$ is an integral multiple of $k$, $I(k)\subset I(l)$, and $I(l)$ is an $A$-set. The argument proceeds to show that, when $N^*=I(l)\cdot\bar{R}$, we get $N^*+X=C(X^*, X)$ where $X^*$ is uniquely the node of $I(l)$ in $N^*$. Then $I(l)=\sum_{n=1}^{l}T^n(C(X^*, q))+E$ and this fact, together with results concerning the interior of $I(l)$, are sufficient to introduce the third stage. The third stage introduces the fourth, and so on. When these inductive steps have all been carried out, we find that (as we progress from $E$ toward the orbit of $p$) the spoke $C(X, E)$ branches into $l/k$ chains at $X$, these in turn branch at the images of $X^*$, and such branching continues indefinitely. This branching occurs only at the designated elements.

The expanding approximation due to Whyburn[13] (and adapted[14] to this type of homeomorphism) insures that the above process will envelop all but the end points of $M$, that is, $M-L$. But $\overline{M-L}=M$; and it can be readily shown that $L$ is the closure of the orbit of $p$, hence perfect, one-dimensional, and uncountable. This ends the discussion of Theorem B.

In conclusion it is evident that the proof of Theorem B does not require the full strength of the elementwise periodicity assumption; a study of the general onto-homeomorphism and the subsequent selection of convenient restrictions will clarify this matter. It may also be remarked that Theorem A may well carry, in such a study, a weight greater than that indicated by its relatively minor role in the proof of Theorem B.

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[13] ATW (4.7) p. 249.

[14] F (4.2).

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# THE EQUIVALENCE OF $n$-MEASURE AND LEBESGUE MEASURE IN $E_n$

ARTHUR SARD

Consider a set $A$ of points in euclidean $n$-space $E_n$. For each countable covering $\{A_i\}$ of $A$ by arbitrary sets consider the sum

$$\sigma = \sum_i c_m \delta(A_i)^m,$$

where $m$ is a fixed positive number, $c_m = \pi^{m/2}/2^m\Gamma[(m+2)/2]$, and $\delta(A)$ is the diameter of $A$. The constant $c_m$ is, for integral $m$, the $m$-volume of a sphere of unit diameter in $E_m$. Let $L_m(A;\alpha)$ be the greatest lower bound of all sums $\sigma$ corresponding to coverings for which $\delta(A_i) < \alpha$ for all $i$ $(\alpha > 0)$. We define the $m$-measure of $A$ as $L_m(A) = \lim_{\alpha \to 0} L_m(A;\alpha)$. We denote the outer Lebesgue measure of $A$ by $|A|$.

We shall show that *$n$-measure and outer Lebesgue measure are equal*: $L_n(A) = |A|$. A statement on this matter by W. Hurewicz and H. Wallman is true but misleading: these authors assert that $L_n(A)/c_n$ and $|A|$ may be unequal.[1]

F. Hausdorff has introduced an $m$-measure $L_m^S(A)$ defined as is $L_m(A)$ except that coverings by spheres are used instead of coverings by arbitrary sets. He has shown[2] that $L_n^S(A) = |A|$. However $L_m(A)$ and $L_m^S(A)$ are unequal in general, as A. S. Besicovitch has shown[3] for $m = 1$, $n = 2$. S. Saks[4] and others define $m$-measure as $L_m(A)/c_m$.

Our proof, which is an obvious extension of Hausdorff's proof, depends on two known theorems.

THEOREM I. *Of all sets in $E_n$ having a given diameter, the $n$-sphere has the greatest outer Lebesgue measure.*[5]

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[1] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941, p. 104.

[2] F. Hausdorff, *Dimension und äusseres Mass*, Math. Ann. vol. 79 (1919) p. 163.

[3] A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points*, Math. Ann. vol. 98 (1928) pp. 458–464. R. L. Jeffery, *Sets of k-extent in n-dimensional space*, Trans. Amer. Math. Soc. vol. 35 (1933) p. 634.

[4] S. Saks, *Theory of the integral*, Warsaw, 1937, pp. 53–54.

THEOREM II. *Suppose that to each point $x$ of a set $A$ in $E_n$ there corresponds a set of closed $n$-spheres centered at $x$ of arbitrarily small positive diameter. Then for any given $\epsilon>0$, a countable number of the spheres cover $A$ and are such that the sum of their Lebesgue measures is at most $|A|+\epsilon$.*[6]

We now prove that

$$|A| \leqq L_n(A) \leqq L_n^S(A) \leqq |A|.$$

For any countable covering $\{A_i\}$ of $A$,

$$|A| \leqq \sum_i |A_i| \leqq \sum_i c_n\delta(A_i)^n$$

by Theorem I. Hence $|A| \leqq L_n(A;\alpha)$ for all $\alpha$ and $|A| \leqq L_n(A)$.

The definitions imply that $L_n(A) \leqq L_n^S(A)$.

Finally, given $\epsilon>0$ and $\alpha>0$, assign to each point $x$ of $A$ the set of all closed spheres centered at $x$ and of positive diameter less than $\alpha$. Then by Theorem II a countable number of these spheres $\{S_i\}$ cover $A$ and are such that

$$\sum_i |S_i| = \sum_i c_n\delta(S_i)^n \leqq |A| + \epsilon.$$

Hence $L_n^S(A;\alpha) \leqq |A|$ and $L_n^S(A) \leqq |A|$.

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[5] W. H. and G. C. Young, *The theory of sets of points*, Cambridge, 1906, pp. 293–294. L. Bieberbach, *Über eine Extremaleigenschaft des Kreises*, Jber. Deutschen Math. Verein. vol. 24 (1915) pp. 247–250. T. Kubota, *Über die konvex-geschlossenen Mannigfaltigkeiten im n-dimensionalen Raume*, Science Reports, Tôhoku Imperial University, vol. 14 (1925) p. 98. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik vol. 3 (1934) pp. 76 and 107. W. Feller, *Some geometric inequalities*, Duke Math. J. vol. 9 (1942) pp. 889–892. The diameter of an arbitrary set $B$ equals the diameter of the smallest closed convex set containing $B$.

[6] H. Rademacher, *Eineindeutige Abbildung und Messbarkeit*, Monatshefte für Mathematik und Physik vol. 27 (1916) p. 190. The case $|A|=\infty$ is not excluded.

# A CONVERGENCE THEOREM FOR LEBESGUE-STIELTJES INTEGRALS

FUMIO YAGI

This paper deals with the following theorem.

THEOREM. *Let $g(x)$ be a bounded Borel measurable function defined everywhere on $(-\infty, \infty)$. Let $p_n(x)$ be a sequence of normalized functions,*[1] $p_n \in V$, *such that*

$$(1) \qquad \sum_{n=1}^{\infty} V(p_n) < \infty.$$

*Then $\sum_{n=1}^{\infty} p_n(x)$ is absolutely and uniformly convergent to a normalized function $p(x) \in V$, and*

$$(2) \qquad \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x).$$

We shall first prove this theorem in the following special cases:

(a) $g(x)$ is a bounded piecewise absolutely continuous function[2] in $(-\infty, \infty)$.

(b) $g(x)$ is continuous in a finite interval and vanishes identically outside this finite interval. (It need not necessarily be continuous at the end points of the interval.)

(c) $g(x)$ is a bounded continuous function in $(-\infty, \infty)$.

First let us prove our assertion concerning $p(x)$. Since $p_n(x)$ are normalized, we have[3] $\sum_{n=1}^{\infty} |p_n(x)| \ll \sum_{n=1}^{\infty} P_n \ll \sum_{n=1}^{\infty} V(p_n)$ where $P_n$ is the upper bound of $|p_n(x)|$ for $-\infty < x < \infty$. Because of (1), it follows that the series $\sum_{n=1}^{\infty} p_n(x)$ is absolutely and uniformly convergent. Let $p(x)$ be the limit function. Evidently $p(x)$ is right-continuous and normalized. To show that $p(x) \in V$ it is sufficient to show that $p(x)$ is of bounded variation on $(-\infty, \infty)$. For $\xi > 0$ and any subdivision of $(-\xi, \xi)$, $-\xi = x_0 < x_1 < \cdots < x_{m-1} < x_m = \xi$, we

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[1] $p(x) \in V$ means that $p(x)$ is right-continuous and of bounded variation on the infinite interval $(-\infty, \infty)$. It is normalized if $p(0) = 0$. $V(p_n)$ denotes the total variation of $p_n$ over $(-\infty, \infty)$.

[2] $f(x)$ is piecewise absolutely continuous in $(-\infty, \infty)$ if we can divide $(-\infty, \infty)$ into a finite number of intervals such that in each of these intervals $f(x)$ is absolutely continuous.

[3] "$<<$" is to be read "is dominated termwise by."

have

$$\sum_{i=0}^{m} | p(x_{i+1}) - p(x_i) | \leq \sum_{n=0}^{\infty} \sum_{i=0}^{m} | p_n(x_{i+1}) - p_n(x_i) | .$$

This implies that[4] $V(p; -\xi, \xi) \leq \sum_{n=1}^{\infty} V(p_n; -\xi, \xi) \leq \sum_{n=1}^{\infty} V(p_n)$. Hence $V(p) < \infty$.

Let $x_i$ ($i = 1, 2, \cdots, N$) be the points of discontinuity of $g(x)$ under condition (a), where $x_1 < x_2 < \cdots < x_N$. Then $g(x)$ is absolutely continuous on $x_i < x < x_{i+1}$ and $g(x) = \int_{x_i+0}^{x} g_1'(x)dx + g(x_i + 0)$, where $g_1(x)$ is the absolutely continuous part of $g(x)$ in the Lebesgue decomposition. For the infinite interval we then have

$$\int_{-\infty}^{\infty} dg(x) = \int_{-\infty}^{\infty} g_1'(x)dx + \sum_{i=1}^{N} j(x_i)$$

where $j(x_i)$ is the jump of $g(x)$ at $x = x_i$. Since $g(x)$ is bounded, it is evident that $\int_{-\infty}^{\infty} g(x)dp(x)$ exists. In addition we have

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} | g(x) | \, | dp_n(x) | \leq G \cdot \sum_{n=1}^{\infty} V(p_n) < \infty \tag{3}$$

where $G = \text{u.b.}_{-\infty < x < \infty} | g(x) |$. Hence $\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x)$ exists.

We shall now show that (2) holds for case (a). Let $A$ be any positive number. Then because of the above remark we have

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) &= \sum_{n=1}^{\infty} \Big\{ g(A)p_n(A) - g(-A)p_n(-A) \\ &\qquad - \int_{-A}^{A} p_n(x)dg(x) \Big\} \\ &= g(A)p(A) - g(-A)p(-A) \\ &\qquad - \sum_{n=1}^{\infty} \int_{-A}^{A} p_n(x)g_1'(x)dx - \sum_{n=1}^{\infty} \sum_{i=1}^{N} p_n(x_i)j(x_i). \end{aligned} \tag{4}$$

By the general convergence theorem[5] of Lebesgue, this becomes

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) &= g(A)p(A) - g(-A)p(-A) \\ &\qquad - \int_{-A}^{A} p(x)g_1'(x)dx - \sum_{i=1}^{N} p(x_i)j(x_i) \\ &= \int_{-A}^{A} g(x)dp(x). \end{aligned} \tag{5}$$

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[4] $V(p_n; -\xi, \xi)$ denotes the total variation of $p_n$ over $(-\xi, \xi)$.

[5] E. C. Titchmarsh, *Theory of functions*, Oxford, 2nd ed., p. 345.

Letting $A \to \infty$, we get

$$(6) \qquad \lim_{A\to\infty} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x).$$

Now consider

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \int_{-A}^{A} g(x)dp_n(x) \right| &\ll \sum_{n=1}^{\infty} \int_{-A}^{A} | g(x) | \, | dp_n(x) | \\ &\ll G\cdot \sum_{n=1}^{\infty} \int_{-A}^{A} | dp_n(x) | \\ &\ll G\cdot \sum_{n=1}^{\infty} V(p_n). \end{aligned}$$

Thus $\sum_{n=1}^{\infty}\int_{-A}^{A} g(x)dp_n(x)$ is absolutely and uniformly convergent in $A$. Hence by a known theorem,[6] we have

$$\lim_{A\to\infty} \sum_{n=1}^{\infty} \int_{-A}^{A} g(x)dp_n(x) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x).$$

Combining this with (6), we get

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x) = \int_{-\infty}^{\infty} g(x)dp(x),$$

which concludes case (a).

To prove the theorem for case (b), it is sufficient to show that the expression

$$\left| \sum_{n=1}^{N} \int_{-\infty}^{\infty} g(x)dp_n(x) - \int_{-\infty}^{\infty} g(x)dp(x) \right|$$

can be made arbitrarily small for all sufficiently large values of $N$. Let $[M_1, M_2]$ be the interval within which $g(x)$ is continuous. By the Weierstrass approximation theorem, there exists a sequence of polynomials $g_m(x)$ which converge uniformly to $g(x)$ in $[M_1, M_2]$. Consider the functions defined by

$$f_m(x) = \begin{cases} g_m(x), & M_1 \leqq x \leqq M_2, \\ 0, & \text{otherwise}. \end{cases}$$

We note that $f_m(x)$ for $m=1, 2, \cdots$ all satisfy the hypothesis (a). Hence we have

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[6] See Hobson, *Theory of functions of a real variable*, vol. 2, p. 121.

$$\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_m(x)dp_n(x) = \int_{-\infty}^{\infty} f_m(x)dp(x) \tag{7}$$

for $m=1, 2, \cdots$. Furthermore we have

$$\begin{aligned}\left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \int_{-\infty}^{\infty} g(x)dp(x)\right| \\ \leqq \left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x)\right| \\ + \left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x) - \int_{-\infty}^{\infty} f_m(x)dp(x)\right| \\ + \left|\int_{-\infty}^{\infty} f_m(x)dp(x) - \int_{-\infty}^{\infty} g(x)dp(x)\right|.\end{aligned} \tag{8}$$

Because of (7) it follows that

$$\left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x)\right| < \epsilon_N(m), \tag{9}$$

where $\epsilon_N(m)\to 0$ as $N\to\infty$ for every $m=1, 2, \cdots$. Since $f_m(x)$ converges uniformly to $g(x)$ in $[M_1, M_2]$ we also have

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f_m(x)dp(x) - \int_{-\infty}^{\infty} g(x)dp(x)\right| \\ = \left|\int_{M_1}^{M_2}\{g_m(x) - g(x)\}dp(x)\right| \leqq \epsilon_m\cdot V(p) = \epsilon_m',\end{aligned} \tag{10}$$

where $\epsilon_m = \max_{M_1\leqq x\leqq M_2}|g_m - g|$ and where $\epsilon_m'\to 0$ when $m\to\infty$ since $V(p)$ is finite. Also

$$\begin{aligned}\left|\sum_{n=1}^{N}\int_{-\infty}^{\infty} g(x)dp_n(x) - \sum_{n=1}^{N}\int_{-\infty}^{\infty} f_m(x)dp_n(x)\right| \\ \leqq \left|\sum_{n=1}^{N}\int_{M_1}^{M_2}\{g(x) - g_m(x)\}dp_n(x)\right| \\ \leqq \epsilon_m\cdot\sum_{n=1}^{N}\int_{M_1}^{M_2}\left|dp_n(x)\right| \\ \leqq \epsilon_m\cdot\sum_{n=1}^{\infty} V(p_n) = \epsilon_m''\end{aligned} \tag{11}$$

where $\epsilon_m''\to 0$ when $m\to\infty$, since $\sum_{n=1}^{\infty} V(p_n)$ is finite.

Now let $\delta$ be an arbitrary positive number and take $m$ large enough so that

$$\epsilon_m' + \epsilon_m'' < 2\delta/3. \tag{12}$$

With $m$ fixed, take $N$ large enough so that

$$\epsilon_{N+\nu}(m) < \delta/3, \qquad \nu = 0, 1, 2, \cdots. \tag{13}$$

It follows that [using (12) and (13) in (8)]

$$\left| \sum_{n=1}^{N} \int_{-\infty}^{\infty} g(x)dp_n(x) - \int_{-\infty}^{\infty} g(x)dp(x) \right| < \delta$$

for all sufficiently large $N$ and arbitrary $\delta > 0$.

Coming to case (c), let $\alpha$ be a positive number and define

$$g_\alpha(x) = \begin{cases} g(x), & -\alpha \leqq x \leqq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Since $g_\alpha(x)$ satisfies the conditions of hypothesis (b), we have

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) = \int_{-\infty}^{\infty} g_\alpha(x)dp(x). \tag{14}$$

By dominated convergence we have

$$\lim_{\alpha\to\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp(x) = \int_{-\infty}^{\infty} g(x)dp(x). \tag{15}$$

Let $G = \text{u.b.}_{-\infty<x<\infty} |g(x)|$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) \right| &\ll \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |g(x)| \, |dp_n(x)| \\ &\ll G \cdot \sum_{n=1}^{\infty} V(p_n). \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x)$ converges absolutely and uniformly in $\alpha$, and it follows that

$$\begin{aligned} \lim_{\alpha\to\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) &= \sum_{n=1}^{\infty} \lim_{\alpha\to\infty} \int_{-\infty}^{\infty} g_\alpha(x)dp_n(x) \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g(x)dp_n(x), \end{aligned} \tag{16}$$

the last step being justified by the theorem on dominated convergence. Combining (14), (15) and (16), we obtain (2). This proves the theorem under hypothesis (c).

To complete the proof of the theorem we need the following lemma.

LEMMA. *Let $p_n(x)$ satisfy the conditions of the theorem. Suppose* (2) *is true for all bounded functions $g(x)$ of Baire class less than $\alpha$. Then* (2) *is true for all bounded functions $g(x)$ in the class $\alpha$.*

Let $f_j(x)$ be any convergent sequence in a Baire class $\alpha_1<\alpha$ such that $f_j(x)$ is uniformly bounded. Let $\lim_{j\to\infty} f_j(x)=f^*(x)$. Clearly $f^*(x)$ is also bounded. By hypothesis, we have

$$\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_j(x)dp_n(x) = \int_{-\infty}^{\infty} f_j(x)dp(x) \tag{17}$$

for $j=1, 2, \cdots$. By a theorem on bounded convergence, we have

$$\lim_{j\to\infty}\int_{-\infty}^{\infty} f_j(x)dp(x) = \int_{-\infty}^{\infty} f^*(x)dp(x). \tag{18}$$

Consider also

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\int_{-\infty}^{\infty} f_j(x)dp_n(x)\right| &\ll F\cdot\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\left| dp_n(x)\right|\\ &= F\cdot\sum_{n=1}^{\infty} V(p_n),\end{aligned}$$

where $F$ is u.b.$_{-\infty<x<\infty,\ j=1,2,\cdots}\left|f_j(x)\right|$. Hence $\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_j(x)dp_n(x)$ converges absolutely and uniformly in $j$. Therefore

$$\begin{aligned}\lim_{j\to\infty}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_j(x)dp_n(x) &= \sum_{n=1}^{\infty}\lim_{j\to\infty}\int_{-\infty}^{\infty} f_j(x)dp_n(x)\\ &= \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f^*(x)dp_n(x).\end{aligned} \tag{19}$$

Combining equations (17), (18) and (19), we get

$$\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f^*(x)dp_n(x) = \int_{-\infty}^{\infty} f^*(x)dp(x),$$

which completes the proof of the lemma.

Finally we return to the original hypothesis. It is known[7] that the set of bounded functions in classes 0, 1, 2, $\cdots$ of Baire is the same as the set of bounded Borel measurable functions. Under (c) we have proved the theorem for any bounded continuous function, that is, for bounded functions in class 0 of Baire. By the lemma it holds then for functions in classes 0, 1, 2, $\cdots$ of Baire and hence for all bounded Borel measurable functions. This completes the proof of the theorem.

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[7] For example, see de la Vallée Poussin, *Intégrales de Lebesgue* (Borel Monograph), 1916, pp. 36, 37.

We note in the theorem above that the boundedness of $g(x)$ was sufficient to insure the existence of the integral $\int_{-\infty}^{\infty} g(x)dp(x)$, provided that $p(x)$ satisfied the conditions of this theorem. Now we shall prove a partial converse of this.

THEOREM. *Let $g(x)$ be a given Borel measurable function defined everywhere in $(-\infty, \infty)$, with the property that $\int_{-\infty}^{\infty} g(x)dp(x)$ exists*[8] *whenever $p(x) \in V$. Then $g(x)$ is bounded everywhere in $(-\infty, \infty)$.*

Let $E_n$ be the set of points $x$ for which $n \leqq g(x) < n+1$. We shall show that the family of non-empty sets $E_{n_i}$ is finite.

Suppose the contrary. Then there must exist an infinity of distinct $x_{n_i}$ since $E_{n_i}$ are mutually disjoint. Consider a particular sequence $x_{n_0}, x_{n_1}, x_{n_2}, \cdots$, one $x_{n_i}$ from each non-empty $E_{n_i}$.

At this point let us note that given any sequence of integers $0 < n_1 < n_2 < \cdots$ there exists a number $\xi$ such that $\sum_{i=1}^{\infty} 1/n_i^{\xi}$ diverges and $\sum_{i=1}^{\infty} 1/n_i^{\xi+1}$ converges. In class $L$ put all real numbers $p$ such that $\sum_{i=1}^{\infty} 1/n_i^{p}$ is divergent. Put all others in $R$-class. To the $R$-class belongs the number 2. The number 0 belongs to the $L$-class. If $p$ belongs to $L$, then all numbers less than $p$ also belong to $L$. Thus we have a Dedekind cut. Let $c$ be the number defined by this cut. Then $\sum_{i=1}^{\infty} 1/n_i^{c-1/2}$ diverges, while $\sum_{i=1}^{\infty} 1/n_i^{c+1/2}$ converges. Hence $\xi = c - 1/2$ satisfies the required condition.

With this in mind, let us return to the proof of the theorem. Using the sequence $x_{n_i}$ defined previously let us form the function

$$p(x) = \sum_{i=0}^{\infty} \frac{1}{(n_i + 1)^{\xi+1}} \omega_{n_i}(x),$$

where

$$\omega_{n_i}(x) = \begin{cases} 0, & x < x_{n_i}, \\ 1, & x \geqq x_{n_i}, \end{cases}$$

and $\xi$ is such a number that $\sum_{i=0}^{\infty} 1/(n_i+1)^{\xi+1}$ converges and $\sum_{i=0}^{\infty} 1/(n_i+1)^{\xi}$ diverges.

Since $\omega_{n_i}(x)$ is right-continuous and the series uniformly convergent, $p(x)$ is also right-continuous. Also $p(x)$ is a monotone increasing function. To show that $p(x) \in V$, we make use of the theorem proved earlier. In fact we have

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[8] M. H. Stone, *Linear transformations in Hilbert space*, Amer. Math. Soc. Colloquium Publications, vol. 15, 1932, p. 206.

$$\begin{aligned}\int_{-\infty}^{\infty} | dp(x) | &= \int_{-\infty}^{\infty} dp(x) = \int_{-\infty}^{\infty} d \left\{ \sum_{i=0}^{\infty} \frac{1}{(n_i + 1)^{\xi+1}} \omega_{n_i}(x) \right\} \\ &= \sum_{i=0}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(n_i + 1)^{\xi+1}} d\omega_{n_i}(x) \\ &= \sum_{i=0}^{\infty} \frac{1}{(n_i + 1)^{\xi+1}}.\end{aligned}$$

To complete the proof of this theorem it is sufficient to show that $\int_{-\infty}^{\infty} g(x) dp(x)$ does not exist for this $p(x)$. Let

$$g_M(x) = \begin{cases} g(x), & \text{if } | g(x) | \leq M, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} | g_M(x) | \, | dp(x) | &= \sum_{i=0}^{\infty} \frac{1}{(n_i + 1)^{\xi+1}} \int_{-\infty}^{\infty} | g_M(x) | \, d\omega_{n_i}(x) \\ &= \sum_{i=0}^{\infty} \frac{1}{(n_i + 1)^{\xi+1}} | g_M(x_{n_i}) | \\ &\geq \sum_{n_i < M} \frac{n_i}{(n_i + 1)^{\xi+1}},\end{aligned}$$

the first step being justified by the first theorem. Hence

$$\lim_{M \to \infty} \sum_{n_i < M} \frac{n_i}{(n_i + 1)^{\xi+1}} \leq \int_{-\infty}^{\infty} | g(x) | \, | dp(x) |,$$

which proves that the integral does not exist, since the series $\sum_{i=0}^{\infty} 1/(n_i+1)^{\xi}$ was constructed to be divergent.

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# DECOMPOSITIONS OF A $T_1$ SPACE

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**Introduction.** Several authors have proved theorems of the type: The "structure" of a certain class of transformations defined on a "suitable" space $A$ to a fixed "suitable" space $B$ "determines" the space $A$.

As examples, we have:

Banach [3, p. 170, see also 6, 7, 13]:[1] The Banach space of all real, continuous functions defined on a compact metric space $A$ "determines" $A$.

Eidelheit [5, see also 2, 10, 11]: The ring of all bounded operators on a real Banach space $A$ "determines" $A$.

In the present paper, we prove an analogous theorem (Theorem 2.5). Intuitively, it says that a $T_1$ space $A$ is "determined" by a rather weak ordered system structure of the collection of all continuous mappings of $A$ onto an arbitrary (variable) $T_1$ space $B$. More exactly, it states: If two $T_1$ spaces $A$, $B$ are such that the ordered system of upper semi-continuous decompositions of $A$ is isomorphic to that of $B$, then $A$ and $B$ are homeomorphic.

In §1 we give a discussion of ordered systems which is sufficient for our purposes. In §2 we prove the theorem mentioned above. In §3 we characterize separation and connectedness properties of a $T_1$ space in terms of order properties of its upper semi-continuous decompositions. In §4 we discuss compactness properties of the space and their relations to order properties of the decompositions, and in §5 we give some examples and counter examples.

## 1. Ordered systems.[2]

We assume that the reader is fairly familiar with the nomenclature of ordered systems and lattices.

An *ordered system* is a collection, $M$, of elements, $D$, with an ordering defined in $M$. That is, there is given a binary relation, $>$, which is defined for some pairs of elements in $M$, and which is transitive, reflexive, and proper.

We use *less than* and *greater than* to refer to the ordering in $M$; *contains* will be used only in the point set sense.

Most of the ordered systems we consider will be directed and will contain atoms. Throughout the paper, we use the letters $a$, $b$, $c$ for

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.

[2] For a more complete discussion, see [4, 12, 14].

atoms of ordered systems.

A ***multiplicative system*** is an ordered system in which each pair of elements has an inf; we use $\sigma$-complete and complete in the usual sense.

THEOREM 1.1. *A complete multiplicative system $M$ with a unit is a complete lattice.*

For any non-empty collection $\{D_\alpha\}$ of elements of $M$, define[3]

$$D' = \wedge \{D \mid D > D_\alpha, \text{ all } \alpha\}.$$

Since $M$ has a unit, $\{D \mid D > D_\alpha, \text{ all } \alpha\}$ is not empty, and $D'$ exists; also, from the definition of $\wedge$, $D' > D_\alpha$ for all $\alpha$. Suppose $D'' > D_\alpha$ for all $\alpha$. Then

$$D' = \wedge \{D \mid D > D_\alpha\} = D'' \wedge (\wedge \{D \mid D > D_\alpha, D \neq D''\} < D''$$

so that $D' = \vee D_\alpha$ and $M$ is a complete lattice.

Two ordered systems $M$, $M'$ are ***isomorphic*** if there is a one-to-one, order preserving correspondence between them; that is, if there exists a one-to-one mapping $f(M) = M'$ such that $D' < D''$ is equivalent to $f(D') < f(D'')$.

Some examples of ordered systems which we shall use in the sequel are:

1. The collection of all open neighborhoods of a fixed point of any topological space, ordered by point set inclusion. That is, $U < V$ means $U \subset V$. This system is directed both by $>$ and by $<$ ($<$ is the "interesting" direction), in fact, it is a lattice. It contains a unit, but usually not a zero, and even if it has a zero, it may contain no atoms.

2. The lattice of all closed subsets of a $T_1$ space, again ordered by point set inclusion. Since this lattice is complete, it has a zero and a unit. The closed sets containing just one point are atoms, and these are the only atoms.

**2. The reconstruction of $S$ from $M$.** Throughout the paper, $S$ will denote a topological space which contains an infinite number of points (four points would be sufficient), and in which each subset containing exactly one point is closed; that is, $S$ is a $T_1$ space. By a decomposition of $S$ we shall mean a collection of non-empty, closed, disjoint subsets of $S$ whose union is $S$.

DEFINITION 2.1. *A decomposition of $S$ is upper semi-continuous*

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[3] We use $\vee$, $\wedge$ for sup, inf; $\cup$, $\cap$ for union, intersection, respectively; and $\{p \mid q\}$ for the set of all elements $p$ with property $q$.

*(u.s.c.) if, for any set $X$ of the decomposition and any open set $U\supset X$, there exists an open set $V\supset X$ such that any set of the decomposition which meets $V$ is contained in $U$.*

It is well known that every upper semi-continuous decomposition (u.s.c.d.; the same abreviation will be used for the plural) of $S$ gives rise to a continuous mapping of $S$ onto a $T_1$ space—the Zerlegungsraum of Alexandroff and Hopf [1, p. 61 ff.]. Thus any statement concerning u.s.c.d. may be interpreted as a statement about continuous mappings of $S$ onto a $T_1$ space.

Let $M=\{D\}$ be the ordered system of u.s.c.d. of $S$, where $D'<D''$ means each set of the decomposition $D'$ is contained in some set of the decomposition $D''$. It is easy to see that $M$ has a zero, 0, (the decomposition of $S$ into the single points of $S$) and a unit, $I$, (the decomposition of $S$ into the single set $S$) and is therefore directed both by $>$ and by $<$. However (Example 5.3), $M$ is in general not a multiplicative system.

If $x_a$, $y_a$ are any two distinct points of $S$, the decomposition of $S$ which consists of the set $x_a\cup y_a$ and the single points of $S-(x_a\cup y_a)$ is u.s.c. and is an atom of $M$. Conversely, any atom of $M$ has this form. We shall say that $a$ is *generated* by $x_a$, $y_a$; the generators of an atom are unique, and completely determine the atom. Even though $M$ is not necessarily a lattice, it is clear that $\bigvee a_i$ $(i=1, 2, \cdots, n)$ exists for any finite number of atoms and $a_i\wedge a_j=0$ if $a_i\neq a_j$.

DEFINITION 2.2. *Two atoms $a$, $b$ are concurrent, in symbols, $a\sim b$, if and only if it is false that $a\vee b$ is greater than exactly two atoms.*

THEOREM 2.1. *A necessary and sufficient condition that $a\sim b$ is that $(x_a\cup y_a)\cap(x_b\cup y_b)\neq 0$ ($a$; $b$ generated by $x_a$, $y_a$; $x_b$, $y_b$ respectively).*

NOTE. From this theorem, it will be observed that the relation $\sim$ is reflexive and symmetric but not, in general, transitive.

*Sufficiency.* We may assume $x_a=x_b$. Then either $y_a=y_b$ so that $a=b=a\vee b$ and $a\vee b$ is greater than just one atom, or $y_a\neq y_b$. In the latter case, $a\vee b$ is the decomposition of $S$ into $x_a\cup y_a\cup y_b$ and the single points of $S-(x_a\cup y_a\cup y_b)$. But then there are three distinct atoms, namely $a$, $b$, and the atom generated by $y_a$, $y_b$, which are less than $a\vee b$.

*Necessity.* Suppose $a$ and $b$ are not concurrent, then $a\neq b$ since otherwise $a\vee b=a$ and is greater than just one atom. But we have seen above that if $a\neq b$ and $(x_a\cup y_a)\cap(x_b\cup y_b)\neq 0$, there are three atoms less than $a\vee b$. Therefore, since $a\vee b$ is greater than only two atoms by hypothesis, we must have

$$(x_a \cup y_a) \cap (x_b \cup y_b) = 0.$$

DEFINITION 2.3. *An u.s.c.d., $D$, of $S$ is trivial if, for any two atoms $a$, $b$ less than $D$, there exists an atom $c$ less than $D$ satisfying $a \sim c$, $b \sim c$.*

THEOREM 2.2. *An u.s.c.d., $D$, of $S$ is trivial if and only if it consists of single points of $S$ except for one closed set $X$ (which may be empty, but cannot contain just one point). We say that $D$ is generated by $X$, and use the notation $D_X$ for it.*

*Sufficiency.* We may suppose $X \neq 0$, since otherwise $D=0$ and is trivial. If $a$, $b$ are any two atoms less than $D$, the atom $c$ generated by $x_a, y_b$ (which will be distinct points under suitable labeling) satisfies our requirements.

*Necessity.* If there are two sets $X$, $Y$ of the decomposition $D$, each containing at least two points, say $x_a, y_a \in X$, $x_b, y_b \in Y$, then $a$, $b$ are two atoms less than $D$ such that no atom $c$ less than $D$ satisfies $a \sim c$, $b \sim c$.

DEFINITION 2.4. *An infinite collection of atoms of $M$ is a $C$-set if each pair of atoms of the collection are concurrent.*

Let us order the $C$-sets of $M$ by inclusion, that is $C' > C''$ if each atom in $C''$ is also in $C'$. Then by Zorn's Lemma (or see below) there exist maximal $C$-sets. We denote the collection of all maximal $C$-sets of $M$ by $S'$.

THEOREM 2.3. *There is a one-to-one correspondence between the points of $S$ and the maximal $C$-sets of $S'$.*

Let $x$ be any point of $S$ and define $f(x)$ as the collection of all atoms generated by $x$, $y$ for arbitrary $y \in S - x$. Then, by Theorem 2.1, $f(x)$ is a $C$-set. (It is infinite since $S$ has an infinite number of points.) Moreover, it is a maximal $C$-set, for if $C'$ is any $C$-set containing $f(x)$, and $c' \in C'$, then (Theorem 2.1) one of the generators of $c'$ must be $x$, and $c' \in f(x)$. Thus $f(x)$ is a single-valued transformation of $S$ into $S'$.

The inverse function $f^{-1}$ is single-valued. For, if $f(x) = f(y)$, then any atom of $M$ which has $x$ as a generator also has $y$ as a generator, and $x = y$.

$f(S)$ covers $S'$. For, if $C$ is any maximal $C$-set, and $a$, $b$ are any two distinct atoms in $C$, then $a \sim b$ and we may suppose $x_a = x_b$, $y_a \neq y_b$. Now, if $c$ is any atom in $C$, we must have

$$(x_a \cup y_a) \cap (x_c \cup y_c) \neq 0 \neq (x_b \cup y_b) \cap (x_c \cup y_c)$$

so that (with suitable labeling) either (1) $x_c = x_a = x_b$, or (2) $x_a = y_a$ and $y_c = y_b$. But the latter case is impossible because $C$ must contain an infinite number of atoms whereas if (2) holds no atom of $M-(a \cup b \cup c)$ can be concurrent with each of $a$, $b$, $c$. Thus (1) is the case, and $f(x_a) = C$.

THEOREM 2.4. *If $M$ is the ordered system of all u.s.c.d. of a $T_1$ space $S$, then, using only the order properties of $M$, a $T_1$ space $S'$ may be defined which is homeomorphic to $S$.*

The points of $S'$ will be the maximal $C$-sets of $M$. A subset, $Q$, of $S'$ will be closed if either (1) $Q$ consists of exactly one point of $S'$, or (2) there exists a trivial decomposition, $D$, of $S$ such that $Q$ is the collection of all maximal $C$-sets which contain at least one atom less than $D$.

It is easy to see that $S'$ is a $T_1$ space. We prove that the transformation $f(S) = S'$ defined in the proof of Theorem 2.3 is a homeomorphism. We must only show that $f$ and $f^{-1}$ are closed.

Let $X$ be any closed subset of $S$ which contains more than one point. Then the closed subset of $S'$ given by condition (2) above with $D = D_X$ is exactly $f(X)$.

Conversely, if $X'$ is closed in $S'$ and contains more than one point, and if $D_X$ is the trivial decomposition given by condition (2) above, then $f^{-1}(X') = X$.

Since both $S$ and $S'$ are $T_1$ spaces, $f$ and $f^{-1}$ are closed and $f$ is a homeomorphism.

THEOREM 2.5. *A necessary and sufficient condition that two $T_1$ spaces be homeomorphic is that their ordered systems of u.s.c.d. be isomorphic.*

The proof is immediate from Theorem 2.4.

## 3. Separation properties of $S$.

THEOREM 3.1. *If $S$ is normal, $M$ is a multiplicative system.*

Let $D' = \{X\}$, $D'' = \{Y\}$ be any two elements of $M$, and define

$$D = \{X \cap Y \mid X \in D', Y \in D'', X \cap Y \neq 0\}.$$

Clearly $D$ is a decomposition of $S$; we show that it is u.s.c. If $X \cap Y \in D$ and $U$ is any open set containing $X \cap Y$, then $X - U$, $Y - U$ are closed, disjoint subsets of $S$ so that there exist open, disjoint subsets $W', W''$ containing them, respectively. Now $U \cup W' \supset X$, $U \cup W'' \supset Y$ and, since $D', D''$ are u.s.c., there exist open sets $V' \supset X$, $V'' \supset Y$ such that any set of $D^{(i)}$ which meets $V^{(i)}$ is contained in $U \cup W^{(i)}$ $(i = ', '')$.

Set $V = V' \cap V''$. Then, if $X' \cap Y'$ is any set of $D$ which meets $V$, we have

$$X' \cap V' \neq 0, \quad \text{and} \quad X' \subset U \cup W',$$

$$Y' \cap V'' \neq 0, \quad \text{and} \quad Y' \subset U \cup W'',$$

so that

$$X' \cap Y' \subset (U \cup W') \cap (U \cup W'') = U,$$

and $D$ is u.s.c. Clearly $D = D' \wedge D''$. Hence, $M$ is a multiplicative system.

The converse of Theorem 3.1 does not hold. See Example 5.2.

DEFINITION 3.1. *Two trivial u.s.c.d. of $S$, $D'$, $D''$ are concurrent, in symbols $D' \sim D''$, if some atom less than $D'$ is concurrent with some atom less than $D''$.*

Notice that the trivial decomposition zero is not concurrent with any decomposition, and if $D'$, $D''$ are atoms, Definition 3.1 agrees with Definition 2.2.

THEOREM 3.2. *A necessary and sufficient condition that two nonzero trivial decompositions $D_X$, $D_Y$ be concurrent is that $X \cap Y \neq 0$.*

*Sufficiency.* Let $x \in X \cap Y$, $y \in Y - x$, $z \in X - x$. The atoms generated by $x$, $z$; $x$, $y$ are less than $D_X$, $D_Y$ respectively, and are concurrent, so that $D_X \sim D_Y$.

*Necessity.* If $X \cap Y = 0$, then no atom less than $D_X$ can be concurrent with any atom less than $D_Y$, so that $D_X$ and $D_Y$ are not concurrent.

DEFINITION 3.2. *Two nonzero, trivial decompositions $D'$, $D''$ run over $M$ if each atom of $M$ is concurrent with at least one of $D'$, $D''$.*

THEOREM 3.3. *A necessary and sufficient condition that two nonzero, trivial decompositions, $D_X$, $D_Y$ run over $M$ is that $S - (X \cup Y)$ contains at most one point.*

If $S - (X \cup Y)$ contains two distinct points $x$, $y$, then the atom generated by $x$, $y$ is not concurrent with either $D_X$ or $D_Y$.

If $S - (X \cup Y)$ contains zero or one points, then any atom of $M$ must have one of its generators in $X \cup Y$ and, by Theorem 3.2, must be concurrent with either $D_X$ or $D_Y$.

THEOREM 3.4. *$S$ is a Hausdorff space if and only if, for any atom $a \in M$, there exist trivial decompositions $D'$, $D''$ which run over $M$, such that*

$$D' \sim a, \quad a \sim D'', \quad D' \not> a, \quad a \not< D''.$$

Suppose $S$ is a Hausdorff space, and let $a$ be any atom of $M$. Then $x_a$, $y_a$ have disjoint neighborhoods $U$, $V$, and $X=S-U$, $Y=S-V$ form a closed covering of $S$. The trivial decompositions $D_X$, $D_Y$ satisfy the conditions of the theorem.

Conversely, if $x_a$, $y_a$ are any two distinct points of $S$, let $a$ be generated by $x_a$, $y_a$, and let $D_X$, $D_Y$ be the trivial decompositions satisfying the conditions of the theorem. Then (under suitable labeling)

$$x_a \in X \cap (S - Y), \qquad y_a \in Y \cap (S - X)$$

and $U=S-X$, $V=S-Y$ are neighborhoods of $x_a$, $y_a$ respectively. If $S-(X\cup Y)$ is empty, $U\cap V=0$, if $S-(X\cup Y)$ contains just one point, $z$, then $z$ is an isolated point and $U-z$, $V-z$ are disjoint neighborhoods of $x_a$, $y_a$, so that $S$ is a Hausdorff space.

The proofs of the following theorems are similar to the proof of Theorem 3.4, and will not be given here.

THEOREM 3.5. *$S$ is regular if and only if, for any nonzero trivial decomposition $D$, and any atom $a$ not concurrent with $D$, there exist trivial decompositions $D'$, $D''$ which run over $M$ and such that $D'>a$, $a$ is not concurrent with $D''$, $D'$ is not concurrent with $D$, $D<D''$.*

THEOREM 3.6. *$S$ is normal if and only if, for any two nonzero trivial decompositions $D'$, $D''$, such that $D'$ and $D''$ are not concurrent, there exist trivial decompositions $D'''$, $D''''$ which run over $M$ such that $D'''>D'$, $D'$ is not concurrent with $D''''$, $D'''$ is not concurrent with $D''$, $D''<D''''$.*

THEOREM 3.7. *$S$ has at least one isolated point if and only if one of the following two equivalent conditions is fulfilled.*

(I) *There exists a trivial decomposition $D$, and an atom $a \not< D$ which run over $M$.*

(II) *There exists a nonzero trivial decomposition $D$ and an atom $a \not< D$ such that, for any atom $b$, $b \not< D$ implies $b\sim a$.*

THEOREM 3.8. *$S$ is connected, except possibly for one isolated point, if and only if every two trivial decompositions which run over $M$ are concurrent.*

## 4. Compactness properties of $S$.

THEOREM 4.1. *If $S$ is a compact*[4] *Hausdorff space, then $M$ is a complete lattice.*

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[4] A topological space is compact if every open covering has a finite subcovering or, equivalently, if every collection of closed subsets whose intersection is empty contains a finite subcollection with empty intersection.

By Theorem 1.1, we need only show that $M$ is a complete multiplicative system. Let $\{D_\alpha\}$ be any non-empty collection of u.s.c.d. of $S$ and set

$$D = \left\{ \bigcap_\alpha X_\alpha \,\middle|\, X_\alpha \in D_\alpha, \bigcap_\alpha X_\alpha \neq 0 \right\}.$$

Clearly $D$ is a decomposition of $S$; we show that it is u.s.c. If $X = \bigcap X_\alpha$ is any set of $D$ and $U$ is any open set containing $X$, then each of the sets $X_\alpha - U$ is closed and their intersection is empty. Since $S$ is compact, there must be a finite number of the sets $X_\alpha - U$ whose intersection is empty, say

$$\bigcap_{i=1}^{n} (X_{\alpha_i} - U) = 0.$$

But $S$ is normal, hence there exist open sets $W_i \supset X_{\alpha_i} - U$ ($i=1, 2, \cdots, n$) such that $\bigcap W_i = 0$. Now for each $i$, $U \cup W_i$ is an open set containing $X_{\alpha_i}$ and, since each $D_\alpha$ is u.s.c., there exist open sets $V_i \supset X_{\alpha_i}$ such that any set of $D_{\alpha_i}$ which meets $V_i$ is contained in $U \cup W_i$. Set $V = \bigcap V_i$. Then $V \supset X$, and any set of $D$ which meets $V$ is contained in $U$. For, if $X' = \bigcap X'_\alpha$ meets $V$, then $X'_{\alpha_i} \cap V_i \neq 0$, and $X'_{\alpha_i} \subset U \cup W_i$, $i = 1, 2, \cdots, n$, so that

$$\bigcap_\alpha X'_\alpha \subset \bigcap_{i=1}^{n} X'_{\alpha_i} \subset \bigcap_{i=1}^{n} (U \cup W_i) = U.$$

Thus $D$ is an u.s.c.d. of $S$. Clearly $D = \bigwedge D_\alpha$, and $M$ is a complete lattice.

THEOREM 4.2. *If $S$ is regular, and each set of the u.s.c.d. $D = \{X\}$ is compact, then $D \wedge D'$ exists in $M$ for any $D' = \{Y\} \in M$.*

In the proof of Theorem 3.1, the only use which we made of the normality of $S$ was in concluding that, since $X - U$, $Y - U$ were closed, disjoint subsets of $S$, they could be separated by disjoint open sets $W'$, $W''$. Suppose $X$ is compact and $S$ is regular, then, for each point $x \in X - U$, there exist disjoint open sets $V_x$, $W_x$ containing $x$, $Y - U$ respectively. Since $X$ is compact, a finite number, say $V_{x_i}$ ($i = 1, \cdots, n$) of the $V_x$ cover $X - U$ and

$$W' = \bigcup_i V_{x_i}, \qquad W'' = \bigcap_i W_{x_i}$$

are disjoint open sets containing $X - U$, $Y - U$ respectively.

THEOREM 4.3. *If $S$ is a Hausdorff space and $D'$, $D''$ are two u.s.c.d.*

*of $S$ such that each set of either decomposition is compact, then $D' \wedge D''$ exists in $M$.*

The proof is similar to that for Theorem 4.2 and will not be given.

The author believes that the compactness hypotheses in Theorems 4.2, 4.3 are necessary, but has been unable to construct an example to prove their necessity. However, Example 5.3 shows that the separation hypotheses cannot be dropped.

THEOREM 4.4. *If $S$ is metric, then a necessary and sufficient condition that $M$ be a complete lattice is that $S$ be either discrete or compact.*

By Theorem 4.1, if $S$ is compact metric, $M$ is a complete lattice; if $S$ is discrete metric, then any decomposition of $S$ is u.s.c. and $M$ is a complete multiplicative system, hence, by Theorem 1.1, a complete lattice.

If $S$ is metric but neither discrete nor compact, there exist two sequences of distinct points, $x_i$, $y_i$ such that $\lim x_i = x$, $\{y_i\}$ has no limit point. Then the set

$$Y = x \cup (\bigcup x_i) \cup (\bigcup y_i)$$

is closed. Let $a_i$ be the atom of $M$ generated by $x_i$, $y_i$, then $D = \bigvee a_i$ does not exist in $M$.

For suppose $D \in M$. Consider the u.s.c.d. $D_n$ $(n = 1, 2, \cdots)$ consisting of the sets (1) single points of $S - Y$, (2) the $n$ sets $x_1 \cup y_1, x_2 \cup y_2, \cdots, x_n \cup y_n$, (3) the set $x \cup (\bigcup_{i>n} x_i) \cup (\bigcup_{i>n} y_i)$. Each $D_n > a_i$ for all $i$, hence $D < D_n$ for all $n$. Thus the decomposition $D$ must have the single point $x$ as one of its sets. But then $D$ is not u.s.c. since each neighborhood of $x$ meets all but a finite number of the sets $x_i \cup y_i$ of $D$, whereas the neighborhood $S - (\bigcup y_i)$ of $x$ does not contain any of the sets $x_i \cup y_i$.

In this connection, see also Examples 5.1, 5.2, 5.4.

5. **Examples.** In this section we give several examples which have been referred to in the other sections of the paper. No proofs will be given.

EXAMPLE 5.1. $S$ is the real line. $M$ is not a $\sigma$-complete multiplicative system.

For each $n > 1$, let

$$D_n = \Big\{ 1/2 \cup 2,\ 1/3 \cup 3, \cdots, 1/n \cup n,\ \Big(\bigcup_{i>n} 1/i\Big) \cup 0 \cup \Big(\bigcup_{i>n} i\Big), \text{ and single points of } S \text{ elsewhere} \Big\}.$$

Then $\bigwedge D_n$ does not exist in $M$. See Theorem 4.4.

EXAMPLE 5.2. $S$ is not discrete, not compact, not a Hausdorff space. $M$ is a complete lattice.

Let $S$ be the set of real numbers with "closed set" defined as (1) the whole space or (2) any countable (empty, finite, or denumerable) subset. Any decomposition of $S$ is u.s.c. See Theorems 3.1, 4.4.

EXAMPLE 5.3. $S$ is compact, not a Hausdorff space. $M$ is not a multiplicative system.

Let $S$ be the point set of the $xy$-plane consisting of the two line segments

$$L_i\colon\ y = i,\quad |x| \leqq 1, \qquad i = 1, 2,$$

but the topology of $S$ is *not* its relative topology. The closed subsets of $S$ are any finite union of the sets (1), single points of $S$, or (2), any subset of $S$ which is closed in the topology of the $xy$-plane and contains its projection on $L_i$ ($i=1, 2$). (Intuitively, (2) says: any set which is the union of the "same" two "closed" subsets of the segments $L_1$, $L_2$.) Let

$$a_i = (-1/i, 0), \qquad b_i = (1 - 1/i, 0), \qquad i = 2, 3, \cdots,$$
$$Y = \bigcup (a_i \cup b_i) \cup (0, 0) \cup (0, 1) \cup (1, 0) \cup (1, 1),$$

and set

$$D' = \{\text{single points of } S - Y,\ (0, 0) \cup (1, 0),\ (0, 1) \cup (1, 1),\ (a_i \cup b_i)\ (i = 2, 3, \cdots)\},$$
$$D'' = \{\text{single points of } S - Y,\ (0, 0) \cup (1, 1),\ (0, 1) \cup (1, 0),\ (a_i \cup b_i)\ (i = 2, 3, \cdots)\}.$$

Then $D' \wedge D''$ does not exist in $M$. See Theorems 3.1, 4.1, 4.2, 4.3.

EXAMPLE 5.4. $S$ is the $xy$-plane. $M$ is not a lattice.

Let

$$Y = \bigcup_{m,n>0} (1/n, m/n),$$

and set

$$D' = \{\text{single points of } S - Y,\ (1/n, m/n) \cup (1/n, (m + 1)/n)\ (n = 1, 2, \cdots;\ m = 1, 3, 5, \cdots)\},$$
$$D'' = \Big\{\text{single points of } (S - Y) \cup \Big(\bigcup_n (1/n, 1/n)\Big),\ (1/n, m/n) \cup (1/n, (m + 1)/n)\ (n = 1, 2, \cdots;\ m = 2, 4, 6, \cdots)\Big\}.$$

Then $D'\vee D''$ does not exist in $M$. See Theorems 3.1, 4.1, 4.4.

In a future paper, the author will discuss an interesting related problem which was suggested to him by A. D. Wallace. Namely: Characterize intrinsically those ordered systems which are "$M$'s" for some space $S$ from a specified class, for example $S$ compact Hausdorff.

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PURDUE UNIVERSITY

# THE ADDITIVITY OF THE LEBESGUE AREA

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A triple of continuous functions $T$: $x^i(u^1, u^2)$, $i=1, 2, 3$, defined on a closed square $Q$ $[0 \leq u^1 \leq 1, 0 \leq u^2 \leq 1]$ represents a surface $\mathfrak{S}$ (1.6, 1.17, 1.21).[1] If $r$ is any closed rectangle in $Q$ then we may speak of the triple $T_r$ consisting of the above triple $T$ with its range of definition restricted to $r$. This triple generates a surface $\mathfrak{S}(r)$. If $r_1$ and $r_2$ have no interior points in common, and $r_1+r_2=Q$, it is natural to hope that $L(\mathfrak{S})=L(\mathfrak{S}(r_1))+L(\mathfrak{S}(r_2))$ where $L$ is the symbol used to indicate the Lebesgue area (3.13). This statement is certainly true whenever the Lebesgue area is given by the standard integral formula, since the Lebesgue integral is additive. However, in general it is false. It may be said that this note is concerned with the statement that if the triple is constant on $r_1 \cdot r_2$, the Lebesgue area is additive.

Stated in this fashion the theorem may appear to be new. Actually it is but a very special case of a fundamental problem in the theory of area. The classical conjecture is that the area is additive under the much weaker requirement that the triple be rectifiable on $r_1 \cdot r_2$. No proof of this conjecture has, to our knowledge, appeared. (In this connection see McShane [1, p. 138].)

In relation to existing literature the theorem of this note is included in a more general theorem due to Morrey [1]. In fact, the first case in Morrey's proof [1, p. 314] is essentially the theorem of this paper. A discrepancy in his argument was rectified by Radó and Reichelderfer [1] in an independently interesting discussion of stretching processes.

This result, therefore, is not new.

The point of this treatment is the utter simplicity of the stretching process here employed. The development serves to make a known result more accessible.

THEOREM 1. *If a polyhedron* $\mathfrak{P}$ (3.11, 1.21) *has a quasi-linear* (1.7) *representation* $T$: $x(u)$, $u \in Q$, *such that the image of the side* $s$ $[u^1=1, 0 \leq u^2 \leq 1]$ *is interior to a sphere of diameter less than* $\epsilon$ *about the point* $a$ $[a^1, a^2, a^3]$, *then there exists a polyhedron* $\mathfrak{P}^*$ *with quasi-linear representation* $T^*$: $x^*(u)$, $u \in Q$, *such that*

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[1] The notation and terminology of this note are largely due to Radó [1]; in fact, numbers in parentheses refer the reader to appropriate paragraphs in his paper. Numbers in brackets refer to the bibliography.

(1) $E(\mathfrak{P}^*)=E(\mathfrak{P})$.[2]
(2) $x^*(u)=a$ for $u\in s$.
(3) $\|x^*(u)-x(u)\|<\epsilon$, hence $d(\mathfrak{P}^*, \mathfrak{P})<\epsilon$ (1.1, 1.22).

PROOF. Define $x_\alpha(u^1, u^2)=x(\alpha u^1, u^2)$, $\alpha>1$. Since $T$ is continuous we may choose $\alpha$ so that $\|x_\alpha(u)-x(u)\|<\epsilon$, for $u\in R_\alpha[0\leqq u^1\leqq 1/\alpha, 0\leqq u^2\leqq 1]$, and further the image of $S=Q-R_\alpha$ under $T$ is in a sphere of diameter less than $\epsilon$ about $a$. The representation $x_\alpha(u)$ is certainly quasi-linear in $R_\alpha$.

Define $x^*(u)$ to be equal to $x_\alpha(u)$ on $R_\alpha$. Now $x^*(u)$ is a quasi-linear representation of $\mathfrak{P}$ on $R_\alpha$.

We have yet to define $x^*(u)$ on the rest of $Q$. There is a set of intervals $I_1, \cdots, I_{n-2}$ on $[u^1=1/\alpha, 0\leqq u^2\leqq 1]$ such that $x^*(u)$ is linear on each of them and consecutive $I$'s abut. Their end points, in order, are

$$p_0[1/\alpha, 0],\ p_1, \cdots, p_{n-2}[1/\alpha, 1].$$

Add to this sequence $p_{n-1}[1, 1]$ and $p_n[1, 0]$.

Let $p$ be the point $[(\alpha+1)/2\alpha, -1]$ and consider the segments $(p, p_k)$, $k=1, \cdots, n-1$. Each segment cuts $[1/\alpha\leqq u^1\leqq 1, u^2=0]$ in exactly one point $q_k$. Join $p_k$ to $q_{k+1}$, $k=1, \cdots, n-2$. The rectangle $S$ is now subdivided into $2n-2$ triangles whose vertices are in the set $p_0, \cdots, p_n, q_1, \cdots, q_{n-1}$.

The triple $x^*(u)$ has been defined at the vertices $p_0, \cdots, p_{n-2}$.

Let $x^*(p_{n-1})=x^*(p_n)=a$, and $x^*(q_k)=x^*(p_k)$, $k=1, \cdots, n-1$.

Since the mapping is defined at the vertices of each triangle of the subdivision there is a unique linear extension over each triangle. The extensions will match along the side two triangles may have in common, and hence we have a quasi-linear[3] extension on $S$. Define $x^*(u)$ over $S$ to be this extension.

Now $T^*:x^*(u)$ is quasi-linear on $Q$ and so represents a polyhedron $\mathfrak{P}^*$.

It is important to notice that the map of each of the triangles in $S$ is a segment or a point and hence contributes zero area. Therefore $E(\mathfrak{P}^*)=E(\mathfrak{P})$.

Secondly, $x^*(u)=x^*(p_n)=x^*(p_{n-1})=a$ for $u\in s$.

Finally, $T^*(S)\subset T(S)+$ [the segment from $T(p_{n-1})$ to $a$]. But $T(S)$

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[2] $E(\mathfrak{P})$ is the elementary area of the polyhedron $\mathfrak{P}$, that is, the area in the ordinary geometric sense. It is not necessary, for the purpose of this note, to know that this coincides with the Lebesgue area. The fact that this is so, though sometimes ignored, is by no means obvious (3.20).

[3] It is important to notice that a quasi-linear representation need not be a map which carries each triangle of the subdivision into a nondegenerate triangle. The point set covered by a quasi-linear map may, for example, be a single point.

is contained in the sphere of diameter $\epsilon$ about $a$. Therefore $T^*(S)$ is in the same sphere. Now in view of our choice of $\alpha$ it follows that $\|x^*(u)-x(u)\|<\epsilon$ for $u\in Q$.

REMARK. The theorem may be stated with any rectangle $R$ replacing the square $Q$.

Let us turn at once to see how this stretching process is applied in the proof of the additivity result.

Consider a continuous triple $T$: $x(u)$, $u\in Q$. Let $Q$ be divided by a line $u^1=c$ into two rectangles $R_1$ $[0\leqq u^1\leqq c,\ 0\leqq u^2\leqq 1]$ and $R_2$ $[c\leqq u^1\leqq 1,\ 0\leqq u^2\leqq 1]$. Let $T_i$ be the triple $x(u)$, $u\in R_i$, $i=1$, 2. Now $T$ is a representation of a surface $\mathfrak{S}$, while $T_i$ is a representation of a surface $\mathfrak{S}_i$, $i=1$, 2. It is both well known and easy to see that $L(\mathfrak{S})\geqq L(\mathfrak{S}_1)+L(\mathfrak{S}_2)$. The reverse inequality is usually false. We are interested in a case in which it is true, making the Lebesgue area additive.

THEOREM 2. *If* $x(u)$ *is constant on* $u^1=c$, *then* $L(\mathfrak{S})=L(\mathfrak{S}_1)+L(\mathfrak{S}_2)$.

PROOF. Grant, for the moment, the existence of two sequences of polyhedra $\{{}_1\mathfrak{P}_n\}$ and $\{{}_2\mathfrak{P}_n\}$, having the property that for $i=1$, 2, $E({}_i\mathfrak{P}_n)\to L(\mathfrak{S}_i)$, and each ${}_i\mathfrak{P}_n$ has a quasi-linear representation ${}_ix_n(u)$ on $R_i$ such that $\|{}_ix_n(u)-x(u)\|<1/n$ for $u\in R_i$.

The justification of this will be taken up presently.

It is enough to obtain a sequence of polyhedra $\{\mathfrak{P}_n\}$ with quasi-linear representation $x_n(u)$ on $Q$ such that $E(\mathfrak{P}_n)$ is within $1/n$ of $E({}_1\mathfrak{P}_n)+E({}_2\mathfrak{P}_n)$, and $\mathfrak{P}_n\to\mathfrak{S}$.

As a matter of fact, we are going to obtain the stronger result, $E(\mathfrak{P}_n)=E({}_1\mathfrak{P}_n)+E({}_2\mathfrak{P}_n)$, and $\mathfrak{P}_n\to\mathfrak{S}$.

The theorem will then follow, since by the definition of the Lebesgue area, $L(\mathfrak{S})\leqq\liminf E(\mathfrak{P}_n)=\liminf\,[E({}_1\mathfrak{P}_n)+E({}_2\mathfrak{P}_n)]=\lim E({}_1\mathfrak{P}_n)+\lim E({}_2\mathfrak{P}_n)=L(\mathfrak{S}_1)+L(\mathfrak{S}_2)$.

The process by which the sequence of polyhedra $\{\mathfrak{P}_n\}$ is obtained is based on Theorem 1. Were we to define $x_n(u)$ to be ${}_ix_n(u)$ on $R_i$ the $x_n(u)$ would fail to represent the required sequence of polyhedra only by virtue of the fact that continuity, in general, would be lacking along $u^1=c$. Theorem 1 essentially enables us to alter ${}_1x_n(u)$ and ${}_2x_n(u)$ *slightly as to position* and *not at all as to area* so that they will match along $u^1=c$.

Consider any ${}_i\mathfrak{P}_n$ and its quasi-linear representation ${}_ix_n(u)$, $u\in R_i$. Theorem 1 guarantees the existence of a polyhedron ${}_i\mathfrak{P}_n^*$ with quasi-linear representation ${}_ix_n^*(u)$, $u\in R_i$, such that:

(1) $E({}_i\mathfrak{P}_n^*)=E({}_i\mathfrak{P}_n)$.

(2) ${}_ix_n^*(u)=x(u)$, a constant, on $[u^1=c,\ 0\leqq u^2\leqq 1]$.

(3) $\|{}_ix_n^*(u) - {}_ix_n(u)\| < 2/n$.

Now define $T_n : x_n(u) = {}_ix_n^*(u)$, $u \in R_i$, $i = 1, 2$.

This is a *continuous* triple due to condition (2), and so it is a quasi-linear triple representing a polyhedron $\mathfrak{P}_n$. We shall show that $\{\mathfrak{P}_n\}$ has all the desired properties.

First, $E(\mathfrak{P}_n) = E({}_1\mathfrak{P}_n) + E({}_2\mathfrak{P}_n)$ since the elementary area is additive.

Next, for $i = 1, 2$:

$$\begin{aligned}\|x_n(u) - x(u)\| &= \|{}_ix_n^*(u) - x(u)\| \text{ on } R_i \\ &\leqq \|{}_ix_n^*(u) - {}_ix_n(u)\| + \|{}_ix_n(u) - x(u)\| \text{ on } R_i \\ &\leqq 2/n + 1/n \text{ on } Q.\end{aligned}$$

Therefore $x_n(u)$ converges uniformly to $x(u)$ on $Q$ and hence $\{\mathfrak{P}_n\}$ certainly converges to $\mathfrak{S}$.

It now remains but to consider the statement deferred in the first paragraph of the proof.

Briefly the problem amounts to this: *Given a representation $x(u)$ of a surface $\mathfrak{S}$ on a rectangle $R$ we need to know that there exists a polyhedron $\mathfrak{P}$ whose area is close to the area of $\mathfrak{S}$ and admits of a quasi-linear representation on $R$ which is close to the given representation of $\mathfrak{S}$.*

Given $\epsilon > 0$, by the definition of the Lebesgue area (3.13) there is a polygon $\mathfrak{P}$ such that $d(\mathfrak{P}, \mathfrak{S}) < \epsilon$ and $|E(\mathfrak{P}) - L(\mathfrak{S})| < \epsilon$.

Since $\mathfrak{P}$ is a polyhedron it has a quasi-linear representation $\mathfrak{x}(u)$, $u \in B$. The set $B$ is the closure of a Jordan region with polygonal boundary. The set $B$ may be $R$ but this is by no means certain, or even useful. By the definition of distance (1.17) there is a topological transformation $\tau(R) = B$ such that $\|x(u) - \mathfrak{x}(\tau(u))\| < \epsilon$.

The continuity of $\mathfrak{x}(u)$ guarantees that there is a $\delta > 0$ such that all maps $\mu(R) = B$ satisfying $\|\mu(u) - \tau(u)\| < \delta$ have the property that $\|\mathfrak{x}(\mu(u)) - \mathfrak{x}(\tau(u))\| < \epsilon$. Hence $\|x(u) - \mathfrak{x}(\mu(u))\| < 2\epsilon$.

*We can select $\mu(u)$ so that in addition to satisfying the above inequality it is quasi-linear and topological on $R$.* The proof of this assertion follows as a result of some remarks of Franklin and Wiener [1, pp. 764–766].

LEMMA (FRANKLIN AND WIENER). *Given $\epsilon > 0$ and a topological transformation $\tau(s) = S$ (where $s$ and $S$ are closed squares) we may subdivide $s$ and $S$ into corresponding convex polygons $\cdots$ having the property that any topological transformation $\mu(s) = S$ which maps each polyhedron of $s$ onto the corresponding polyhedron of $S$ is at a distance less than $\epsilon$ from $\tau$.*

The reader will readily make the necessary adjustments from $s$ to $R$ and $S$ to $B$. The convex polygons mentioned above may be chosen as triangles and a $\mu$ may be exhibited which is linear on the triangles of $R$.

Now $\mathfrak{x}(\mu(u))$ is clearly quasi-linear on $R$. Moreover, it is a representation of $\mathfrak{P}$, and so $\mathfrak{P}$ has a quasi-linear representation on $R$ within $2\epsilon$ of the given representation of $\mathfrak{S}$. Since $|E(\mathfrak{P})-L(\mathfrak{S})|<\epsilon$ the proof is complete.

We have incidentally proved the independently important theorem:

THEOREM 3. *Any polyhedron has a quasi-linear representation on a rectangle.*

REMARK. The square $Q$ in the additivity theorem can be replaced by a simply connected Jordan region and its boundary, the dividing line can then be replaced by any crosscut on which the triple is constant.

It is possibly of some interest, in conclusion, to compare the surface stretching done here in Theorem 1 with that of the paper by Radó and Reichelderfer. Briefly, their process stretches a surface *directly towards a fixed point*. Each point of the surface sufficiently close to the fixed point is moved along the ray to the fixed point. The stretching is managed, however, so that the increase in area, though generally positive, is ingeniously kept within bounds. Here the stretching is by no means directly towards the fixed point, and it is precisely the rather circuitous route taken to the point which enables us to keep the added area equal to zero. If a polyhedron undergoes their stretching process it will, in general, no longer be a polyhedron; the stretching done here will preserve the polyhedral character of the surface. On the other hand their stretching process applies to situations which this cannot touch. For example, it will deal with the case in which $r_1$ is a square interior to $Q$, the set $r_2=\overline{Q-r_1}$, and the triple is constant on $r_1\cdot r_2$.

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# SOLUTION OF THE "PROBLÈME DES MÉNAGES"

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The *problème des ménages* asks for the number of ways of seating $n$ husbands and $n$ wives at a circular table, men alternating with women, so that no husband sits next to his wife. Despite the considerable literature devoted to this problem (cf. the appended bibliography), the following simple solution seems to have been missed.

It is convenient first to solve two preliminary problems, perhaps of some interest in themselves.

LEMMA 1. *The number of ways of selecting $k$ objects, no two consecutive, from $n$ objects arrayed in a row is ${}_{n-k+1}C_k$.*

Let $f(n, k)$ be the desired number. We split the selections into two subsets: those which include the last of the $n$ objects and those which do not. The former are $f(n-2, k-1)$ in number (since further selection of the second last object is forbidden); the latter are $f(n-1, k)$ in number. Hence

$$f(n, k) = f(n - 1, k) + f(n - 2, k - 1),$$

and, combining this with $f(n, 1) = n$, we readily prove by induction that $f(n, k) = {}_{n-k+1}C_k$.

LEMMA 2. *The number of ways of selecting $k$ objects, no two consecutive, from $n$ objects arrayed in a circle is ${}_{n-k}C_k n/(n-k)$.*

This differs from the preceding problem only in the imposition of the further restriction that no selection is to include both the first and last objects; and the number of such selections which are otherwise acceptable is $f(n-4, k-2)$. Hence the desired result is $f(n, k) - f(n-4, k-2) = {}_{n-k}C_k n/(n-k)$.

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We now restate the *problème des ménages* in the usual fashion by observing that the answer is $2n!u_n$, where $u_n$ is the number of permutations of $1, \cdots, n$ which do not satisfy any of the following $2n$ conditions: 1 is 1st or 2nd, 2 is 2nd or 3rd, $\cdots$, $n$ is $n$th or 1st. Now let us select a subset of $k$ conditions from the above $2n$ and inquire how many permutations of $1, \cdots, n$ there are which satisfy all $k$; the answer is $(n-k)!$ or 0 according as the $k$ conditions are compatible or not. If we further denote by $v_k$ the number of ways of selecting $k$ compatible conditions from the $2n$, we have, by the familiar argument of inclusion and exclusion, $u_n = \sum(-1)^k v_k (n-k)!$. It remains to evaluate $v_k$, for which purpose we note that the $2n$ conditions, when arrayed in a circle, have the property that only consecutive ones are not compatible. It follows from Lemma 2 that $v_k = {}_{2n-k}C_k 2n/(2n-k)$, and hence

$$u_n = n! - \frac{2n}{2n-1}\,{}_{2n-1}C_1(n-1)! + \frac{2n}{2n-2}\,{}_{2n-2}C_2(n-2)! - \cdots .$$

From this result it follows without difficulty that $u_n/n! \to e^{-2}$ as $n \to \infty$.

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# THEORY OF INTERSECTION OF TWO PLANE CURVES

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1. **Introduction.** This paper is concerned with the study of the projective differential geometry of two plane curves intersecting at an ordinary point. The method used is similar to the one given by the author[1] in a previous paper.

In §2 we show the existence of two projective invariants $I$, $J$ determined respectively by the fourth order of the two curves at the point of intersection.

In §3 by introducing a certain covariant triangle of reference associated to the two curves at the point of intersection, we reduce the invariants $I$, $J$ to simpler form and give them geometrical characterizations.

Finally, in §4 a covariant point is suitably chosen for the unit point of the coordinate system in order to obtain the canonical power series expansions of the two curves at the point of intersection. With reference to the vanishing or nonvanishing of the two invariants $I$, $J$ we have four different types of expansions. The absolute invariants in the expansions of each type are interpreted geometrically in terms of certain double ratios.

2. **Derivation of invariants.** Let us establish a projective coordinate system in a plane, in which a point has nonhomogeneous coordinates $x$, $y$ and homogeneous coordinates $x_1$, $x_2$, $x_3$, connected by the relations $x=x_2/x_1$, $y=x_3/x_1$. The context will show in any instance which coordinates are being used. Then let us consider two curves $C$, $\bar{C}$ in the plane which intersect at an ordinary point $O$ with distinct tangents $t$, $\bar{t}$. If we choose the point $O$ to be the vertex $(1, 0, 0)$ of the triangle of reference, and the tangents $t$, $\bar{t}$ to be the sides $y=0$, $x=0$ of the triangle, then the power series expansions of the curves $C$, $\bar{C}$ in the neighborhood of the point $O$ may be written in the form

$$\begin{cases} C: & y = a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + \cdots, \\ \bar{C}: & x = \bar{a}_1y^2 + \bar{a}_2y^3 + \bar{a}_3y^4 + \bar{a}_4y^5 + \cdots, \qquad a_1\bar{a}_1 \neq 0. \end{cases} \tag{1}$$

Let us now make a most general transformation of coordinates which shall leave the point $O$ and the tangents $t$, $\bar{t}$ unchanged. This

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[1] C. C. Hsiung, *Projective differential geometry of a pair of plane curves*, to appear in the Duke Mathematical Journal.

transformation is expressed in terms of the nonhomogeneous coordinates by the equations

$$(2)\qquad \begin{cases} x = B_2x^*/(1 + A_2x^* + A_3y^*), \\ y = C_3y^*/(1 + A_2x^* + A_3y^*). \end{cases}$$

The effect of this transformation on equations (1) is to produce two other equations of the same form each of whose first three coefficients, indicated by stars, are given by the formulas

$$(3)\qquad \begin{cases} C_3\overset{*}{a}_1 = B_2^2a_1, \\ C_3(A_2\overset{*}{a}_1 - \overset{*}{a}_2) = B_2^2(2A_2a_1 - B_2a_2), \\ C_3(A_2^2\overset{*}{a}_1 - A_3\overset{*}{a}{}_1^2 - A_2\overset{*}{a}_2 + \overset{*}{a}_3) = B_2^2(3A_2^2a_1 - 2A_3a_1\overset{*}{a}_1 \\ \qquad\qquad - 3A_2B_2a_2 + B_2^2a_3); \\ B_2\overset{*}{\bar{a}}_1 = C_3^2\bar{a}_1, \\ B_2(A_3\overset{*}{\bar{a}}_1 - \overset{*}{\bar{a}}_2) = C_3^2(2A_3\bar{a}_1 - C_3\bar{a}_2), \\ B_2(A_3^2\overset{*}{\bar{a}}_1 - A_2\overset{*}{\bar{a}}{}_1^2 - A_3\overset{*}{\bar{a}}_2 + \overset{*}{\bar{a}}_3) = C_3^2(3A_3^2\bar{a}_1 - 2A_2\bar{a}_1\overset{*}{\bar{a}}_1 \\ \qquad\qquad - 3A_3C_3\bar{a}_2 + C_3^2\bar{a}_3). \end{cases}$$

Solving the first four of equations (3) for $A_2$, $A_3$, $B_2$, $C_3$ and subsituting the results in the fifth, it is easily seen that

$$(4)\qquad I = (1/a_1(a_1\bar{a}_1^2)^{1/3})(a_1\bar{a}_2/\bar{a}_1 + a_2^2/a_1^2 - a_3/a_1)$$

is an invariant. In a way similar to the foregoing we have from the last one of equations (3) another invariant

$$(5)\qquad J = (1/\bar{a}_1(a_1^2\bar{a}_1)^{1/3})(\bar{a}_1a_2/a_1 + \bar{a}_2^2/\bar{a}_1^2 - \bar{a}_3/\bar{a}_1),$$

which can be obtained also immediately by interchanging $a_i$, $\bar{a}_i$ $(i=1, 2, 3)$ from equation (4).

3. **The canonical triangle and geometrical characterizations of the invariants** $I$, $J$. By suitable choice of the other two vertices of the triangle of reference, expansions (1) can be simplified. For this purpose we consider the pencil of conics having contact of the third order with the curve $C$ at the point $O$. The equation of a general conic of this pencil is readily found to be

$$(6)\qquad a_1^2x^2 + a_2xy - a_1y + ky^2 = 0,$$

where $k$ is arbitrary. The coordinates of the pole $O_1$ of the tangent $I$, $x=0$, with respect to this conic are

$$(a_2,\ a_1,\ 0). \tag{7}$$

Similarly, the pole $O_2$ of the tangent $t$, $y=0$, with respect to any conic having contact of the third order with the curve $\bar{C}$ at the point $O$ has the coordinates

$$(\bar{a}_2,\ 0,\ \bar{a}_1). \tag{8}$$

It is convenient to call the points $O_1$, $O_2$ *the associate points* of the curves $C$, $\bar{C}$ at the point $O$. If these points be taken for the vertices $(0, 1, 0)$, $(0, 0, 1)$ of the triangle of reference, then $a_2=\bar{a}_2=0$ and therefore equations (1) become

$$\begin{cases} C: & y = a_1x^2 + a_3x^4 + a_4x^5 + \cdots, \\ \bar{C}: & x = \bar{a}_1y^2 + \bar{a}_3y^4 + \bar{a}_4y^5 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0, \tag{9}$$

while the invariants $I$, $J$ take the simpler form

$$I = -\,a_3/a_1^2(a_1\bar{a}_1^2)^{1/3}, \tag{10}$$

$$J = -\,\bar{a}_3/\bar{a}_1^2(a_1^2\bar{a}_1)^{1/3}. \tag{11}$$

Thus the coordinate system except the unit point has been determined, and the triangle $OO_1O_2$ will be called *the canonical triangle* of the curves $C$, $\bar{C}$ at the point $O$.

We are now in position to characterize geometrically the invariants $I$, $J$ by expressing them in terms of certain double ratios. To this end let us consider the conic which passes through the associate point $O_2$ and has contact of the third order with the curve $C$ at the point $O$. Hereafter we shall call this conic *the associate conic* of the curve $C$ at the point $O$. From equation (6) its equation with respect to the canonical triangle

$$a_1x^2 - y = 0 \tag{12}$$

follows immediately. Similarly, we obtain the equation of the associate conic of the curve $\bar{C}$ at the point $O$,

$$\bar{a}_1y^2 - x = 0. \tag{13}$$

The two associate conics (12), (13) intersect, besides in $O$, in three points $P_i$ $(i=1, 2, 3)$ with coordinates

$$\left(\epsilon^i/(a_1^2\bar{a}_1)^{1/3},\ \epsilon^{2i}/(a_1\bar{a}_1^2)^{1/3}\right), \tag{14}$$

where $\epsilon^3=1$. Let $P_k'$ be the intersections of the lines $OP_k$, $P_iP_j$, where $i$, $j$, $k=1, 2, 3$ and are all distinct. It is evident that $P_1'P_2'$ is the unique real one among the three lines $P_i'P_i''$. Its equation may be easily found to be

$$(15)\qquad 1-2(a_1^2\bar{a}_1)^{1/3}x-2(a_1\bar{a}_1^2)^{1/3}y=0,$$

and we shall call it *the real associate line* of the curves $C$, $\bar{C}$ at the point $O$. This real associate line intersects the tangent $\bar{t}$, $x=0$, in the point $R$,

$$(16)\qquad (0,\ 1/2(a_1\bar{a}_1^2)^{1/3}).$$

On the other hand we consider a general cubic curve which has contact of the fourth order with the curve $C$ at the point $O$ and a node at $O$ with $t$, $\bar{t}$ for nodal tangents. A simple calculation suffices to give the equation of this nodal cubic curve, namely,

$$(17)\qquad a_1^3x^3+a_2xy^2-a_1^3xy+ky^3=0,$$

where $k$ is arbitrary. The line containing the three inflexions of the cubic (17) has the equation

$$(18)\qquad y=a_1^2/a_2,$$

which is intersected by the tangent $\bar{t}$, $x=0$, in the point $S$,

$$(19)\qquad (0,\ a_1^2/a_2).$$

From equations (10), (16), (19) it follows at once that the double ratio of the four points $O$, $O_2$, $R$, $S$ is equal to

$$(20)\qquad (OO_2,\ RS)=-I/2.$$

Hence we obtain the following geometrical characterization of the invariant $I$.

*Let two plane curves $C$, $\bar{C}$ intersect at an ordinary point $O$ with distinct tangents $t$, $\bar{t}$. Let $O_2$ be the associate point of the curve $\bar{C}$ at the point $O$, and $C_2$ a general cubic curve which has contact of the fifth order with the curve $C$ at the point $O$ and a node at $O$ with $t$, $\bar{t}$ for nodal tangents. If the tangent $\bar{t}$ intersects the real associate line of the curves $C$, $\bar{C}$ at $O$ in point $R$, and intersects the line containing the three inflexions of the nodal cubic curve $C_2$ in point $S$; then the invariant $I$ associated with the point $O$ of the curves $C$, $\bar{C}$ is, except for sign, equal to two times the double ratio $(OO_2,\ RS)$.*

The other invariant $J$ can be characterized geometrically in a similar manner.

**4. Canonical power series expansions.** In this section we shall complete the characterization of the coordinate system by suitably choosing a covariant point for the unit point, and obtain the canonical power series expansions of the curves $C$, $\overline{C}$ in the neighborhood of the point $O$. It will be convenient to discuss four cases with reference to the vanishing or nonvanishing of the two invariants $I$, $J$.

*Case* I. $IJ\neq 0$. Here we can choose for the unit point the intersection of the line of inflexions (18) and the similar one

$$x = \bar{a}_1^2/\bar{a}_3, \tag{21}$$

so that $a_3=a_1^2$, $\bar{a}_3=\bar{a}_1^2$. Thus we arrive at *the canonical power series expansions of the curves* $C$, $\overline{C}$ *in the neighborhood of the point* $O$,

$$\begin{cases} C: & y = a_1x^2 + a_1^2x^4 + \cdots, \\ \overline{C}: & x = \bar{a}_1y^2 + \bar{a}_1^2y^4 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0, \tag{22}$$

*in which the coefficients* $a_1$, $\bar{a}_1$ *can be expressed in terms of the two invariants* $I$, $J$, *namely,*

$$a_1 = -I/J^2, \qquad \bar{a}_1 = -J/I^2. \tag{23}$$

*Case* II. $I=0$, $J\neq 0$. In this case we observe that in the pencil (17) of nodal cubic curves there is a unique one having contact of the fifth order with the curve $C$ at the point $O$. Making use of the first of equations (9) and noticing that $a_3=0$ we find at once the equation of this cubic, namely,

$$a_1^4x^3 + a_4y^3 - a_1^3xy = 0. \tag{24}$$

The join $m$ of the two real intersections of the cubic curve (24) and the associate conic (13) of the curve $\overline{C}$ at $O$ is given by the equation

$$a_1\bar{a}_1^{1/3}x + (a_4 - a_1^3\bar{a}_1)^{1/3}y = 0. \tag{25}$$

If the unit point of the coordinate system be taken at the intersection of the lines (21), (25), then $a_4=a_1^3(\bar{a}_1-a_1)$, $\bar{a}_3=\bar{a}_1^2$, and therefore *the canonical power series expansions of the curves* $C$, $\overline{C}$ *in the neighborhood of the point* $O$ are reached:

$$\begin{cases} C: & y = a_1x^2 + a_1^3(\bar{a}_1 - a_1)x^5 + \cdots, \\ \overline{C}: & x = \bar{a}_1y^2 + \bar{a}_1^2y^4 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0. \tag{26}$$

Accordingly, the invariant $J$ becomes

$$J = -1/(a_1^2\bar{a}_1)^{1/3}. \tag{27}$$

In §2 we have shown that there are only two independent invariants determined by the neighborhoods up to and including the fourth order of the curves $C$, $\bar{C}$ at the point $O$, one of them vanishing now. In order to exhaust the two independent coefficients in the canonical expansions (26) it is sufficient for us to find another independent invariant of the curves $C$, $\bar{C}$ at the point $O$. For this purpose let $r$ be the join of the two real intersections of the two associate conics (12), (13), whose equation is found to be

$$a_1^{1/3}x - \bar{a}_1^{1/3}y = 0. \tag{28}$$

From equations (25), (28) the following double ratio is immediate:

$$\Delta \equiv (t\bar{t}, mr) = (\bar{a}_1/a_1)^{1/3}. \tag{29}$$

Thus the two coefficients $a_1$, $\bar{a}_1$ in expansions (26) have been completely expressed in terms of the invariant $J$ and the double ratio $\Delta$, namely,

$$a_1 = -1/J\Delta, \qquad \bar{a}_1 = -\Delta^2/J. \tag{30}$$

*Case* III. $I \neq 0$, $J=0$. Similarly, with the rôles of the curves $C$, $\bar{C}$ in Case II interchanged we obtain *the desired canonical power series expansions for this case,*

$$\begin{cases} C: & y = a_1x^2 + a_1^2x^4 + \cdots, \\ \bar{C}: & x = \bar{a}_1y^2 + \bar{a}_1^3(a_1 - \bar{a}_1)y^5 + \cdots, \end{cases} \qquad a_1\bar{a}_1 \neq 0, \tag{31}$$

*in which the two coefficients* $a_1$, $\bar{a}_1$ *can be expressed also in terms of the invariant* $I$ *and a double ratio similar to* (29).

*Case* IV. $I=0$, $J=0$. It is useful for this case to consider the cubic curve which has contact of the fifth order with the curve $\bar{C}$ at the point $O$ and a node at $O$ with $t$, $\bar{t}$ for nodal tangents. The equation of this cubic can be written immediately by interchanging $x$, $y$ and $a_i$, $\bar{a}_i$ $(i=1, 4)$ from equation (24). The result is

$$\bar{a}_4x^3 + \bar{a}_1^4y^3 - \bar{a}_1^3xy = 0. \tag{32}$$

The line $n$ joining the associate point $O_2$ to the real intersection, other than $O$, of the cubic curve (32) and the associate conic (12) of the curve $C$ at $O$ is given by the equation

$$x = (1/a_1\bar{a}_1)(a_1\bar{a}_1^2 - \bar{a}_4/\bar{a}_1)^{1/3}. \tag{33}$$

If the unit point of the coordinate system be taken at the intersection of the lines (25), (33), then $a_4=a_1^3\ (\bar{a}_1-a_1)$, $\bar{a}_4=a_1\bar{a}_1^3(1-a_1^2\bar{a}_1)$, and therefore we reach *the required canonical power series expansions,*

$$(34)\qquad \begin{cases} C: & y = a_1x^2 + a_1^3(\bar{a}_1 - a_1)x^5 + \cdots , \\ \bar{C}: & x = \bar{a}_1y^2 + a_1\bar{a}_1^3(1 - a_1^2\bar{a}_1)y^5 + \cdots , \end{cases} \qquad a_1\bar{a}_1 \neq 0.$$

In order to interpret geometrically the coefficients in the canonical expansions (34), let us assume $s$ to be the line joining the associate point $O_2$ to the real intersection, other than $O$, of the two associate conics (12), (13), whose equation is found to be

$$(35)\qquad x_1 - (a_1^2\bar{a}_1)^{1/3}x_2 = 0.$$

From equations (33), (35) we obtain at once the following double ratio,

$$(36)\qquad D \equiv (\bar{l},\ O_1O_2,\ n,\ s) = (a_1^2\bar{a}_1)^{1/3}.$$

Thus the two coefficients $a_1$, $\bar{a}_1$ in expansions (34) have been completely expressed in terms of the two double ratios $\Delta$, $D$, namely,

$$(37)\qquad a_1 = D/\Delta, \qquad \bar{a}_1 = D\Delta^2.$$

Some of the results of this section are contained in the following summary.

*Let two plane curves $C$, $\bar{C}$ intersect at an ordinary point $O$ with distinct tangents $t$, $\bar{t}$. The projective differential geometry of the curves $C$, $\bar{C}$ in the neighborhood of the point $O$ can be studied by means of expansions* (22), (26), (31), (34) *of four different types, all of whose coefficients are absolute invariants of the curves. The covariant coordinate system for each type of expansions has the following geometric description. The vertex* (1, 0, 0) *of the triangle of reference is the point $O$ under consideration, and the other vertices* (0, 1, 0), (0, 0, 1) *are the associate points $O_1$, $O_2$ of the curves $C$, $\bar{C}$ at the point $O$. Let $C_3(\bar{C}_3)$ be a general one of a pencil of cubic curves which have contact of the fourth order with the curve $C(\bar{C})$ at the point $O$ and a node at $O$ with $t$, $\bar{t}$ for nodal tangents, $K_3(\bar{K}_3)$ the unique cubic of this pencil having contact of the fifth order with the curve $C(\bar{C})$ at the point $O$, and $l(\bar{l})$ the line of inflexions of the nodal cubic curve $C_3(\bar{C}_3)$. Further, let $m(\bar{m})$ be the join of the two real intersections of the cubic $K_3(\bar{K}_3)$ with the associate conic $\bar{C}_2(C_2)$ of the curve $\bar{C}(C)$ at $O$, and $n$ the line joining the associate point $O_2$ to the real intersection, other than $O$, of the cubic $\bar{K}_3$ with the associate conic $C_2$. Then the unit point of the covariant coordinate system for each type of expansions is respectively one of the intersections of the four pairs of lines*: (i) *$l$ and $\bar{l}$,* (ii) *$\bar{l}$ and $m$,* (iii) *$l$ and $\bar{m}$, and* (iv) *$m$ and $n$.*

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# A TRANSFORMATION OF JONAS SURFACES

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It is well known that when an analytic surface $S$ is referred to its asymptotic net $(u, v)$ the homogeneous point coordinates $x^i(u, v)$ $(i=1, 2, 3, 4)$ of a generic point on $S$ can then be normalized, so that they satisfy the differential equations,

$$\left\{\begin{aligned} x_{uu} &= \beta x_v + p x, \\ x_{vv} &= \gamma x_u + q x, \end{aligned}\right. \tag{1}$$

where the coefficients $\beta$, $\gamma$, $p$, $q$ satisfy the conditions of integrability,

$$\left\{\begin{aligned} &(\beta_v + 2p)_v = (\beta\gamma)_u + \beta\gamma_u, \qquad (\gamma_u + 2q)_u = (\beta\gamma)_v + \gamma\beta_v, \\ &\qquad (p_v + \beta q)_v + \beta_v q = (q_u + \gamma p)_u + \gamma_u p. \end{aligned}\right. \tag{2}$$

The conjugate net $\Omega$ of $S$ defined by

$$C du^2 + D dv^2 = 0,$$

has equal point invariants when and only when[1]

$$(\log (C/D))_{uv} - (\gamma(C/D))_v + (\beta(D/C))_u = 0. \tag{3}$$

The necessary and sufficient condition that $\Omega$ should have equal tangential invariants is obtained from (3) by replacing $\beta$, $\gamma$ by $-\beta$, $-\gamma$ respectively. If $\Omega$ has equal invariants, both point and tangential, then it is a Jonas net, and $S$ then becomes a Jonas surface.[2] For a Jonas net we have thus the following relations:

$$(\log (C/D))_{uv} = 0, \qquad (\gamma(C/D))_u - (\beta(D/C))_v = 0.$$

By a suitable transformation of asymptotic parameters, leaving the asymptotic net unaltered, the above equations reduce to

$$\beta_u = \gamma_v, \qquad C = D.$$

Hence a Jonas net on a Jonas surface $S$ may be represented by the equation

$$du^2 - dv^2 = 0, \tag{4}$$

and the surface is characterized by

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[1] Cf. G. Fubini-E. Čech, *Geometria Proiettiva Differenziale*, vol. 1, Bologna, Zanichelli, 1927, p. 105.

[2] Cf. Fubini-Čech, ibid. p. 106.

$$\beta_u = \gamma_v. \tag{5}$$

The main object of this note is to prove the following theorem:

THEOREM. *The projection in a fixed plane of a Jonas net of a Jonas surface is a plane net with equal point invariants, and stands for the projection of the asymptotic net of another Jonas surface.*

A point $P_\theta$ with the coordinates $\theta = rx + sx_u + tx_v + lx_{uv}$ is fixed in space if

$$r = l_{uv} - l\beta\gamma, \qquad s = -l_v, \qquad t = -l_u, \tag{6}$$

where $l$ satisfies the system of equations

$$\begin{cases} l_{uu} = -\beta l_v + (\beta_v + p)l, \\ l_{vv} = -\gamma l_u + (\gamma_u + q)l. \end{cases} \tag{7}$$

A point $P_y$ on the straight line $P_\theta P_x$ is evidently given by $y = \lambda\theta + x$. In virtue of (1), we find by differentiation that

$$\begin{cases} y = (1 + \lambda r)x + \lambda s x_u + \lambda t x_v + \lambda l x_{uv}, \\ y_u = \lambda_u r x + (1 + \lambda_u s)x_u + \lambda_u t x_v + \lambda_u l x_{uv}, \\ y_v = \lambda_v r x + \lambda_v s x_u + (1 + \lambda_v t)x_v + \lambda_v l x_{uv}, \\ y_{uv} = \lambda_{uv} r x + \lambda_{uv} s x_u + \lambda_{uv} t x_v + (1 + \lambda_{uv} l)x_{uv}, \\ y_{uu} = (\lambda_{uu} r + p)x + \lambda_{uu} s x_u + (\lambda_{uu} t + \beta)x_v + \lambda_{uu} l x_{uv}, \\ y_{vv} = (\lambda_{vv} r + q)x + (\lambda_{vv} s + \gamma)x_u + \lambda_{vv} t x_v + \lambda_{vv} l x_{uv}. \end{cases} \tag{8}$$

In order that $P_y$ be in a fixed plane, it is necessary and sufficient that

$$\begin{cases} y_{uu} = A y_u + B y_v + C y, \\ y_{uv} = A' y_u + B' y_v + C' y, \\ y_{vv} = A'' y_u + B'' y_v + C'' y. \end{cases} \tag{9}$$

Substituting (8) in (9) and reducing, we obtain

$$\begin{array}{lll} A = 0, & B = \beta, & C = p, \\ A' = (\log l)_v, & B' = (\log l)_u, & C' = \beta\gamma - l_{uv}/l, \\ A'' = \gamma, & B'' = 0, & C'' = q, \end{array}$$

and the conditions for the parameter $\lambda$

$$\lambda_{uu} = \beta\lambda_v + p\lambda,$$

$$\lambda_{vv} = \gamma\lambda_u + q\lambda,$$

$$r\lambda + s\lambda_u + t\lambda_v + l\lambda_{uv} + 1 = 0.$$

Thus we have a plane net given by the equations

$$(10)\qquad \begin{cases} y_{uu} = \beta y_v + p y, \\ y_{vv} = \gamma y_u + q y, \\ y_{uv} = (\log l)_v y_u + (\log l)_u y_v + (\beta\gamma - l_{uv}/l) y. \end{cases}$$

The curves of this net are the perspectives on the fixed plane of the asymptotic curves of a Jonas surface $S$ obtained by projecting from the centre $P_\theta$. In order to obtain the perspectives on the same fixed plane of the Jonas net $\Omega$, we have to use the transformation

$$\bar{u} = u - v, \qquad \bar{v} = u + v,$$

so that

$$(11)\qquad \begin{cases} y_{\bar{u}} = (y_u - y_v)/2, \\ y_{\bar{v}} = (y_u + y_v)/2, \end{cases}$$

namely,

$$(11')\qquad \begin{cases} y_u = y_{\bar{u}} + y_{\bar{v}}, \\ y_v = y_{\bar{v}} - y_{\bar{u}}. \end{cases}$$

Putting

$$\log l = \theta, \qquad \beta\gamma - l_{uv}/l = c,$$

we find after a simple calculation that

$$(12)\qquad \begin{cases} y_{\bar{u}\bar{u}} = (1/4)(\gamma - \beta + 2\theta_u - 2\theta_v) y_{\bar{u}} \\ \qquad + (1/4)(\gamma + \beta - 2\theta_v - 2\theta_u) y_{\bar{v}} + (1/4)(p + q - 2c) y, \\ y_{\bar{v}\bar{v}} = (1/4)(\gamma - \beta - 2\theta_u + 2\theta_v) y_{\bar{u}} \\ \qquad + (1/4)(\gamma + \beta + 2\theta_v + 2\theta_u) y_{\bar{v}} + (1/4)(p + q + 2c) y, \\ y_{\bar{u}\bar{v}} = -(1/4)(\beta + \gamma) y_{\bar{u}} + (1/4)(\beta - \gamma) y_{\bar{v}} + (1/4)(p - q) y, \end{cases}$$

which represent the perspectives of the Jonas net $d\bar{u}^2 - d\bar{v}^2 = 0$. Since

$$-(\beta + \gamma)_{\bar{u}} = (\beta - \gamma)_{\bar{v}},$$

this net is of equal point invariants and therefore asymptotic. That is, it may be regarded as the perspectives of the asymptotic curves of a certain surface $Q$. The projective linear element of the surface $Q$ is easily found to be

$$(\bar{\beta}\, d\bar{u}^3 + \bar{\gamma}\, d\bar{v}^3)/2\, d\bar{u}\, d\bar{v},$$

where[3]

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[3] The projective linear element of a plane net has been defined by E. Čech. Cf. Fubini-Čech, *Introduction à la géométrie projective différentielle des surfaces*, 1931, chap. 10.

$$\bar{\beta} = (\gamma + \beta - 2\theta_v - 2\theta_u)/4, \qquad \bar{\gamma} = (\gamma - \beta - 2\theta_u + 2\theta_v)/4.$$

The relation $\beta_u = \gamma_v$ gives, however, the similar relation

$$\bar{\beta}_{\bar{u}} = \bar{\gamma}_{\bar{v}}.$$

Hence $Q$ is also a Jonas surface, which completes the proof.

As a special case of the theorem we have the result:

*Any Jonas net of a Jonas surface $S$ is perspective to the asymptotic net of another Jonas surface $Q$ from a fixed point. Conversely, if a Jonas net of a Jonas surface is perspective to the asymptotic net of another surface $Q$, then $Q$ is also a Jonas surface.*

THE NATIONAL UNIVERSITY OF CHEKIANG

# A NEW DEFERMENT PROCEDURE FOR MATHEMATICIANS

In April, 1943, a deferment procedure for mathematicians, involving the National Committee on Physicists and Mathematicians, was authorized under Activity and Occupation Bulletin No. 35. This was explained fully in a memorandum distributed to chairmen of departments of mathematics on May 11, 1943, and reproduced in the July issue of this BULLETIN and in the June-July issue of the *American Mathematical Monthly*. By this procedure employers were advised to send the original copy of form 42A to the National Committee for evaluation. From the Committee, the form was returned to the local board with an appropriate statement by the Committee concerning the advisability of the deferment. The National Committee was also empowered to appeal cases of registrants for whom occupational deferment was not granted by the local board. Under this arrangement, the Committee has performed its duties well and effectively.

Activity and Occupation Bulletin No. 35, which authorizes the activities of the National Committee, is to be rescinded in the near future, over the strong protests of representatives of mathematics and physics. The Committee will therefore cease to exist and will no longer be available to advise local boards on problems connected with the deferment of mathematicians. The new procedure for persons engaged in the occupations defined as critical in Local Board Memorandum No. 115, revised August 16, 1943, is described in Local Board Memorandum 115B which is reproduced at the end of this memorandum. *Only those cases in which occupational deferment is refused by the local board will become involved in this procedure.*

In the August 16 revision of Local Board Memorandum 115, a list of critical occupations, which includes mathematicians, was set up.

The following represents a summary of the occupational deferment procedure for the critical occupations:

1. The employer continues, as in the past, to present his case for deferment to the local board through the use of Form 42A.

2. The local board will arrive at a classification for the registrant on the basis of the evidence submitted to it by the employer. It may, if it so desires, consult with the local United States Employment Service Office concerning classification. (LBM 115 and LBM 115-C)

3. The local board will then make its decision. If occupational deferment is granted, no further action is involved during the period covered by the deferment. If, however, deferment is refused, and the registrant is placed in I-A, the case is continued as follows.

4. *If no appeal is taken* within the 10-day period allowed for this purpose, the local board is *directed* to refer the case to the local (local with respect to the registrant's local board) office of the United States Employment Service and a 30-day stay of induction is granted to permit action by this office.

5. *If an appeal is taken* and the local board's decision is reversed, the registrant will be reclassified, and the case is closed for the duration of the deferment period. If, however, the I-A classification made by the local board is supported by the appeal board, the case ***must be*** referred to the local U.S.E.S. office.

6. When the local U.S.E.S. office receives a case from the local board either directly (4, above), or after appeal (5, above), it may certify to the local board either that

a. the registrant should be deferred in his present position, or

b. that they have succeeded in placing the registrant in a new position.

In the first case the local board is directed to reopen the classification of the registrant, and presumably will grant occupational deferment. In the second case, a further period of 10 days is given the registrant for the filing of a new Form 42A requesting occupational deferment in his new position, and on the basis of which the local board shall reopen the classification.

7. If the U.S.E.S. fails to present any notification to the local board during the period allowed, or if it certifies that the registrant is needed in new employment, but does not succeed in placing the man in such employment, the local board may proceed with the induction of the registrant.

8. The local offices of the U.S.E.S. can refer cases reaching them and involving persons in "critical" professional and scientific occupations to the National Roster of Scientific and Specialized Personnel. The details of the relation between the U.S.E.S. offices and the Roster have not been fully defined.

The following suggestions are made to facilitate the proper functioning of the procedure herein outlined.

(1) It becomes of even greater importance than heretofore that an employer's original presentation of a case to the local board for occupational deferment or continuation of deferment be made as strong as possible. No opportunity for strengthening a case should be overlooked.

(2) The employer having once requested occupational deferment should be prepared to appeal every case in which such deferment is refused by the local board. This again implies that the original presen-

tation should be as strong as possible, so as to stand up under an appeal.

(3) If an appeal does not secure deferment, the employer should take it upon himself to see that the case is referred to the local U.S.E.S. office, as directed by Local Board Memorandum 115-B. It is possible that the local board may not be familiar with the requirement that they do this.

(4) The employer should take steps to insure that a case which has reached the U.S.E.S. office is actually referred by it to the National Roster.

(5) It should be noted that there is no clear statement as to whether an appeal may be taken *after* the action of the U.S.E.S. Presumably an appeal could be taken at this time, but it is implied in LBM 115-B that the proper time for an appeal is immediately (within 10 days) after the local board's original I-A classification.

(6) In case the procedures here listed have been followed and misclassification of important personnel nevertheless results, the National Roster should be informed regarding the particulars of the case.

The Secretary of the Society will be interested in learning of the experiences of department chairmen with the new procedure. Such information will give our representatives in Washington and the War Policy Committee a basis for judging the effectiveness of the plan.

J. R. KLINE,
*Secretary*

October 20, 1943

National Headquarters

SELECTIVE SERVICE SYSTEM

Washington, D. C.

# LOCAL BOARD MEMORANDUM NO. 115-B

ISSUED: 9/1/43

SUBJECT: REGISTRANTS IN CRITICAL OCCUPATIONS

1. *Importance of critical occupations.*—Attached to Local Board Memorandum No. 115, amended August 16, 1943, is a List of Critical Occupations. Only 149 occupations are included on the list, and it is estimated that there are not more than 400,000 registrants of military age throughout the country who have the skills required of the critical occupations. This number is exceedingly small in comparison to the total working force in industry. It can be safely said that the critical occupations as listed by the War Manpower Commission represent the highest levels of skills within industry and that, therefore, all production necessary to the war effort is directly dependent upon the most efficient use within industry of persons qualified in these occupations.

2. *Need for extending every consideration for occupational deferment to registrants in critical occupations.*—It is of the utmost importance that registrants (1) who have the necessary qualifications, (2) who are utilizing them to the fullest extent in a critical occupation in war production or in support of the war effort, and (3) whose removal from their present employment would have an adverse effect upon the maintenance of required production schedules, be given the most serious consideration for extended occupational deferment before being reclassified out of a deferred classification into a class available for service.

3. *Referral to United States Employment Service.*—(a) Local boards are directed that whenever, after the most careful consideration, they conclude a registrant who is alleged to be in a critical occupation in war production or in support of the war effort is not entitled to occupational deferment on the basis of his present employment, to refer the registrant's name and present occupation, together with a statement of his skills and qualifications and place of present employment, to the local office of the United States Employment Service in the area in which the local board is located.

(b) This reference is mandatory and will be accomplished—

(1) If no appeal is taken, upon the expiration of the period afforded the registrant to take an appeal following classification of the registrant in Class I-A, or

(2) If an appeal is taken, upon the return of the file to the local board with the decision that the registrant is continued in Class I-A on appeal.

(c) In all such cases, local boards will delay the issuance of an Order to Report for Induction to such registrant for 30 days from the date of referral to the United States Employment Service.

(d) If, during such 30-day period, the United States Employment Service certifies to the local board that—

(1) The registrant possesses and is fully employing the qualifications required of a critical occupation in war production or in support of the war effort and his removal from his present employment would adversely affect the maintenance of his employer's required production, or

(2) The United States Employment Service has succeeded in placing the registrant in a critical occupation in war production or in support of the war effort with another employer who requires the registrant's skills and qualifications and will fully utilize such skills and qualifications,

the local board will delay the issuance of an Order to Report for Induction to such registrant for a further period of 10 days from the receipt of such certification so as to permit the filing of a new Form 42A requesting the occupational deferment of the registrant.

(e) In the event that either a Form 42A is filed by a new employer with whom the registrant has been placed or certification is received from the United States Employment Service stating that the registrant is necessary in his present employment, the local board will reopen the classification of the registrant and will consider the new evidence as a basis for further occupational deferment.

(f) If, however, no notification is received from the United States Employment Service prior to the expiration of the original 30-day period following referral or, in the event certification is made, that the registrant is needed in new employment but a Form 42A is not received before the expiration of the 10-day period allowed for the filing of a new claim for deferment, the local board may proceed with the induction of the registrant.

4. *Cooperation by United States Employment Service.*—The War Manpower Commission is informing the United States Employment Service of the need for expediting investigation of the cases of any registrants qualified for critical occupations which are referred to it by local boards. Even though a registrant is not employed in the area in which his local board is located, reference by the local board will, nevertheless, be to the local United States Employment Service office in the local board area, and any further reference of the case to the United States Employment Service office in the area in which the registrant is employed will be undertaken and accomplished by the United States Employment Service.

5. *Registrants between ages of 18 and 25 in critical occupations.*—Men who are utilizing their full skills in critical occupations when the necessity for their continued retention is clearly demonstrated by their employer should be considered as registrants who are utilizing the highest skills or professional qualifications, which is the exception to the general policy expressed in Local Board Memorandum No. 158-A, requiring the release of registrants between the ages of 18 and 25 during the first 6 months of a replacement schedule. In the lower age brackets, however, employers should be required to supply complete evidence that the registrant's qualifications and skills are exceptional, are being fully utilized, and cannot be replaced.

LEWIS B. HERSHEY,
*Director*

## DEFERMENT OF A.S.T.P. TEACHERS

The following is a copy of an undated notice sent through Service Commands to presidents of institutions participating in or scheduled to participate in the Army Specialized Training Program. It is not known as yet whether the instructions given therein are modified or altered by the provisions of LBM 115-B.

## NOTICE CONCERNING DEFERMENT OF ASTP TEACHERS

### (Sent through Service Commands)

To Presidents of Institutions Participating in or Scheduled to Participate in the Army Specialized Training Program:

It is of the utmost importance that qualified and irreplaceable teachers who are at institutions under contract to the Army Specialized Training Program and who are providing instruction under the program be permitted to continue this work.

The requirements of the Army Specialized Training Program indicate that very serious disruption is probable if the responsible officials of these institutions do not sufficiently emphasize to Selective Service Authorities the necessity for the occupational deferment of these teachers.

As a first step, it is requested that all teachers in institutions under contract to the Army Specialized Training Program register their qualifications with the National Roster of Scientific and Specialized Personnel, Washington, D. C. The National Roster is concerned with the proper utilization of professional personnel and is an important source of information in this matter to the Selective Service System.

The Selective Service Form 42-A, completely filled out, should be used for individual teachers for whom deferment is considered necessary, to furnish the appropriate local board with specific information reflecting the considered judgment of the institution with respect to qualifications, degree of training, and experience required to engage in the profession; and to inform the board either (1) that no replacement can be obtained and that replacement training time is too great to afford relief, or (2) that no replacement can be obtained immediately, in which case the time required for replacement should be indicated. Special attention should be devoted to furnishing complete information concerning the capacities and duties of the individual teacher. In cases where teachers can be replaced, it should be emphasized that the replacement should not break into the term during which the incumbent teacher is giving instruction.

It is essential that all possible means of procuring replacement be employed, including the transfer of non-vulnerable teachers who are engaged in civilian training, the conversion of teachers from cognate fields or from administrative duties, the recruitment of teachers not now engaged in essential activities, and in general the maximum practical use of existing teaching personnel.

Where deferment is not obtained at the local board level, appeal should be made within ten days to the area or district appeals board and if necessary the case should be called to the attention of the State Director with a request for a Presidential appeal.

You are requested to adhere strictly to all Selective Service procedures regarding occupational deferment. You are also requested to advise the Military Commandant of your institution immediately when any irreplaceable teacher is classified by Selective Service authorities as available for military service.

In the event that the Military Commandant at the institution considers the deferment of the teacher to be essential to the success of the Army Specialized Training Program, representations to that effect may be made by him to the State Director of the Selective Service System. Such representations are not to be made, however, unless all appeal procedure provided under the Selective Service Act is being pursued and unless in fact the particular teacher is irreplaceable in the institution.

In the event that the deferment is denied after all appeal steps through the state level have been complied with, the institution should immediately transmit the matter to the Labor Branch, Industrial Personnel Division, Headquarters, Army Service Forces, for representations at the Washington level.

In the case of the irreplaceable teacher whose induction into the Army has taken place, causing serious jeopardy to the Army Specialized Training Program, institutions may submit a full statement of the facts to the Labor Branch, Industrial Personnel Division, Headquarters, Army Service Forces, Washington, D. C., for consideration of methods for remedial action.

# COMMITTEE ON AVAILABLE TEACHERS OF COLLEGIATE MATHEMATICS

The Committee on Available Teachers of Collegiate Mathematics, established by the War Policy Committee of the American Mathematical Society and the Mathematical Association of America, has been in existence since the beginning of April, 1943. During this time it has received and answered numerous inquiries from colleges and universities needing teachers of mathematics, as well as from teachers who were free to accept appointments.

A total of 149 persons have registered with the Committee. Of this number, 129 have at least the master's degree. The Committee has answered requests for teachers of mathematics from 67 different institutions. To these institutions a total of 348 names, representing 111 different persons, have been suggested. According to the Committee's records, 57 of the persons registered are no longer available. It should be pointed out that of the remaining candidates some have indicated that they are available only for summer or part-time teaching, others have had only secondary school teaching experience, while others are available for appointment in restricted geographical areas.

It is anticipated that the demand for teachers will increase considerably during the next two or three months. On the other hand, the number of available well-qualified candidates who have registered with the Committee for such appointments has been reduced to such an extent that the remaining supply has become quite inadequate to meet the expected demand.

For this reason, the committee requests that (a) individual teachers report their availability and (b) departments of mathematics inform the Committee at the earliest possible date of their needs during the next half year, giving as full details as they can concerning the qualifications expected, the salary offered and other pertinent facts relating to their vacancies, and of members who are free to fill temporary positions in other institutions.

The effectiveness of the work of this Committee will be enhanced if registrants will report positions which they have accepted and if inquiring institutions will report appointments that have been made.

Committee on Available Teachers,
110 Bennett Hall, University of Pennsylvania
W. D. Cairns
Arnold Dresden
J. R. Kline

September 2, 1943.

# COMPLEX METHODS IN THE THEORY OF FOURIER SERIES

A. ZYGMUND

**1. General remarks.** Two new ideas which greatly influenced the theory of Fourier series in this century are the Lebesgue integral and the applications of complex functions. The original impetus due to the discoveries of Lebesgue would have been spent long ago, but for the fact that its combination with complex methods opened entirely new prospects for trigonometric series.

The essential tool of Lebesgue theory is the fact that the integral is differentiable almost everywhere and that the derivative is equal to the integrand almost always. Most of the fundamental results of the theory of trigonometric series which were based on that fact had been known, roughly, before 1920. Although some important results have been discovered since then, the progress of purely real methods in the last twenty odd years has been relatively slow and limited to isolated problems. It seems quite likely that the structure of real functions must be investigated in more detail before purely real methods can resume their progress. On the other hand, it seems that the complex variable approach to many problems of the theory is the most natural one and may even be of considerable help in the analysis of the structure of real functions.

Every trigonometric series

$$\tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) \tag{1}$$

is the real part of the power series

$$\tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu - ib_\nu)z^\nu \tag{2}$$

on the unit circle $z=e^{i\theta}$. The imaginary part of the series (2) for $z=e^{i\theta}$ is the series

$$\sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta) \tag{3}$$

and is called the *conjugate* of (1).

Similarly, the harmonic function

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An address delivered before the New Brunswick meeting of the Society on September 12, 1943; received by the editors September 12, 1943.

$$u(r, \theta) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)r^\nu \tag{4}$$

associated with the series (1) is the real part of the analytic function $\phi(z)$, $z = re^{i\theta}$, defined by the series (2). The harmonic function

$$v(r, \theta) = \sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta)r^\nu \tag{5}$$

associated with the series (3) is conjugate to the function $u(r, \theta)$ and is the imaginary part of the function $\phi(z)$.

Thus the problems of trigonometric series may be treated as problems (boundary value problems) of the theory of analytic functions. By complex methods in the theory of trigonometric series we however mean something more special, namely the application of the methods of analytic functions and in particular of the fact that the latter form a field. Elementary operations performed on analytic functions lead to analytic functions, as does also the operation of taking a function of a function. Nothing like that holds for harmonic functions, since even the square of a harmonic function need not be harmonic. Thus dealing directly with analytic functions instead of with their real parts gives obvious advantage.

The complex methods in trigonometric series have been systematically developed in the last quarter century, although some isolated applications can be traced back to an earlier period. Roughly speaking, in the development of complex methods we may discern three major trends:

(a) The method of the classes $H^p$,
(b) The method of conformal representation,
(c) The Littlewood-Paley method,

and it is the purpose of this talk to say a few words about each of these methods. It goes without saying that in a talk like this the presentation may be only very sketchy and must be limited to a discussion of a few particular results.

2. **Classes $H^p$ of analytic functions.** One of the important problems of the theory of trigonometric series is to establish conditions under which a given trigonometric series (1) is a Fourier series. In other words: when is there an integrable function $f(x)$ such that the coefficients $a_n$, $b_n$ are given by the familiar formulas

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx dx? \tag{6}$$

In the theory of Fourier series we often consider besides the most general integrable functions (by "integrable" we always mean $L$-integrable) classes of more special functions, for example, continuous, bounded, of the Lebesgue class $L^p$, $p \geqq 1$, and so on, and we may ask, in addition, under what condition does $f$ belong to one of those classes. The Fourier character of the series (1) may be easily detected by means of the harmonic function (4) associated with the series. For functions of the class $L^p$, $p > 1$, we have a very simple test: a necessary and sufficient condition that (1) is the Fourier series of a function $f$ of the class $L^p$ is that the integral

$$\int_0^{2\pi} |u(r, \theta)|^p d\theta \tag{7}$$

be bounded for $0 \leqq r < 1$. For $p = 1$ this result is no longer true: a necessary and sufficient condition for the boundedness of the integral

$$\int_0^{2\pi} |u(r, \theta)| d\theta \tag{8}$$

is that there exist a function $F(x)$, $0 \leqq x \leqq 2\pi$, of bounded variation and such that

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos nx dF(x), \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} \sin nx dF(x). \tag{9}$$

Of course, if $F(x)$ is absolutely continuous and $F'(x) = f(x)$, these formulas reduce to (6), but in general it is not true. The series (1) with coefficients (9) is called a *Fourier-Stieltjes series*. The series (1) is a Fourier-Stieltjes series if and only if the integral (8) is bounded for $0 \leqq r < 1$. A necessary and sufficient condition that (1) should be an ordinary Fourier series is slightly more complicated: it is

$$\lim_{r, r' \to 1} \int_0^{2\pi} |u(r, \theta) - u(r', \theta)| d\theta = 0. \tag{10}$$

(For the proofs of all these results see, for example, Evans [2][1] or Zygmund [33]; in the sequel, the latter book will be quoted TS.)

Let us now consider any function

$$\phi(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu \tag{11}$$

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[1] Numbers in brackets refer to the references listed at the end of the paper.

regular for $|z| < 1$. If the integral

$$\int_0^{2\pi} |\phi(re^{i\theta})|^p d\theta \tag{12}$$

(analogous to (7)) is bounded for $0 \leqq r < 1$, the function $\phi(z)$ is said to belong to the class $H^p$ ($H$ stands for Hardy); $p$ is here any positive number. The case $p=2$ is of special interest since in this case the integral (12) is easily expressible in terms of the coefficients $c_\nu$ by means of the Parseval formula

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi(re^{i\theta})|^2 d\theta = \sum_{\nu=0}^{\infty} |c_\nu|^2 r^{2\nu}.$$

Hence the function $\phi(z)$ belongs to the class $H^2$ if and only if the series $\sum |c_\nu|^2$ converges. No result of this kind holds for $p \neq 2$, and this makes $H^2$ a central class ("central" in more than one sense), whose properties are the easiest to study. The fact that if $\sum |c_\nu|^2$ is finite then the series

$$\sum_{\nu=0}^{\infty} c_\nu e^{i\nu\theta}$$

is the Fourier series (with respect to the system $\{e^{i\nu\theta}\}$) of a function of the class $L^2$ is the classical Riesz-Fischer theorem.

Of course, given any function $\phi(z)$ regular in $|z| < 1$ and of the class $H^p$ we might set

$$\phi^p(z) = \psi^2(z) \tag{13}$$

so that the boundedness of the integral (13) is equivalent to that of

$$\int_0^{2\pi} |\psi(re^{i\theta})|^2 d\theta,$$

that is to say to the fact that $\psi$ is of the class $H^2$, but the formula (13) defines a function $\psi$ regular in $|z| < 1$ only if $\phi$ has no zeros there. If $\phi(z)$ does have zeros, the argument has to be modified slightly, and following F. Riesz [26] (TS, [6]) we may proceed as follows. Let $z_1, z_2, \cdots$ ($|z_\nu| < 1$) be the zeros (counted according to their multiplicity) of the function $\phi(z) \not\equiv 0$, of the class $H^p$. It may be shown that the product

$$\prod_{\nu=1}^{\infty} |z_\nu| \tag{14}$$

converges. In other words, the sum $\sum(1-|z_\nu|)$ is finite. Conversely, given any sequence, finite or infinite, of numbers $z_\nu$, $|z_\nu|<1$, such that the product (14) converges, there is a function $B(z)$, regular and bounded in $|z|<1$ (and so, in particular belonging to every class $H^p$), having zeros at the points $z_\nu$ and only there. If, for example, the points $z_\nu$ are all different from the origin, the function $B(z)$ may be defined as the product ("Blaschke product")

$$B(z) = \prod_\nu \frac{z - z_\nu}{z - z_\nu^*}\cdot\frac{1}{|z_\nu|} \tag{15}$$

where $z_\nu^* = 1/\bar{z}_\nu$ is the point conjugate to $z_\nu$ with respect to the circumference $|z|=1$. (If $\phi$ has a zero of order $k$ at the origin, we have to insert the factor $z^k$ on the right of (15).) It may be easily shown that $|B(z)|<1$ for $|z|<1$. Thus the function $\phi(z)/B(z)=\psi(z)$ is regular for $|z|<1$, does not vanish there, and we have the decomposition

$$\phi(z) = B(z)\psi(z).$$

If $\phi(z)$ has no zeros we set $B(z)\equiv 1$. It is an important fact that the function $\psi$ also belongs to $H^p$ (more precisely, if for the function $\phi$ the integral (12) does not exceed a constant $M$ for all $r<1$, the function $\psi$ has the same property). Since we may write

$$\phi(z) = \psi(z) + (B(z) - 1)\psi(z) = \psi(z) + \psi_1(z)$$

say, and since $B(z)-1$ does not vanish in $|z|<1$ and is absolutely less than 2 there, we get the following decomposition theorem: *every function of the class $H^p$ may be represented as a sum of two functions of the class $H^p$ which have no zeros in* $|z|<1$. Since the functions of the class $H^p$ and without zeros are reducible to functions of the class $H^2$, whose properties are particularly simple, those properties may be extended to classes $H^p$. In particular, we get the following fundamental theorem (in which by a non-tangential path we mean any continuous curve approaching a point $z_0$, $|z_0|=1$, from inside the unit circle and contained between two chords through $z_0$ of that circle).

*Suppose that $\phi(z)$ is of the class $H^p$. Then for almost every point $e^{i\theta}$ on* $|z|=1$,

$$\lim_{z\to e^{i\theta}} \phi(z) \tag{16}$$

*exists and is finite provided that $z$ approaches $e^{i\theta}$ along any non-tangential curve. Moreover, if $\phi(e^{i\theta})$ denotes the limit* (16),

(17) $$\int_0^{2\pi} |\phi(re^{i\theta}) - \phi(e^{i\theta})|^p d\theta \to 0,$$

(18) $$\int_0^{2\pi} |\phi(re^{i\theta}) - \phi(r'e^{i\theta})|^p d\theta \to 0;$$

*as $r$ and $r'$ tend to* 1.

It must be added that for $p>1$ this result had been known before the decomposition theorem was proved (for then, by what was said before, the series $\sum c_\nu e^{i\nu\theta}$ is a Fourier series, and so may be studied directly by familiar methods), but in the case $0<p\leqq 1$ it brings to light some new facts.

The most important case here is that of $p=1$. Suppose that $\phi(z)$ belongs to $H$ (that is $H^1$), say

(19) $$\int_0^{2\pi} |\phi(re^{i\theta})| d\theta < M$$

for $0\leqq r<1$. The function $\phi(z)$ is a (complex-valued) harmonic function. Thus (by what was said before) the series

(20) $$\sum_{\nu=0}^{\infty} c_\nu e^{i\nu\theta} = \sum_{\nu=0}^{\infty} c_\nu(\cos \nu\theta + i \sin \nu\theta)$$

is a Fourier-Stieltjes series. On the other hand, since (19) implies (18) with $p=1$, we see that the series (20) is an ordinary Fourier series. Thus it turns out that for the trigonometric series (20) which are generated by power series on the circle of convergence, the distinction between ordinary Fourier series and Fourier-Stieltjes series disappears. If we take into account the familiar fact [TS, 16] that Fourier-Stieltjes series are obtained by differentiating formally Fourier series of functions of bounded variation we obtain the following result.

*If the series*

(21) $$\sum_{\nu=0}^{\infty} C_\nu e^{i\nu\theta}$$

*is the Fourier series of a function of bounded variation, this function must be absolutely continuous. In particular, if $\Phi(z)$ is regular in $|z|<1$, continuous in $|z|\leqq 1$, and if $\Phi(e^{i\theta})$ is of bounded variation, then $\Phi(e^{i\theta})$ is absolutely continuous* (F. and M. Riesz [27]).

This fact is of great importance for the theory of conformal mapping. Another result which may be obtained by means of the decomposition theorem, and which shows the difference that exists between ordinary trigonometric series and the series (20), is the following theorem.

*If* (21) *is the Fourier series of a function of bounded variation the series* $\sum|C_\nu|$ *converges* (Hardy and Littlewood [5]).

There exist such simple proofs of this result that one of them may be reproduced here (See Hardy, Littlewood and Pólya [7]). The problem reduces to showing that if the function $\phi(z)=\sum c_\nu z^\nu$ is of the class $H$, the series $\sum|c_\nu|/(\nu+1)$ converges. We may assume that $\phi(z)$ has no zeros in $|z|<1$, so that $\phi=\psi^2$ where $\psi(z)=\sum d_\nu z^\nu$ is of the class $H^2$. If we use the fact that $\sum|d_\nu|^2<+\infty$, the convergence of $\sum|c_\nu|(\nu+1)^{-1}$ follows from the fact that

$$\begin{aligned}\sum_{\nu=0}^{\infty}(\nu+1)^{-1}|c_\nu| &= \sum_{\nu=0}^{\infty}(\nu+1)^{-1}\Big|\sum_{k+l=\nu} d_k d_l\Big| \\ &\leqq \sum_{\nu=0}^{\infty}(\nu+1)^{-1}\sum_{k+l=\nu}|d_k d_l| \\ &= \sum_{k,l=0}^{\infty}|d_k||d_l|/(k+l+1)\end{aligned}$$

and from the familiar theorem of Hilbert asserting the convergence of the series $\sum x_k x_l/(k+l+1)$ if $\sum x_k^2$ is finite.

The following theorem due to Hardy and Littlewood [6; TS, p. 249] throws additional light on the behavior of the functions of the class $H^p$.

*Suppose that* $\phi(z)$ *is of the class* $H^p$, *and let*

$$\phi^*(\theta) = \sup_{0\leqq r<1}|\phi(re^{i\theta})|.$$

*Then* $\phi^*(\theta)$ *is of the class* $L^p$, *and*

$$\int_0^{2\pi}\{\phi^*(\theta)\}^p d\theta \leqq C_p\int_0^{2\pi}|\phi(e^{i\theta})|^p d\theta,$$

*where* $C_p$ *depends on* $p$ *only.*

Hence every function of the class $H^p$ has a majorant which depends on the argument $\theta$ only and which is of the class $L^p$. An analogous result holds for harmonic functions, and is an important tool in many problems of the theory of Fourier series. A result similar to the theorem just stated holds if instead of $\phi^*(\theta)$ we consider the upper bound of $\phi(z)$ in a sector with vertex at the point $e^{i\theta}$ and directed to the interior of the unit circle.

Let $\phi(z)=u+iv$ be a function regular for $|z|<1$. It is natural to ask for the relation between the behavior of the two integrals

$$\int_0^{2\pi} |u(re^{i\theta})|^p d\theta, \qquad \int_0^{2\pi} |v(re^{i\theta})|^p d\theta.$$

The fundamental theorem of M. Riesz [28; TS, p. 147] asserts that, *if $p>1$, these two integrals are simultaneously bounded or simultaneously unbounded*. Thus, *the series conjugate to the Fourier series of a function of the class $L^p$ is also the Fourier series of a function of the class $L^p$*. The result is not true if $p=1$, that is to say for functions merely integrable.

How the method of the complex variable may be applied to problems of real functions is illustrated by the following example which is by now quite familiar, but deserves mention here. Let $f(\theta)$ be a function of period $2\pi$, and let (3) be the series conjugate to the Fourier series of $f(\theta)$. If $f(\theta)$ is sufficiently regular, for example if it has a continuous derivative, or even if $f(\theta)$ only satisfies a Lipschitz condition of positive order, the series (3) converges to the sum

$$\tilde{f}(\theta) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta+t)\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt = -\frac{1}{\pi}\lim_{\epsilon\to+0}\left(\int_{-\pi}^{-\epsilon}+\int_{\epsilon}^{\pi}\right),$$

which is called the function *conjugate* to $f(\theta)$. Under the conditions imposed on $f$, the integral converges, even absolutely. The problem is whether this integral converges, at least almost everywhere, for the most general function $f$ integrable $L$? That this is so for $f$ continuous was shown already by Fatou [3] and even that rather special result is far from obvious. For the most general integrable $f$ it is easy to show that almost everywhere the existence of $\tilde{f}(\theta)$ is equivalent to the existence of the radial limit of the function $\tilde{f}(r, \theta)$ conjugate to the Poisson integral of $f$. If we use rather elementary facts from the theory of the complex variable, we may easily show that the radial limit of $\tilde{f}(r, \theta)$ exists almost everywhere, and this proves that the integral $\tilde{f}(\theta)$ exists almost everywhere (Privaloff [24], Plessner [19]; TS, p. 145).

Although there exist purely real proofs of the existence of $\tilde{f}(\theta)$ (Besicovitch [1], Titchmarsh [30], Marcinkiewicz [14]), the one sketched above is the simplest, and seems to lead to the roots of the matter.

In the case that $f(\theta)$ is the characteristic function of a measurable set $E$, the existence of the function $\tilde{f}(\theta)$ has a certain geometric significance. It shows that in the neighborhood of almost every point of

$E$ that set has a certain symmetry of structure (for otherwise the integral defining $\tilde{f}(\theta)$ would be divergent). This property is not easily deducible from the familiar properties of measurable sets (for example, from the theorem on the points of density), for otherwise we should have a simple real function proof of the existence of $\tilde{f}(\theta)$.

One might argue that the study of the conjugate functions is strictly speaking outside the scope of the theory of Fourier series. But this is not so, and there seems to be a close relation between the behavior of the partial sums $S_n(\theta)$ of a Fourier series and certain conjugate functions. Let us consider instead of $S_n(\theta)$ the modified partial sums $S_n^*(\theta)$ differing from $S_n(\theta)$ in that only half of the last term is taken. Hence

$$S_n^*(\theta) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{n-1} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) + \tfrac{1}{2}(a_n \cos n\theta + b_n \sin n\theta).$$

For $S_n^*(\theta)$ we have a formula similar to the classical Dirichlet formula,

$$S_n^*(\theta) \doteq \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta + t)\frac{\sin nt}{2\tan(t/2)}\,dt$$

(TS, p. 21) and it may be written formally

$$\begin{aligned} S_n^*(\theta) &= \frac{\cos n\theta}{\pi}\int_{-\pi}^{\pi} f(\theta + t)\sin n(\theta + t)\tfrac{1}{2}\cot(t/2) \\ &\quad - \frac{\sin n\theta}{\pi}\int_{-\pi}^{\pi} f(\theta + t)\cos n(\theta + t)\tfrac{1}{2}\cot(t/2)\,dt \\ &= -\,g_n(\theta)\cos n\theta + h_n(\theta)\sin n\theta, \end{aligned}$$

say, where $g_n(\theta)$ and $h_n(\theta)$ denote functions conjugate to $f(\theta)\sin n\theta$ and $f(\theta)\cos n\theta$. From this formula and from the slightly strengthened form of the theorem of M. Riesz just stated, the following fact (also due to M. Riesz) follows easily: For every function $f(\theta)$ of the class $L^p$, $p>1$,

$$\int_0^{2\pi} |f(\theta) - S_n(\theta)|^p d\theta \to 0 \qquad \text{as } n \to +\infty.$$

It is a curious fact that in order to prove this result from the theory of Fourier series (in the narrow sense of the word) we have to use properties of conjugate functions. No other proof seems to have been discovered so far.

The theory of the functions $H^p$ has an analogue for functions regular in a half-plane. The latter theory (developed mainly by Hille and Tamarkin [8]) has its applications mostly in the domain of Fourier integrals, and for this reason we omit its discussion here. Also to the domain of Fourier integrals belong the complex methods developed by Paley and Wiener [18].

3. **The method of conformal representation.** We first recall familiar facts. Let $\Delta$ be any domain in the $\zeta$-plane limited, say, by a simple Jordan curve $\Gamma$. There is a function $\zeta=h(z)$ defined and regular in the unit circle

$$(D) \qquad |z|<1$$

and mapping $D$ conformally onto $\Delta$. The function $h(z)$ may be extended continuously to the closed domain $D+C$, where $C$ is the circumference $|z|=1$, and gives a one-one correspondence between $D+C$ and $\Delta+\Gamma$. If we add some normalizing conditions, the function $h(z)$ is unique.

Let us assume from now on that the curve $\Gamma$ is rectifiable. For point sets situated on a rectifiable curve we have, of course, a theory of measure analogous to that of Lebesgue. In particular, we may speak of sets of measure 0, of "almost everywhere," and so on.

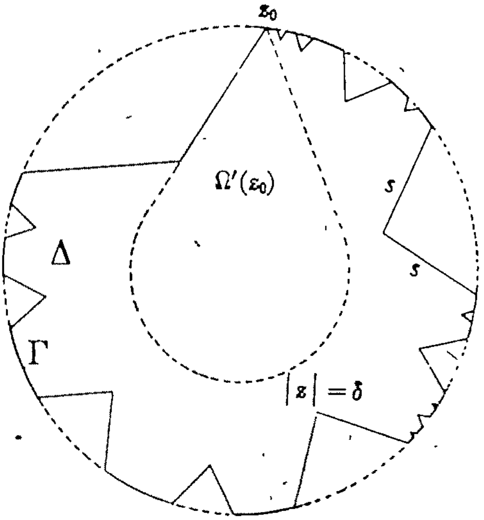
In our case, since $\Gamma$ is of finite length, the function $\phi(z)$ must be of bounded variation on the circumference $C$. This means, as we know, that $\phi(z)$ must be absolutely continuous on $C$. In particular, it transforms every set of measure 0 on $C$ into a set of measure 0 on $\Gamma$. The converse is also true, the sets of measure 0 on $\Gamma$ correspond to sets of measure 0 on $C$. In other words, in our case, the sets of measure 0 on the boundaries are invariants of conformal mapping. This result was first proved by F. and M. Riesz [27].

There is another property, obtained by Privaloff [24], of the mapping function $h(z)$: the mapping is conformal at almost every point of the boundary. This is to be understood as follows. If we exclude a certain point set of measure 0 on $C$, each of the remaining points $z_0$ has the property that if $C'$ and $C''$ are any two paths leading from the interior of $D$ to the point $z_0$ and making angle $\alpha$, the images $\Gamma'$, $\Gamma''$ of $C'$, $C''$ make angle $\alpha$ at the point $\zeta_0=\phi(z_0)$.

In order to see how the above results can be applied, let us sketch the proof of a theorem of Privaloff [23]. Suppose that a function $\phi(z)$ $=u+iv$ is regular in $|z|<1$, and that there is a point set $E$ on the circumference $|z|=1$ with the following property: for every $z_0=e^{i\theta_0}$ belonging to $E$ there is a circular sector $\Omega(z_0)$ with vertex at $z_0$, lying except for $z_0$ entirely inside $C$, and such that the real part $u(z)$ of

$\phi(z)$ is bounded in $\Omega(z_0)$. Then, *the function* $\phi(z)$ *has a nontangential limit at almost every point* $z_0$ *of* $E$.

A special case of this result worth a separate statement is: *if the real part* $u(z)$ *of a regular function* $\phi(z)$, $|z|<1$, *has a nontangential limit at every point* $z_0$ *of a set* $E$ *of the circumference* $|z|=1$, *the imaginary part* $v(z)$ *has the same property at almost every point of* $E$.



Let us assume, to fix the ideas, that $\Omega(z_0)$ is symmetrical with respect to the radius $(0, z_0)$. Without loss of generality we may also assume that the function $u(z)$ is bounded *uniformly* in all the $\Omega(z_0)$, $z_0 \in E$. For the upper bound $M(z_0)$ of $u$ in $\Omega(z_0)$ is a function of $z_0$ and is finite on $E$, so it follows that if we reject from $E$ a subset of arbitrarily small measure, $M(z_0)$ will be bounded on the rest of $E$. By a similar argument we may assume that the angles of the sectors $\Omega(z_0)$ are the same, or even that all these sectors are congruent. Let $|z|=\delta$ be the circle tangent to the rectilinear sides (or their continuations) of the sectors $\Omega(z_0)$. Without loss of generality, we may replace each $\Omega(z_0)$ by the domain $\Omega'(z_0)$ limited by the two tangents from $z_0$ to $|z|=\delta$, and by the more distant arc of the circle. One more remark: we may assume that $E$ is a perfect set.

Let now $\Delta$ denote the sum of all the domains $\Omega'(z_0)$ for $z_0 \in E$; $\Delta$ has a starlike shape (see the figure). It is connected. The points of its frontier $\Gamma$, which is a simple Jordan curve, may belong either (a) to the set $E$, (b) to a number of arcs (nonexistent on the figure) of the circle $|z| = \delta$, (c) to the denumerable set of segments $s$ belonging to the boundaries of some $\Omega'(z_0)$. Since the regions $\Omega'(z_0)$ are all congruent, it is clear that the total length of the segments $s$ does not exceed a fixed multiple (depending only on the shape of the $\Omega$'s) of the total length of the intervals contiguous to $E$. Thus the curve $\Gamma$ is rectifiable.

The function $u(z)$ is bounded in $\Delta$. Hence, if $z=h(w)$ maps $\Delta$ conformally onto the unit circle $|w| < 1$, the function

$$u(z) = u(h(w)) = u_1(w)$$

is harmonic in the circle $|w| < 1$. It is the real part of the function

$$\phi(h(w)) = \phi_1(w)$$

regular in $|w| < 1$. The real part of $\phi_1(w)$ being bounded, the latter function is of the class $H^2$, and so has a nontangential limit at almost every point of the circumference $|w| = 1$. If we go back from $\phi_1(w)$ to the function $\phi(z)$, and take into account the results of F. and M. Riesz and of Privaloff mentioned above, we see that $\phi(z)$ has a nontangential limit at almost every point of $\Gamma$, in particular almost everywhere in $E$. This completes the proof.

Without changing the idea of the proof we may generalize this result considerably (see Plessner [21]). We prefer, however, to give a different application.

Let $\phi(z)$ be a function regular in $|z| < 1$ and of the class $H^2$, and let $\Omega'(z) = \Omega'(\theta)$ have the same meaning as before. Lusin proved that then the integral

$$I(\theta) = \iint_{\Omega'(\theta)} |\phi'(z)|^2 d\omega \tag{22}$$

($d\omega$ an element of area) is finite for almost every $\theta$. This integral represents the area of the domain (generally non-schlicht) obtained from $\Omega'(\theta)$ by the mapping $w = \phi(z)$. By an argument similar to the above we may show (see Marcinkiewicz and Zygmund [15]) that, *if $\phi(z)$ is any function regular in $|z| < 1$, and if it has a nontangential limit at every point $e^{i\theta}$ of a set $E$ of positive measure, then the integral* (22) *is finite almost everywhere in $E$.* Recently, Spencer [29] proved a converse of the above result, namely that *the finiteness of the integral $I(\theta)$ in a set of $\theta$ of positive measure implies the existence of the nontangential limit at almost every point of that set.* Thus we see that almost everywhere the existence of the nontangential limit is equivalent to the finiteness of a certain area.

The method of conformal representation, as presented above, seems to have originated with Golubeff [4]. If we are to use it, we must know the behavior of the function in certain sectorial domains from which we build up a new domain whose boundary has "many" points in common with the boundary of the domain in which the function is defined. It would not work if we wanted, for example, to prove that the existence of the radial limit of $u(z)$ along a set of positive measure of radii implies the existence of the radial limit of $v(z)$ along almost every radius of the set, and the problem itself is open ($u$ and $v$ here are the real and the imaginary part of a function regular in $|z| < 1$). In other words, we do not know whether the Abel summability of a trigonometric series does or does not imply (almost everywhere) the Abel summability of the conjugate series.

For some other methods of summability, in particular for ordinary convergence, the problem is solved. A very important step in this direction was first made by Kuttner [12], who showed that *if a Fourier series converges in a set of positive measure, the conjugate series converges almost everywhere in that set.* This result was later on extended to the most general trigonometric series, and to the Cesàro summability. The existing proofs (see Plessner [22], Marcinkiewicz and Zygmund [16, 17]) are quite difficult and are essentially based on complex methods.

The problem of the convergence of power series (or trigonometric series) leads naturally to the problem of the distribution of the partial sums of divergent series. It turns out that under certain conditions the distribution of those partial sums displays a very simple geometric character.

We shall say that a sequence of points $\{s_n\}$ is of *circular structure* with respect to the point $s$, if the derived set of the sequence $\{s_n\}$ consists of a certain number (finite, denumerable, or nondenumerable, but in any event a closed set) of circles with center at the point $s$. If for the sake of simplicity we confine our attention to power series

$$\sum_{\nu=0}^{\infty} c_\nu z^\nu = \phi(z) \tag{23}$$

with bounded coefficients, we have the following result (Marcinkiewicz and Zygmund [17]).

*Suppose that*

$$\lim_{r \to 1} \phi(re^{i\theta}) = s(\theta)$$

*exists for every $\theta$ of a set $E$ of positive measure. Then, for almost every $\theta$ of $E$, the sequence $\{s_\nu(e^{i\theta})\}$ of the partial sums of the series* (23) *is of circular structure with respect to the point $s(\theta)$. In particular, if the coefficients $c_\nu$ tend to* 0, *then for almost every $\theta \in E$ the derived set of the sequence $\{s_\nu(e^{i\theta})\}$ is a circle (finite or infinite) with center at the point $s(\theta)$.*

Quite recently new applications of conformal mapping to the theory of trigonometric series were obtained by F. Wolf. He combined conformal mapping with certain extensions of the Phragmèn-Lindelöf principle. I would like to mention here one of the important results he obtained.

The problem whether a given function can be represented by more than one trigonometric series (not necessarily a Fourier series) may be reduced, by subtracting these series, to the following problem: can a trigonometric series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) \tag{24}$$

which does not vanish identically represent zero? This "problem of uniqueness" of trigonometric series has various aspects. It is a classical fact that in the case of convergent series it admits of a positive solution: if the series (24) converges to 0 at every point, the coefficients $a_\nu$ and $b_\nu$ all vanish. The next step is to consider summable series. The case of Abel summability is of particular importance since it means studying the radial behavior of the harmonic function

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)r^\nu \tag{25}$$

associated with the series (24). Rajchman [25] proved that if the function (25) tends to 0 along every radius and if in addition $|a_\nu| + |b_\nu| \to 0$, the series (24) vanishes identically. Verblunsky [31] replaced the last condition by $|a_\nu| + |b_\nu| = O(\nu)$. If we consider only radial limits, we cannot go much further since there are trigonometric series with coefficients $O(\nu)$, summable $A$ everywhere, and yet not vanishing identically: the series

$$\sum_{\nu=1}^{\infty} \nu \sin \nu\theta$$

is an instance in point. Thus we must impose new restrictions if we wish to obtain positive results. Wolf [32] proved that:

*If* (i) $u(r, \theta)$ *is harmonic for* $r<1$, (ii) *for every* $\theta_0$, $u(r, \theta)\to 0$ *as* $(r, \theta)\to(1, \theta_0)$ *along any nontangential path,* (iii) $|u(r, \theta)| \leq \exp A/(1-r)^m$, *where* $A$ *and* $m$ *are independent of* $r$ *and* $\theta$, *then the series* (24) *vanishes identically.*

4. **The Littlewood-Paley method.** This method (see Littlewood and Paley [13]) cannot be easily described without going into technical details, and by necessity I shall have to be rather brief here. Let me start with some of the results they achieved. Suppose that $f(\theta)$ is an integrable function of period $2\pi$, and let $S_n(\theta)$ denote the $n$th partial sum of the Fourier series of $f(\theta)$. It is a familiar fact that $S_n(\theta)$ may diverge at some points even if $f(\theta)$ is a continuous function. Whether there exists a continuous function $f$ such that $S_n(\theta)$ diverges everywhere, or at least almost everywhere, is still an open problem. For $f$ merely integrable, the divergence of $S_n(\theta)$ may actually occur everywhere, as shown by Kolmogoroff [9; TS, p. 175]. It is a curious fact that so far it has not been possible to construct a similar example for functions of any class $L^p$, $p>1$, and the problem seems to be much more difficult there.

Of course, from a certain point of view the function $f(\theta)$ is adequately represented by the Fejér means of its Fourier series, but the behavior of the partial sums $S_n$ is of considerable intrinsic interest. For functions of the class $L^p$, $p>1$, Paley and Littlewood proved the following result (the special case $p=2$ was solved earlier by Kolmogoroff [10; TS, p. 257]).

*Whatever the sequence of positive integers* $n_1, n_2, \cdots$ *satisfying an inequality*

$$n_{k+1}/n_k > q > 1,$$

*the partial sums* $S_{n_k}(\theta)$ *converge to* $f(\theta)$ *at almost every point.*

The result is false for $p=1$ (although it holds for power series of the class $H$; see Zygmund [34]). Its significance consists in the fact that the sequence $n_k$ is independent of the function $f$.

Another result which Littlewood and Paley proved concerns the *convergence factors* of Fourier series. It is a familiar fact due to Hardy that if (24) is the Fourier series of an integrable function, the series

$$\sum_{\nu=2}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)/\log \nu$$

converges for almost every $\theta$. If $f$ is of integrable square, even the series

$$(26) \qquad \sum_{\nu=2}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)/(\log \nu)^{1/2}$$

converges almost everywhere (Kolmogoroff and Seliverstoff [11], Plessner [20]; TS, p. 252). Littlewood and Paley proved the intermediate result: if $f \in L^p$, $1 < p < 2$, the series

$$\sum_{\nu=2}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)/(\log \nu)^{1/p}$$

converges almost everywhere. Nothing is known about the case $p > 2$. In particular, about convergence factors for continuous functions we know no more than for functions of the class $L^2$, that is to say that the series (26) is convergent almost everywhere. The latter fact is equivalent to saying that the partial sums $S_n(\theta)$ of the Fourier series satisfy the relation

$$(27) \qquad S_n(\theta) = o(\log n)^{1/2}$$

at almost every point. It may be that the estimate (27) is the best result for the Fourier series of continuous functions.

The Littlewood-Paley theory is based on the use of the function $g(\theta)$ defined by the formula

$$g(\theta) = \left( \int_0^1 (1 - r) \left| \phi'(re^{i\theta}) \right|^2 dr \right)^{1/2}$$

where $\phi(z)$ is any function regular in $|z| < 1$. The function $g(\theta)$ has no simple geometric interpretation, but has some connection with the integral $I(\theta)$ (see (22)) whose geometric significance and relation to the existence of the nontangential limit was already mentioned. Let

$$s(\theta) = I^{1/2}(\theta),$$

so that $s(\theta)$ is of dimension 1. It may be shown that $s(\theta)$ is, effectively, a majorant of $g(\theta)$; more precisely, $g(\theta) \leqq C_\alpha s(\theta)$ where $C_\alpha$ depends only on the angle $\alpha$. The results obtained by Littlewood and Paley for the function $g(\theta)$ when interpreted in terms of the function $s(\theta)$ may be stated as follows: *If* $\phi(z) \in H^p$, $p > 1$, *then*

$$\int_0^{2\pi} s^p(\theta) d\theta \leqq A_{p,\alpha} \int_0^{2\pi} \left| \phi(e^{i\theta}) \right|^p d\theta,$$

$$\int_0^{2\pi} \left| \phi(e^{i\theta}) \right|^p d\theta \leqq B_{p,\alpha} \int_0^{2\pi} s^p(\theta) d\theta.$$

Here $A_{p,\alpha}$ and $B_{p,\alpha}$ depend on $p$ and $\alpha$ only; in addition, for the validity of the second inequality we have to assume that $\phi(0)=0$. Thus the integrability of $s(\theta)$ imitates that of $|\phi(e^{i\theta})|$.

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28. M. Riesz, *Sur les séries conjuguées*, Math. Zeit. vol. 27 (1927) pp. 218–244.

29. D. C. Spencer, *A function-theoretic identity*, Amer. J. Math.

30. E. C. Titchmarsh, *On conjugate functions*, Proc. London Math. Soc. vol. 29 (1928) pp. 49–80.

31. S. Verblunsky, *On the theory of trigonometric series*. I, Proc. London Math. Soc. vol. 34 (1932) pp. 441–456.

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Mt. Holyoke College

# THE SUMMER MEETING IN NEW BRUNSWICK

The forty-ninth Summer Meeting of the Society and the twenty-fifth Colloquium were held at New Jersey College for Women, Rutgers University, on Sunday and Monday, September 12–13, 1943. The Mathematical Association of America met on Saturday, September 11, and the Institute of Mathematical Statistics met on Sunday and jointly with the Society on Monday morning. About three hundred fifty persons attended the meetings among whom were the following two hundred forty members of the Society:

C. R. Adams, E. B. Allen, C. B. Allendoerfer, Warren Ambrose, T. W. Anderson, H. E. Arnold, K. J. Arnold, L. A. Aroian, W. L. Ayres, A. V. Baez, F. E. Baker, D. H. Ballou, Valentin Bargmann, J. L. Barnes, P. T. Bateman, B. M. Bennett, Stefan Bergman, Lipman Bers, J. H. Bigelow, Garrett Birkhoff, G. D. Birkhoff, B. H. Bissinger, E. E. Blanche, W. A. Blankinship, E. M. Boggs, T. A. Botts, J. W. Bower, A. H. Bowker, S. E. Brasefield, H. W. Brinkmann, B. H. Brown, C. T. Bumer, L. H. Bunyan, Herbert Busemann, Hobart Bushey, J. H. Bushey, S. S. Cairns, J. W. Calkin, Achille Capecelatro, M. E. Carlen, W. B. Carver, C. R. Cassity, Abraham Charnes, M. D. Clement, A. B. Coble, H. R. Cooley, T. F. Cope, Byron Cosby, Richard Courant, A. P. Cowgill, C. C. Craig, E. L. Crow, H. B. Curry, J. H. Curtiss, E. H. Cutler, J. F. Daly, D. R. Davis, F. F. Decker, C. L. Dolph, Arnold Dresden, R. F. Dressler, William H. Durfee, M. L. Elveback, Benjamin Epstein, Paul Erdös, G. M. Ewing, W. K. Feller, F. G. Fender, F. A. Ficken, W. W. Flexner, Tomlinson Fort, R. M. Foster, F. H. Fowler, J. S. Frame, Hans Fried, Bernard Friedman, K. O. Friedrichs, Orrin Frink, R. E. Fullerton, A. S. Galbraith, G. N. Garrison, R. E. Gaskell, Abe Gelbart, B. P. Gill, S. H. Gould, H. S. Grant, J. W. Green, Lewis Greenwald, C. C. Grove, V. G. Grove, E. J. Gumbel, Theodore Hailperin, D. W. Hall, P. R. Halmos, G. E. Hay, G. A. Hedlund, M. H. Heins, Aaron Herschfeld, Fritz Herzog, T. H. Hildebrandt, Einar Hille, T. R. Hollcroft, Harold Hotelling, S. E. Hotelling, E. M. Hull, Witold Hurewicz, H. D. Huskey, L. C. Hutchinson, R. E. Johnson, B. W. Jones, Edward Kasner, R. B. Kershner, J. R. Kline, E. G. Kogbetliantz, Arthur Korn, W. D. Lambert, K. W. Lamson, H. G. Landau, V. V. Latshaw, Solomon Lefschetz, Joseph Lehner, Marguerite Lehr, D. C. Lewis, Marie Litzinger, J. N. B. Livingood, Simon Lopata, E. R. Lorch, R. T. Luginbuhl, P. H. McGrath, E. J. McShane, C. C. MacDuffee, G. W. Mackey, Saunders MacLane, H. F. MacNeish, W. G. Madow, M. P. Martin, W. T. Martin, A. E. Meder, F. H. Miller, K. S. Miller, L. W. Miller, E. B. Mode, Deane Montgomery, R. K. Morley, Richard Morris, D. S. Morse, E. J. Moulton, F. D. Murnaghan, W. R. Murray, C. A. Nelson, P. B. Norman, C. O. Oakley, L. F. Ollmann, P. S. Olmstead, F. W. Owens, Gordon Pall, C. W. Pflaum, R. S. Phillips, Everett Pitcher, Hillel Poritsky, Willy Prager, M. H. Protter, Hans Rademacher, J. F. Randolph, C. H. Rawlins, G. E. Raynor, O. H. Rechard, C. J. Rees, Mina Rees, C. F. Rehberg, R. G. D. Richardson, D. E. Richmond, C. E. Rickart, J. F. Ritt, H. E. Robbins, M. S. Robertson, R. E. Root, J. H. Rosenbloom, P. C. Rosenbloom, A. E. Ross, J. B. Rosser, S. G. Roth, Raphael Salem, Hans Samelson, F. E. Satterthwaite, R. H. Scanlan, Henry Scheffé, S. A. Schelkunoff, I. J. Schoenberg, Abraham Schwartz, H. M. Schwartz, G. E. Schweigert, Seymour Sherman, Max Shiffman, J. A. Shohat,

D. T. Sigley, L. L. Smail, D. M. Smiley, M. F. Smiley, T. L. Smith, W. M. Smith, Ernst Snapper, Andrew Sobczyk, R. D. Specht, E. P. Starke, H. W. Steinhaus, R. W. Stokes, R. R. Stoll, R. E. Street, R. C. Strodt, W. C. Strodt, Ethel Sutherland, J. L. Synge, Otto Szász, J. D. Tamarkin, J. I. Tracey, C. A. Truesdell, H. F. Tuan, A. W. Tucker, J. W. Tukey, D. F. Votaw, Abraham Wald, G. L. Walker, R. J. Walker, A. D. Wallace, R. M. Walter, G. C. Webber, K. W. Wegner, A. P. Wheeler, P. M. Whitman, Hassler Whitney, G. T. Whyburn, J. E. Wilkins, S. S. Wilks, W. L. G. Williams, Jacob Wolfowitz, Bertram Yood, J. W. T. Youngs, Oscar Zariski, Antoni Zygmund.

The Colloquium Lectures on the subject *Existence theorems in the calculus of variations* were delivered by Professor E. J. McShane of the University of Virginia on Sunday morning and Monday morning and afternoon. Acting President C. C. MacDuffee presided at the first and third lectures and Vice President J. D. Tamarkin at the second.

On Sunday at 2:00 P.M., Professor Antoni Zygmund of Mount Holyoke College gave an address entitled *The complex method of the theory of trigonometric series*. Professor G. D. Birkhoff presided.

At 3:15 P.M., Sunday, Professor F. D. Murnaghan of Johns Hopkins University gave an address entitled *Finite deformations of an elastic solid*. The discussion was led by Professor J. L. Synge of Ohio State University. Dean R. G. D. Richardson presided.

There was a business meeting Monday morning at which the Society voted to suspend for the duration of the war the first sentence of Article VIII, Section 1, of the By-Laws which reads "The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following."

The presiding officers for the sessions of short papers were: Applied Mathematics, Sunday morning, Professor Hobart Bushey; Analysis, Sunday morning, Professor Hassler Whitney; Probability and Statistics (a joint session with the Institute of Mathematical Statistics), Monday morning, Professor C. C. Craig; Algebra, Geometry, and Topology, Monday morning, Professor G. A. Hedlund; Applied Mathematics, Monday afternoon, Professor F. D. Murnaghan; Analysis and Logic, Monday afternoon, Professor J. B. Rosser.

Headquarters for the meeting were in the Little Arcade on Jameson Campus. Rooms in the dormitories were made available to those attending the meetings.

On Saturday afternoon Dean Margaret T. Corwin received the mathematicians and their guests at her home.

A photograph was taken at 1:15 P.M., Sunday.

On Sunday evening there was a dinner for members of the Society, the Association, and the Institute and their guests. Professor Richard Morris was toastmaster. Dean Corwin welcomed the guests on behalf of President R. C. Clothier of Rutgers University. Dean R. G. D. Richardson of Brown University spoke of the importance of applied mathematics in the war effort. Professor G. D. Birkhoff of Harvard University urged mathematicians and scientists to maintain a proper balance of values during the emergency. Professor W. L. Ayres of Purdue University presented a resolution of thanks to the President and Dean of New Jersey College for Women of Rutgers University, the local committee and all who assisted them for their excellent arrangements and cordial hospitality.

Following the dinner, members of the Department of Music of New Jersey College for Women gave an excellent musicale.

The Council met on Sunday, September 12, at 9:00 P.M. in the lounge at Jameson Campus.

The Secretary announced the election of the following twenty-eight persons to membership in the Society:

Professor Ernest Willard Anderson, Iowa State College;
Dr. Melvin Avrami, Columbia University;
Mr. Ralph Herbert Beard, New York Telephone Company;
Mr. J. Oliver Brown, Hampton Institute, Hampton, Va.;
Dr. Wei-Zang Chien, New York, N. Y.;
Mr. Murray Ellis, University of Chicago;
Mr. Eugene Fately, U. S. Army, Camp Murphy, Fla.;
Dr. Casper Goffman, Westinghouse Electric and Manufacturing Company, Pittsburgh, Pa.;
Mr. Clarence Frederic Hall, Rensselaer Polytechnic Institute;
Dr. Robert Griffith Helsel, Ohio State University;
Professor Gerald Boone Huff, Southern Methodist University;
Mr. Solomon Hurwitz, Brooklyn College;
Mrs. Florence Jacobson, University of North Carolina;
Mr. Robert Kates, University of Chicago;
Dr. Ervand G. Kogbetliantz, Lehigh University;
Dr. Zbigniew Lepecki, University of Paraná, Paraná, Brazil;
Mr. Simon Lopata, Columbia University;
Mr. Elbert D. Miller, Yuba Junior College, Marysville, Calif.;
Mr. Kenneth S. Miller, Yonkers, N. Y.;
Dr. Alfonso Gándara Nápoles, Institute of Mathematics, National University of Mexico;
Dr. Paul Felix Nemenyi, University of Colorado;
Dr. Leonard Nicholl, Kay-Fries Chemicals, West Haverstraw, N. Y.;
Mr. Robinson Henry Parson, Houston College for Negroes;
Mr. George Paul Roland, Department of Public Works, Albany, N. Y.;
Dr. Fritz Rothberger, Acadia University;
Dr. Feodor Theilheimer, Trinity College, Hartford, Conn.;
Mr. Milton Albert Treuhaft, Westinghouse Radio Division, Baltimore, Md.;
Professor Wyman Loren Williams, University of South Carolina.

The following appointments by the President were reported: as representative of the Society at the meeting in tribute to Nicholas Copernicus in New York City on May 24, 1943, Professor C. C. MacDuffee; as a Committee on Arrangements for the Annual Meeting in Chicago on November 26–27, 1943, Professors L. R. Ford (Chairman), W. L. Ayres, W. B. Carver, A. S. Householder, R. G. Sanger, and H. S. Wall; as liaison officer of the Society in connection with the editorial management of the Quarterly of Applied Mathematics, Professor Einar Hille; as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings, to replace Professor M. H. Stone during his absence from the country, Professor J. F. Ritt.

It was announced that a report was sent in July to the Carnegie Corporation regarding the grant to Mathematical Reviews. Attention was called to the fact that the number of Mathematical Reviews subscribers for 1943 is 1239. This subscription list is undoubtedly larger than that of any mathematical journal which is not provided by a mathematical society to its members as one of the privileges of membership.

Certain invitations to give invited addresses were announced: Professor Reinhold Baer for the 1943 annual meeting in Chicago; Dr. Alfred Tarski, Professor M. A. Zorn, and Professor Theodore von Kármán for the November, 1943, meeting in Pasadena; Dr. D. H. Hyers and Professor R. M. Robinson for the 1944 spring meeting in Berkeley; Professors J. L. Synge and I. S. Sokolnikoff for the 1944 spring meeting in Chicago.

It was reported that fifteen microfilm reading machines had been sent to the London Mathematical Society for distribution to scientific organizations in Great Britain. It was also reported that a microfilm reading machine and copies of the Semicentennial volumes had been donated to the library of the American Philosophical Society.

On recommendation of the Colloquium Editorial Committee, the Council voted to invite Professor Tibor Radó to deliver a series of Colloquium Lectures in 1945.

Dr. Warren Weaver was appointed representative on the National Research Council for a period of three years beginning July, 1944.

Professor E. J. Moulton and Dean W. G. Simon were reappointed representatives on the Council of the American Association for the Advancement of Science for 1944.

Dean R. G. D. Richardson was appointed as an additional member of the Committee on Addresses in Applied Mathematics. The other members of this committee are Professor Richard Courant (Chairman), Mr. R. M. Foster, and Professor Harold Hotelling.

The Secretary reported the foundation of the Mexican Mathematical Society on June 30, 1943. (The inaugural ceremonies were originally planned for April 9, 1943, as reported on page 517 of the July Bulletin.) Professor Alfonso Nápoles Gándara is the president of the new organization. Greetings were sent to the group on behalf of the American Mathematical Society.

The Council adopted resolutions on the deaths of the following persons who have been prominent in the work of the Society:

### HOWARD HAWKES MITCHELL

The Council of the American Mathematical Society records its deep sense of loss in the death on March 13, 1943, of Howard Hawkes Mitchell, Professor of Mathematics at the University of Pennsylvania.

Professor Mitchell was prominent in all phases of the Society's activity. He was a member of the Council from 1920 to 1922 and Vice President from 1932 to 1933. He was made an Associate Editor of the Transactions of the Society in 1920 and in 1925 was elected to the Editorial Committee of that journal. He served as a member of this Committee with distinction until 1930. His keen insight and excellent mathematical intuition fitted him particularly for the exacting duties of an Editor and, in every way, he maintained the high standards of the Transactions. He was most sympathetic in the handling of the work of younger mathematicians and was a valuable source of inspiration to them for their further research.

He was a kindly gentleman with broad interests and high ideals. In every manner he exemplified the finest in American mathematics and it is with deep emotion that we present this expression of our feelings on his passing.

### WILLIAM FOGG OSGOOD

William Fogg Osgood, thoroughly representative of the mathematicians of his time, was self-disciplined in method, painstaking in detail, rigorous and clear in arriving at his results. For him mathematics was a way of discovery, and each individual function was an object worth knowing in all of its details. The method of strict analysis was a keen tool whose fineness he reverenced, and no labor was too great for him if he could create an end product that was smoothly machined in every part. At a time when scholarship and technique were of the highest repute, he was of the best of artist craftsmen. Some of his ideas were instrumental in forging new concepts. Yet he was innately conservative, because he felt that he had at his command all the tools which he needed in investigating those things which were vital to him.

He had his share in building the Harvard Division of Mathematics to an outstanding world position in teaching and research. But perhaps it is as a teacher of the first course in calculus that most of his pupils remember him. His classes in Mathematics 2 contained men of all ages, from the young freshman or sophomore who was stepping abroad in mathematics for the first time, to the experienced teacher who spent his sabbatical leave in learning how to solve the teaching difficulties which had plagued him. Osgood carried his mathematical straightforwardness fully into his instruction. His aim was to teach ideas, and to make clear the difficult ones, so that they might serve as a basis of further knowledge. The way of mathematics is long and difficult and makes great demands on will and conscience. Osgood wanted no false starts. His purpose sank deep into his students' minds, and they have never ceased to be grateful.

## CLARA ELIZA SMITH

The Council of the American Mathematical Society desires to record its grateful appreciation of the service of Clara Eliza Smith, Professor of Mathematics at Wellesley College, deceased on May 12, 1943. Professor Smith's devotion to the Society was such that in the dozen years beginning with 1922, when it was necessary greatly to increase the membership of the Society in order to find financial support for its publications, she labored faithfully and unobtrusively as a member of the Committee on Membership, doubling the roll of members. Much of the success of that committee was due to her laborious and painstaking assembling of data regarding teachers of college mathematics. On behalf of the Society, the Council wishes to acknowledge her loving interest in its welfare.

## EDWARD BURR VAN VLECK

June 7, 1863–June 2, 1943

The American Mathematical Society notes with sorrow the death of Professor E. B. Van Vleck, whom it honored as an erstwhile active participant and leader in its affairs, and as a noted mathematical scholar and investigator.

Professor Van Vleck's membership in this Society extended over more than a half century, dating from the year 1893, when he was fresh from the attainment of his doctorate. Over the course of many years he was a frequent attendant at the Society's meetings, and marked many of these occasions by the presentations of results of his investigations. The mathematical literature of those years is dotted with papers of his on infinite series, on functions defined by ordinary differential equations, on continued fractions, on point sets, on functional equations, on the roots of polynomials and on other subjects.

In recognition of his productive mathematical work Professor Van Vleck was variously and highly honored by this Society. The posts which were bestowed upon him include those of: Member of the Council, 1902–1904; Colloquium lecturer (The Boston Colloquium), 1903; Editor of the Transactions, 1905–1910; President, 1913–1914. Honors bestowed upon him by other sources include the Doctorate of Laws (Clark University), 1909; Election to the National Academy, 1911; the chairmanship of Section A of the American Association for the Advancement of Science, 1912; the Doctorate of Mathematics and Physics (University of Groningen), 1914; the Doctorate of Science (University of Chicago), 1916; Decoration by the French Government as "Officer de l'instruction publique," 1920; the Doctorate of Laws (Wesleyan University), 1925.

In his person Professor Van Vleck was modest and kindly. He was a sensitive man of broad cultural interests. As a teacher he stood on all occasions and steadfastly for the highest standards of achievement and of scholarly integrity.

The American Mathematical Society is indebted to Professor Van Vleck for his participation and counsel, which came to it during those important years when its membership was to be counted by hundreds rather than the present day's thousands. It gratefully acknowledges this, and mourns his departure from this life.

## HENRY SEELY WHITE

Inasmuch as Henry Seely White, ninth President of the Society, 1903–1904, died on May 20, 1943; and

Inasmuch as he was very much one of a small number of enthusiastic young Americans, trained in Europe, who, near the close of the last century, determined to give their lives to bringing American mathematical study and research up to the standard attained in Europe; and

Inasmuch as a most necessary element in accomplishing this purpose was founding and cherishing our Society, guiding its meetings and caring for its publications; and

Inasmuch as he was an unselfish and helpful friend to all who came to him for counsel or encouragement, were they contemporaries or youthful pupils; and

Inasmuch as his interest in mathematics never slackened, so that his most sustained piece of writing, a treatise on cubic curves, appeared near the close of his career; be it therefore

Resolved that the Society hereby place on record its deep sense of loss in his death.

The Council passed unanimously the following resolution: that it is the sense of the Council that the roster of scientific meetings of the Society should be continued at at least the same level as in preceding years. The Council designated the November 26–27, 1943, meeting in Chicago as the Annual Meeting of the Society. At this meeting the Bôcher Prize for papers in analysis published during the period of 1938–1942 will be awarded and Professor Marston Morse will deliver his retiring presidential address on *New settings for topology in analysis*. Times and places of meetings during 1944 were set as follows: February 26–27 in New York City; October 28 in New York City.

The committee on Mathematical Surveys reported the publication of the two monographs, *The theory of rings* by Professor Nathan Jacobson and *The problem of moments* by Professors J. A. Shohat and J. D. Tamarkin. The Council voted to continue the new series by publishing other books of the same general character as these two monographs.

Professor MacDuffee, Acting Chairman of the War Policy Committee, reported for the committee the following items: (1) Professors Stone and Morse have been representing the American Mathematical Society and the Mathematical Association of America in Washington on matters affecting the use of mathematically-trained and qualified personnel, working in close cooperation with Dean H. L. Dodge, Director of the Office of Scientific Personnel. (2) The Rockefeller Foundation has made a grant of $2500 to defray the expenses of the War Policy Committee. Of this amount $1,000 has been appropriated to the support of the Office of Scientific Personnel. (3) A subcommittee of the War Policy Committee to gather information as to the actual effect of the mathematical aspects of the Army and Navy training programs, and to formulate suggestions for their improvement, has been established. Professor W. L. Hart is Chairman and the other members are Professors C. R. Adams, H. M. Bacon, B. H. Brown, H. J. Ettlinger, C. V. Newsom, G. B. Price, and W. M. Whyburn.

(4) The War Policy Committee has recognized the desirability of gathering information concerning the war activities of mathematicians but at present sees no way of carrying out a plan for doing so. Members of the Society and Association are urged to cooperate in making this undertaking possible at a later date. By keeping a careful and detailed record of his own activities and experience during the war period, each mathematician will contribute to the preservation of the data without which a record of this extraordinarily significant period in our history would be impossible. (5) The Committee on Available Teachers of Collegiate Mathematics has been functioning since April, 1943. A report on the activities of this Committee appears elsewhere in this Bulletin.

Titles and cross references to the abstracts of papers read are given below. The papers were read as follows: papers 1–8 in the section for Applied Mathematics on Sunday morning; papers 9–15 in the section for Analysis on Sunday morning; papers 16–20 in the section for Probability and Statistics on Monday morning; papers 21–26 in the section for Algebra, Geometry, and Topology on Monday morning; papers 27–32 in the section for Applied Mathematics on Monday afternoon; papers 33–36 in the section for Analysis and Logic on Monday afternoon; and papers 37–88, whose abstract numbers are followed by the letter *t*, were read by title. Professor Lee was introduced by Professor Leonard Carlitz, Mr. Handelman by Professor Willy Prager, Dr. Merrill by Professor E. B. Allen, Mr. Hsiung and Mr. Wang by Professor J. A. Shohat, and Mr. Salzer by Professor A. N. Lowan. Paper 3 was presented by Professor Hay, paper 7 by both authors, paper 11 by Dr. Herzog, paper 12 by Professor Martin, paper 14 by Dr. Fried, papers 19 and 20 by Professor Wald, and paper 24 by Professor Pall.

1. L. C. Hutchinson: *Free vibrations in a rectangular rod.* Preliminary report. (Abstract 49-11-284.)

2. E. G. Kogbetliantz: *Detailed quantitative interpretation of maps of gravitational anomalies with the aid of mathematical analysis.* Preliminary report. (Abstract 49 9-223.)

3. G. E. Hay and Willy Prager: *On plane rigid frames containing curved members.* (Abstract 49-9-221.)

4. Garrett Birkhoff: *Reversibility paradox and camber.* (Abstract 49-11-277.)

5. Arthur Korn: *Compressibility with respect to vibrations of high frequency.* (Abstract 49-11-285.)

6. Stefan Bergman: *Solutions of linear partial differential equations of fourth order.* (Abstract 49-9-218.)

7. D. W. Dudley and Hillel Poritsky: *The geometry of cutting and hobbing of worms and gears.* (Abstract 49-11-279.)

8. Otto Szász: *On uniform convergence of Fourier series.* (Abstract 49-9-215.)

9. C. E. Rickart: *An abstract Radon-Nikodym theorem.* (Abstract 49-11-270.)

10. B. H. Bissinger: *Generalizations of continued fractions.* (Abstract 49-11-252.)

11. Fritz Herzog and B. H. Bissinger: *Generalization of Borel's and F. Bernstein's theorems on continued fractions.* (Abstract 49-11-253.)

12. R. H. Cameron and W. T. Martin: *Transformations of Wiener integrals under translations.* (Abstract 49-9-204.)

13. Tomlinson Fort: *The weighted vibrating string and its limit.* (Abstract 49-9-207.)

14. Paul Erdös and Hans Fried: *On the connection between gaps in power series and the roots of their partial sums.* (Abstract 49-11-258.)

15. P. C. Rosenbloom: *Sequences of polynomials, especially sections of power series.* (Abstract 49-9-210.)

16. Jacob Wolfowitz: *Asymptotic distributions of ascending and descending runs.* (Abstract 49-7-197.)

17. E. J. Gumbel: *On the plotting of statistical observations.* (Abstract 49-9-238.)

18. W. K. Feller: *On a general class of "contagious" distributions.* (Abstract 49-11-301.)

19. Abraham Wald and Jacob Wolfowitz: *An exact test for randomness in the non-parametric case based on serial correlation.* (Abstract 49-11-303.)

20. H. B. Mann and Abraham Wald: *On the statistical treatment of linear stochastic difference equations.* (Abstract 49-11-302.)

21. William H. Durfee: *Congruence of quadratic forms over valuation rings.* (Abstract 49-9-200.)

22. Edward Kasner: *Motion in a resisting medium.* (Abstract 49-9-234.)

23. V. G. Grove: *A general theory of surfaces and conjugate nets.* (Abstract 49-9-230.)

24. R. E. O'Connor and Gordon Pall: *On integral quadratic forms in many variables.* (Abstract 49-11-247.)

25. H. L. Lee: *The sum of the kth power of polynomials of degree m in a Galois field.* Preliminary report. (Abstract 49-9-201.)

26. H. D. Huskey: *On polyhedra and polyhedral path surfaces.* (Abstract 49-9-240.)

27. C. A. Truesdell: *The differential equations of the membrane theory of shells of revolution.* (Abstract 49-11-294.)

28. G. H. Handelman: *On a principle of M. A. Sadowsky.* Preliminary report. (Abstract 49-11-282.)

29. A. E. Ross: *A note on forced oscillations of a conducting sphere.* (Abstract 49-11-292.)

30. S. H. Gould: *The Rayleigh-Ritz method for higher eigenvalues.* (Abstract 49-9-220.)

31. R. E. Gaskell: *On moment balancing in structural dynamics.* (Abstract 49-11-281.)

32. L. L. Merrill: *The mathematical determination of cooling rates during arc welding.* (Abstract 49-11-287.)

33. J. E. Wilkins: *Multiple integral problems in parametric form in the calculus of variations.* (Abstract 49-11-275.)

34. F. A. Ficken: *On two criteria for convexity in a symmetric Banach space.* (Abstract 49-5-134.)

35. Theodore Hailperin: *A set of axioms for logic.* (Abstract 49-9-236.)

36. Andrew Sobczyk: *On the extension of linear transformations.* (Abstract 49-9-214.)

37. A. A. Albert: *Quasigroups.* I. (Abstract 49-9-199-*t*.)

38. E. F. Beckenbach and R. H. Bing: *Conformal minimal varieties.* (Abstract 49-7-182-*t*.)

39. Stefan Bergman: *Fundamental solutions of partial differential equations of the second order.* (Abstract 49-11-251-*t*.)

40. L. M. Blumenthal: *Metric study of generalized elliptic spaces.* Preliminary report. (Abstract 49-11-304-*t*.)

41. L. M. Blumenthal: *New formulations of some imbedding theorems.* (Abstract 49-11-305-*t*.)

42. R. P. Boas: *Almost periodic functions of exponential type.* (Abstract 49-7-183-*t*.)

43. Herbert Busemann: *On spaces in which two points determine a geodesic.* (Abstract 49-9-228-*t*.)

44. Vincent Cowling and Walter Leighton: *On convergence regions for continued fractions.* (Abstract 49-7-184-*t*.)

45. M. M. Day: *Uniform convexity.* IV. (Abstract 49-9-205-*t*.)

46. John DeCicco: *Kasner's pseudo-angle.* (Abstract 49-9-229-*t*.)

47. Nelson Dunford and D. S. Miller: *On the ergodic theorem.* (Abstract 49-11-257-*t*.)

48. C. J. Everett: *Closure operators and galois theory in lattices.* (Abstract 49-7-180-*t*.)

49. G. M. Ewing: *Existence theorems for multiple integral variation problems.* Preliminary report. (Abstract 49-9-206-*t*.)

50. F. A. Ficken: *Note on the existence of scalar products in normed linear spaces.* (Abstract 49-5-133-*t*.)

51. G. E. Forsythe: *Note on equivalent potential temperature.* (Abstract 49-9-219-*t*.)

52. D. W. Hall: *A note on primitive skew curves.* (Abstract 49-11-307-*t*.)

53. M. H. Heins: *On a problem of Walsh concerning the Hadamard three circles theorem.* (Abstract 49-11-261-*t*.)

54. M. R. Hestenes: *The isoperimetric problem of Bolza in parametric form.* (Abstract 49-11-263-*t*.)

55. F. B. Hildebrand: *On the stress distribution in cantilever beams.* (Abstract 49-9-222-*t*.)

56. Carl Holtom: *Permanent configurations in the n-body problem.* (Abstract 49-11-283-*t*.)

57. C. C. Hsiung: *An invariant of intersection of two surfaces.* (Abstract 49-9-231-*t*.)

58. C. C. Hsiung: *Projective invariants of some particular pairs of space curves.* (Abstract 49-9-232-*t*)

59. C. C. Hsiung: *Theory of intersection of two plane curves.* (Abstract 49-9-233-*t*.)

60. Nathan Jacobson: *An extension of Galois theory to non-separable and non-normal fields.* (Abstract 49-11-245-*t*.)

61. Irving Kaplansky: *Solution of the "problème des ménages."* (Abstract 49-7-181-*t*.)

62. Brockway McMillan: *Random point patterns.* Preliminary report. (Abstract 49-11-265-*t*.)

63. Szolem Mandelbrojt: *Quasi-analyticity and analytic continuation. A general principle.* (Abstract 49-9-208-*t*.)

64. Morris Marden: *On the stream function of axially symmetric flows.* (Abstract 49-9-225-*t*.)

65. A. D. Michal: *An analogue of the Maupertuis-Jacobi "least" action principle for dynamical systems of infinite degrees of freedom.* (Abstract 49-11-288-*t*.)

66. A. D. Michal: *Physical models of some curved differential-geometric metric spaces of infinite dimensions.* I. *Wave motion as a study in geodesics.* (Abstract 49-11-289-*t*.)

67. D. S. Miller: *On Carathéodory and Gillespie linear measure.* (Abstract 49-9-209-*t*.)

68. H. T. Muhly: *Independent integral bases and a characterization of regular surfaces.* (Abstract 49-7-195-*t*.)

69. H. E. Newell: *On the asymptotic forms of the solutions of a linear matric differential equation in the complex domain.* (Abstract 49-11-269-*t*.)

70. Isaac Opatowski: *Cantilever beam of circular cross section and constant strength under the action of its own weight.* (Abstract 49-11-290-*t*.)

71. W. V. Parker: *Limits to the characteristic roots of a matrix.* (Abstract 49-9-202-*t*.)

72. M. O. Peach: *Simplified technique for constructing orthonormal functions.* (Abstract 49-11-291-*t*.)

73. Harry Pollard: *A new criterion for completely monotonic functions.* (Abstract 49-7-190-*t*.)

74. Raphael Salem: *A singularity of the Fourier series of continuous functions.* (Abstract 49-9-211-*t*.)

75. H. E. Salzer: *Table of first two hundred squares expressed as a sum of four tetrahedral numbers.* (Abstract 49-9-203-*t*.)

76. Henry Scheffé: *On a measure-theoretic problem arising in the theory of non-parametric tests.* (Abstract 49-9-239-*t*.)

77. H. M. Schwartz: *Riemann integration with respect to additive interval functions.* Preliminary report. (Abstract 49-11-271-*t*.)

78. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* I. (Abstract 49-9-212-*t*.)

79. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* II. (Abstract 49-9-213-*t*.)

80. Seymour Sherman: *Finite aspect ratio effect in the Glauert-Prandtl regime.* (Abstract 49-9-226-*t*.)

81. W. J. Thron: *A general theorem on convergence regions for continued fractions* $b_0+K(1/b_n)$. (Abstract 49-7-191-*t*.)

82. W. J. Thron: *Convergence regions for the general continued fraction.* (Abstract 49-7-192-*t*.)

83. A. D. Wallace: *A structural property of transformations.* (Abstract 49-9-241-*t*.)

84. F. T. Wang: *On Riesz summability of Fourier series.* III. (Abstract 49-9-217-*t*.)

85. Alexander Weinstein: *On the bending of a clamped plate.* (Abstract 49-9-227-*t*.)

86. Paul White: *On r-regular convergence.* (Abstract 49-9-242-*t*.)

87. Hassler Whitney: *On the extension of differentiable functions.* (Abstract 49-7-193-*t*.)

88. D. V. Widder: *Positive temperatures on an infinite rod.* (Abstract 49-11-295-*t*.)

T. R. Hollcroft,
*Associate Secretary*

# BOOK REVIEWS

*Development of the sciences.* Second series. By Oystein Ore, Frank Schlesinger, Henry Margenau, John Arrend Timm, Chester Ray Longwell, Lorande Loss Woodruff, Walter Richard Miles, John Farquhar Fulton. New Haven, Yale University Press; London, Humphrey Milford and Oxford University Press, 1941. 10+336 pp. $3.00.

Keyed to the present interest in scientific advance, these chapters form a short introduction to science, its methods and ideas, background and trends. The presentation is not technical but designed for the general reader. The material was first offered during the winter and spring of 1940 as a series of public lectures sponsored by the Yale University chapter of Gamma Alpha. It comprises discussions by eight well known Yale scientists, representing the fields of mathematics, astronomy, chemistry, physics, geology, biology, psychology, and medicine. Each of the first seven lectures traces the development of one of the basic sciences from its beginnings to its most recent results. The last lecture shows the interdependence of these various sciences as illustrated by specific examples in the history of medicine.

The preceding statements are quoted literally from the jacket of the book under review. Since the book comprises less than three hundred and fifty pages, it is obvious that the presentation is extremely concise. But the reviewer found that it makes interesting and stimulating reading, and feels that the book cannot fail to convey to the general reader an overwhelming impression of the power of the human mind engaged in unselfish and purposeful endeavors. The chapter on mathematics, written by O. Ore, comes up in every way to the well known standards of depth of thought and clarity of presentation of its author. This chapter should be very valuable to graduate students of mathematics who, as a rule, have little help in obtaining an integrated picture of mathematics from the historical point of view. It is to be hoped that this chapter may be made available to this particular group of readers in some mathematical journal, possibly in somewhat extended form.

TIBOR RADÓ

*The methodology of Pierre Duhem.* By Armand Lowinger. New York, Columbia University Press, 1941. 184 pp. $2.25.

Pierre Duhem (1861–1916) was an outstanding French physicist whose work was mainly concerned with thermodynamics, but he was one of those scientists who not only walk on scientific paths but want to understand the history of discoveries and the different methods used in scientific proceedings.

The greater part of the present book is a well ordered report on the writings of Duhem on methods in theoretical physics. These investigations have two aspects. On the one hand, Duhem tries to describe objectively the actual methods; on the other hand, he cannot suppress his temperament and speaks, appreciatingly or blamingly, about the different methods and gives prescriptions about the aims and the proceedings. He belongs to that group of physicists of the latter half of the last century who could be called methodical ascetics. They asserted that the theoretical physicist has only the task to *describe* and order the physical phenomena but not to *explain* them. They tried to suppress the natural demand to find out *causes* for the phenomena, a demand which requires the connecting of the phenomena through intermediate tensions and phases which are not directly observed. A more detailed analysis would show that it is impossible to avoid in the description such intermediate, invisible, "theoretical" phases of the pair, cause and effect.

Another object of dislike for Duhem are the "models" in physics, for instance the electrical models of the English school of physicists. He admits that the models sometimes suggest valuable ways of progress but he does not like them. They are not abstract enough, too near to the tempting but forbidden field of "explanation." He would, probably, have been astonished to see the progress of modern molecular physics made with the help of the atom model. From the logical point of view there is no precise distinction between representation through mathematical symbols and through models.

Duhem is somewhat subjective in his views and indulges in dangerous generalizations and summaries in describing the different kinds of minds. There is the broad but feeble mind (mainly English) which uses models, and the narrow but strong mind using the method of abstract reasoning (French and German). Later, after the first world war had started, he discovers also some characteristics of the "German mind." It is "geometric"; it treats all things by the axiomatic method and gives birth to theories which deny obvious truth, for instance, the various non-Euclidian geometries. This assertion is a good example of the innate tendency of the human understanding to order and simplify the wealth of phenomena, outside physics as well as in it, and of the manner in which prejudice, antipathy, and sympathy help to achieve this order.

The last chapter gives critical remarks and conclusions of the author. Some of the remarks are given above. I do not quite share his opinion that the "methodologist" has only to describe the methods used by the physicists or the scientists in general. Even a non-expert may see sometimes, by general considerations, ways which the researchman should go. Plato and Aristotle were no mathematicians, Bacon no physicist in a proper sense, but, undoubtedly, they furthered the development of mathematics and physics. That Aristotle was an impediment to the development of physics was the fault of "experts" who adhered in a slavish way to his physical theories. The outsider *sometimes* sees more of the general landscape of science, where the scientific workers go the toilsome ways which lead to discoveries.

In his conclusions the author presents certain "indefeasible facts" of the scientific situation, for instance that in modern physics metaphysical explanatory theories are excluded. His last "conclusion" is somewhat metaphysical and not easily accepted, namely that science approaches "asymptotically" a perfectly adequate account of reality. And it is given as an "indefeasible fact of the human situation—that the Human Spirit is one and that the different activities in which it expresses itself must in the end arrive at the same conclusion." It is difficult for me to imagine common conclusions which will be reached "asymptotically" by the Human Spirit in its metaphysical, social and scientific activities.

Only one little critical remark: the mathematical reader will be astonished to find on page 94, quoted from Poincaré, an erroneous description of the method of mathematical induction.

M. DEHN

*Transients in linear systems, studied by the Laplace transform.* Vol. I. By M. F. Gardner and J. L. Barnes, New York, Wiley; London, Chapman and Hall, 1942. 9+389 pp. $5.00.

This book is remarkable in that it is perhaps the first serious attempt to present the theory of the Laplace transform to the mathematics or engineering student at an early stage of his studies. For the complete understanding of this transform it is necessary to have mastered the theory of functions of a complex variable. This fact has hitherto barred many a student from the use of a valuable analytic tool until rather late in his mathematical career. The present work shows conclusively that this delay is unnecessary.

The authors are able to place the fundamental idea of the method before the reader even in the first chapter. The essence of the matter consists in replacing by the Laplace transform one function space by another in such a way that differential and integral operations on the functions of the first space become algebraic operations on the functions of the second. Then an integrodifferential equation becomes an algebraic equation. Let us illustrate by finding that solution of the differential equation $dy/dx - y = x$ for which $y(0) = 0$. One first defines the Laplace transform $Y(s)$ of the required solution $y(x)$:

$$Y(s) = \int_0^\infty e^{-sx} y(x) dx.$$

Integration by parts shows that

$$\int_0^\infty e^{-sx} y'(x) dx = -y(0) + sY(s),$$

so that the given differential equation transforms into

$$sY(s) - y(0) - Y(s) = s^{-2}.$$

After introducing the boundary condition $y(0) = 0$ one easily solves the algebraic equation and finds that

$$Y(s) = (s - 1)^{-1} - s^{-1} - s^{-2}.$$

It now only remains to find the function $y(x)$ which has this function for its transform. This is easily read off from a table of transforms and is found to be $y(x) = e^x - x - 1$.

A critical examination of the above solution will show that certain assumptions have been made, notably that the unknown function $y(x)$ and its derivative have Laplace transforms. The authors take the perfectly acceptable point of view that such assumptions are justified by the results. It is always possible to check directly the solution of a differential system. If it is correct, no further investigation of the method is necessary. If the brilliant success arouses the curiosity of the student, so much the better. He can learn why it all works when he is better equipped with mathematical tools. One outstanding advantage of the method is brought out in the above example. Due to the fact that the transform is unilateral (the integration runs from 0 to $+\infty$ and not from $-\infty$ to $+\infty$), one-point boundary conditions are brought automatically into the solution. Thus no general solution of the differential equation with arbitrary constants is involved.

There is a very welcome second chapter in which the physical background of the electrical and mechanical problems treated is discussed in great detail. Most comparable texts assume that the student is already familiar with the physics of the situation, and if he is not, he must look elsewhere for help. This chapter is independent of the Laplace method and could be used in connection with any course where practice in setting up differential systems from physical problems is required.

It is natural that the authors should wish, in a book of this character, to keep the proof of theorems a secondary matter. Thus in chapter 3 the Laplace transform is introduced by analogy with the more familiar Fourier series and integrals. Even in chapters 4 and 5, where a more formal approach is undertaken, demonstrations of theorems are either omitted entirely (Theorems 1, 2, 3 and 4) or sketched briefly. This procedure seems not to detract from the essential clarity of the presentation. The remainder of the book, with the exception of chapter 8, elaborates further and illustrates profusely the method of solving integrodifferential and difference equations.

The reviewer's chief adverse criticism of the book has to do with chapter 8. Here the authors seem to abandon the general plan of the book. Some twenty theorems of a more advanced nature about the Laplace transform are developed. Demonstrations are given. No use of the results is made in the present volume. In several cases the theorems are incorrectly stated, in others the proofs are wrong or at least inadequate. One of the more important errors stems from indiscriminate interchange of limit and integral operations (though elsewhere the authors are careful about this). Another is based on the incorrect assumption that a Laplace integral defines an analytic function which must have a singularity on the axis of convergence. No doubt the chapter was included to prepare the way for a second volume on partial differential equations and related material which is to follow. The flaws in the chapter need not impair essentially the usefulness of the present volume. For, the student who is looking merely for technique may omit the chapter, the brilliant student may read it with circumspection. It may be worth while to call attention to one other misstatement in chapter 4. Here it is stated that a function of a complex variable which has a first derivative at a point necessarily has all higher derivatives there also (compare the function $|z|^2$).

In conclusion it may be well to summarize the special features of the book. First and foremost, it places a vital tool well within the grasp of a student with meager mathematical training. It dispells the mystery of the Heaviside operational calculus by showing the real basis of that technique. It gives complete physical background for the integrodifferential and difference equations it solves. It has a wealth of illustrative examples, done in the text, and many problems for the student at the ends of the chapters (no answers). There is a very extensive table of Laplace transforms, as useful as a table of integrals in a calculus course. In chapter 1 and in appendix B a valuable comparison of the Laplace method with other possible techniques is given. Historical notes on the mathematical theory appear in appendix C. Finally there is one of the most extensive bibliographies on the subject yet to appear.

D. V. WIDDER

*A treatise on projective differential geometry.* By Ernest Preston Lane. Chicago, University of Chicago Press, 1942. 9+466 pp. $6.00.

Since the appearance of the author's earlier volume, *Projective differential geometry of curves and surfaces* (University of Chicago Press, Chicago, 1932), significant contributions to the field of projective differential geometry have been made by geometers in various parts of the world. In the preface the author does not claim that the present treatise is exhaustive, but states that it represents the fruit of ten years of study and investigation and gives an account of the author's experience with those portions of the subject which interested him most. It is with respect to those portions of the subject, then, that the reviewer interprets the author's statement, appearing earlier in the paragraph, that "the present volume integrates the new material with the old and gives a connected exposition of the theory to date." The treatise reports results of studies made by many workers in the field, and expounds a wide range of topics. It is also notable, however, that some of the newer topics that have attracted rather general interest have not been mentioned. Parts of the proofs, involving calculations that are difficult for the uninitiated, are so frequently left to the care of the reader that only the more advanced students of the subject can use the volume successfully as a textbook. The treatise is properly designed to serve as a reference book for the research worker in the field. Geometric concepts and results are consistently described in a lucid graphic manner.

It is satisfying to observe that the present volume devotes considerably more attention to the methods of Wilczynski on the study of curves by means of linear differential equations than does the earlier volume. It is disappointing, however, to find no mention made of Stouffer's simplifications of Wilczynski's methods of determining (i) canonical power series expansions for the local equations of plane and space curves, and (ii) the geometric characterizations of the vertices of the associated reference triangles and tetrahedra. About three times as much space is afforded the theories of curves as was allotted to this subject in the earlier volume. A similarly expanded amount of space is afforded the subject of surfaces in the present volume. The metric and affine applications are omitted. The Fubini method of the use of differential forms is omitted. The theorems stated in the earlier volume as exercises appear without proofs at the ends of the sections as supplementary theorems. Topics in the study of conjugate nets are treated much more fully than in the former volume. In particular, three types of sequences of Laplace are treated extensively in the present volume, whereas the subject of sequences of Laplace received scant attention in the earlier volume. A twenty page chapter is devoted to the study of plane nets. Grove's studies of the neighborhoods of singular points of surfaces are not mentioned. Their omission represents a distinct loss. A brief treatment of Fubini's theory of W-congruences is given. This theory could have been brought more nearly up to date had the contributions of Jonas (1937) and Fubini (1940) been added.

The subjects of plane and space curves in hyper- and ordinary space are treated in the first three chapters. The next three chapters deal with surfaces in ordinary projective space. Two chapters are devoted to the study of conjugate nets and one is devoted to the theory of plane nets. Transformations of surfaces and surfaces and varieties in hyperspace are studied in the last two chapters. The exposition is uniformly excellent throughout the volume. No attempt will be made to review each chapter by topics, but to give an indication of the nature of the relationship of the material included in the treatise to other material in the literature a topical review is here given of the subject matter contained in the two chapters on surfaces in ordinary space.

Chapters V and VI contain expositions of fundamental topics in the theory of analytic surfaces in ordinary three-dimensional space. The parametric curves on each surface under consideration have been selected to be the asymptotic curves. The form of the fundamental system of defining differential equations is deduced from a consideration of the curvilinear differential equation of the asymptotic curves of a surface. Integrability conditions are discussed and a study is made of the effects on the differential equations of the most general transformations of variables which leave the surface undisturbed and preserve the parametric character of the asymptotic curves. Use is then made of special transformations of proportionality factor, from original homogeneous coordinates $x$ to new coordinates $\bar{x}$, to obtain reductions of the system of differential equations to canonical forms. The first canonical form results from the choice of coordinates $\bar{x}_i$ defined by $\bar{x}_i = x_i/x_4$. The second and third canonical forms are Wilzynski's and Fubini's, respectively. Both are obtained by making a transformation of the form $x_i = \lambda \bar{x}_i$. For Wilczynski's canonical form $\lambda = e^{\theta/2}$, and for Fubini's canonical form $\lambda = (e^{\theta}/\beta\gamma)^{1/2}$, wherein $\theta$, $\beta$, $\gamma$ are coefficients of the original differential equations of the surface. The canonical forms of Wilczynski and Fubini have been used extensively by various geometers. In order to arrive at Fubini's canonical form it is necessary that the surface under consideration be not ruled. No mention is made in the present volume of Grove's important class of canonical forms. This class not only contains Fubini's canonical form but also contains other canonical forms of similar geometric interest which, in marked contrast with Fubini's canonical form, exist whether or not the surface is ruled. A brief but interesting account is given in the present treatise of a method of calculation of an expansion which represents one nonhomogeneous coordinate $z$ as a power-series in the two nonhomogeneous coordinates $x$ and $y$ of a point $X$ of a surface. A great expenditure of labor is involved in the calculation of the terms of the fifth and sixth degrees of this power-series. The theory and method of procedure are adequately explained; the tedious details of the calculations are appropriately omitted. The author has contributed much to the development of such power-series expansions. A canonical form for the differential equations of a ruled surface is obtained in a direct and simple manner. The relations between the coefficients of these equations and those on which Wilczynski based his theory of ruled surfaces are then established. At this point properties of the curves of sections by planes through an asymptotic tangent are deduced. The flecnode curves are briefly considered. The quadrics of Darboux are defined and their equations with reference to a local coordinate system are obtained. The quadric of Lie and the asymptotic osculating quadrics of Bompiani are also defined and their local equations are calculated. Bompiani's characterization of the quadric of Wilczynski as a special asymptotic osculating quadric is deduced. A somewhat similar characterization of the quadric of Fubini is reached. Reciprocal congruences are defined and studied, and their developables and focal surfaces are determined. Although G. M. Green pioneered in this subject of reciprocal congruences by his study of congruences in the relation $R$, no reference is made to Green's work. The author's presentation appears to be essentially an interpretation of Green's work in terms of a different system of differential equations than was employed by Green. The equations and expressions are simpler in the forms in which they were originally expressed by Green, in terms of the coefficients of Wilczynski's canonical form. Considerable attention is given to the geometric characterization of various canonical lines, but no reference is made to a known unified geometric characterization of the canonical pencil.

Chapter VI opens with the introduction of the point-to-plane correspondence in which each point of a surface corresponds to the tangent plane to the surface at the point. This correspondence has been thoroughly studied by the author and Fubini and Čech. The axis, ray, associate axis, and associate ray are defined at a point in connection with a conjugate net, and relations among these are established. The author's studies of a general pencil of conjugate nets and the associated ray-point cubic and ray-conic are presented. The cones which correspond to the ray-point cubic and the ray-conic by the point-to-plane correspondence mentioned above are determined. These are the axis plane cone and the axis quadric cone, respectively. Hypergeodesics are defined on a surface by means of a curvilinear differential equation of the second order. Several properties of hypergeodesics are enumerated. A cone of the third class is determined as the envelope at a point of the surface of all of the osculating planes of the hypergeodesics of a family passing through that point. The cone has three cusp planes which intersect in a line called the cusp-axis of the system of hypergeodesics at the point. It is shown that the curves of a pencil of conjugate nets on a surface form a family of hypergeodesics. Other families of hypergeodesics are discussed, among which are the projective geodesics and the union curves of a congruence $\Gamma_1$ on a surface. The projective geodesics are the extremal curves of the invariant integral

$$\int_{u_0}^{u} (\beta\gamma v')^{1/2} du$$

known as the projective arc length of the curve along which it is calculated.

The property of the projective normal which Fubini discovered is established, namely, at a point on a surface the cusp-axis of the projective geodesics is the projective normal. The quadric of Moutard for a direction at a point of a surface is defined and its local equation is calculated. The author describes some properties of the transformations of Čech and presents his own construction for the point $P_k$ which corresponds to a given plane in the general transformation of

Čech at a point $P_x$ of the surface $S$. The author fails to mention the generalization of this transformation by Fubini and Čech. A simple construction is known for the elements which correspond in this more general transformation. The pangeodesics are defined as the extremal curves of the integral invariant

$$\int_{u_0}^{u}\left(\frac{\beta+\gamma v'^2}{v'}\right)du.$$

The geometric characterizations of the pangeodesics obtained by Lane and Segre appear in the volume. Formulas for the differentiation of local point coordinates are deduced and it is proved that with the use of these formulas the ordinary theory of envelopes may be applied in finding the envelope of a family of surfaces $S_1$ defined by $f(x_1, x_2, x_3, x_4, u, v)=0$ in which $x_1, \cdots, x_4$ are the local coordinates of a variable point on the surface $S_1$ and $u, v$ are the curvilinear coordinates of the point $x$ on the surface $S$. An application of this theory of envelopes is given in the determination of the local coordinates of the vertices of the tetrahedron of Demoulin. Results of L. Green, B. Segre, Su, and Thomsen and Mayer, on the respective subjects, the envelope of the quadric of Moutard, projective curves, projectively minimal curves, and projectively minimal surfaces, are reported as supplementary theorems. Important concepts which might have been included are those of projective curvatures and torsions of a curve on a surface, with respect to differential forms of the surface. New characterizations of the curves of Darboux, the curves of Segre, and of the Darboux-Segre pencil of conjugate nets could have been given. Also simple geometric characterizations of the important integral invariants

$$\int_{u_0}^{u}(\beta\gamma v')^{1/2}du, \qquad \int_{u_0}^{u}\left(\frac{\beta+\gamma v'^2}{v'}\right)du,$$

are known, but none are given.

The author has on former occasions called the attention of differential geometers to the need of an intrinsic tensor calculus for the study of a variety $V_k$ immersed in a projective space $S_n$, where $n>k>1$ and $k\neq n-1$. In the present treatise he has neglected to announce that Grove has recently developed such a tensor calculus. Grove has shown that certain tensors arising in the theory of the geometry of paths on a variety $V_k$ can be expressed in terms of tensors arising in the study of the variety from the point of view of classical projective differential geometry. The works of E. Cartan on projective differential geometry have also been overlooked. Cartan has used his method of moving reference systems with outstanding effectiveness to study both classical projective differential geometry and the generalized projective differential geometry of spaces with a projective connection.

To form a fair estimate of the treatise, in view of numerous omissions of important topics, we must keep in mind that the aim of the author was not to write an exhaustive treatise on the subject but to present an account of his experience with those portions of the subject which interested him most.

The style of the book is attractive, the typography is clear and restful to the eye, and the proofreading has been carefully done. The research worker will find the volume informative, suggestive, and stimulating.

P. O. BELL

## NOTES

The Indian Mathematical Society will hold its next conference at Annamalai University at Annamalainagar about the end of December, 1943. In connection with the conference, it is proposed to have a mathematical exhibition intended to illustrate the richness and variety of the subject and the wide range of its applicability to life situations.

Miss Louise Adams has been appointed to an assistant professorship at High Point College, High Point, North Carolina.

Assistant Professor C. K. Alexander of Occidental College, Los Angeles, California, has been promoted to an associate professorship.

Associate Professor H. F. Archibald of Keuka College, Keuka Park, New York, has been promoted to a professorship.

Dr. H. G. Ayre of Western Illinois State Teachers College has been promoted to an associate professorship.

Assistant Professor H. M. Bacon of Stanford University has been promoted to an associate professorship.

Dr. G. A. Baker of the University of California at Davis has been promoted to an assistant professorship.

Associate Professor A. C. Berry of Lawrence College, Appleton, Wisconsin, has been promoted to a professorship.

Assistant Professor J. W. Blincoe of the University of Tennessee has been appointed to a professorship at Randolph-Macon College.

Dr. J. O. Blumberg of the University of Pittsburgh has been promoted to an assistant professorship.

Mr. W. H. Bradford of the John McNeese Junior College of the Louisiana State University has been promoted to an assistant professorship.

Mr. R. L. Calvert of the Utah State Agricultural College has been promoted to an assistant professorship.

Assistant Professor C. E. Clark of Bradley Polytechnic Institute has been appointed to an associate professorship at Emory University.

Dr. J. B. Coleman has been appointed to a professorship at the University of Richmond.

Dr. M. W. Dehn of Illinois Institute of Technology has been appointed to a professorship at St. John's College, Annapolis, Maryland.

Dr. R. W. Gibson of the University of Illinois has been appointed to an assistant professorship at Kansas State College.

Dr. Edward Helly of Monmouth Junior College has been appointed visiting lecturer at the Illinois Institute of Technology.

Dr. Fritz Herzog of Cornell University has been appointed to an assistant professorship at Michigan State College.

Mrs. Phyllis H. Hutchings has been appointed assistant professor of astronomy and mathematics at Whitman College, Walla Walla, Washington.

Associate Professor Nathan Jacobson of the University of North Carolina has been appointed to an associate professorship at Johns Hopkins University.

Dr. S. A. Jennings of the University of British Columbia has been promoted to an assistant professorship.

Dr. Lois Kiefer of the University of Illinois has been appointed head of the department of mathematics at State Teachers College, Silver City, New Mexico.

Professor M. G. Moore of Tri-State College, Angola, Indiana, has been appointed to an assistant professorship at Bradley Polytechnic Institute.

Dr. Haim Reingold of the Illinois Institute of Technology has been promoted to an assistant professorship.

Assistant Professor M. A. Sadowsky of the Illinois Institute of Technology has been promoted to an associate professorship.

Dr. Robert Schatten has been appointed to an assistant professorship at the University of Vermont.

Dr. C. L. Seebeck of the University of North Carolina has been appointed to an assistant professorship at the University of Alabama.

Dr. W. S. Snyder of Illinois Institute of Technology has been appointed to an assistant professorship at the University of Tennessee.

Dr. C. J. Thorne of Washington University has been appointed to an assistant professorship at Louisiana State University.

Assistant Professor T. L. Wade of the University of Alabama has been appointed head of the department of mathematics at Florida State College for Women.

Assistant Professor L. R. Wilcox of the Illinois Institute of Technology has been promoted to an associate professorship.

The following appointments to instructorships are announced: Bard College, Columbia University: Dr. H. B. Mann; Harvard University: Mr. R. E. Lane; Illinois Institute of Technology: Mr. Furio Alberti, Dr. Ruth Ballard, Mr. P. L. Browne, Mr. H. J. Miser; Indiana University: Dr. Marion D. Wetzel; Massachusetts Institute of Technology: Mr. G. N. Raney; Michigan State College: Dr. B. H. Bissinger; University of Minnesota: Dr. Louis Garfin; University of Pennsylvania: Mr. R. T. Luginbuhl; University of Rochester: Dr. H. S. Kieval; Syracuse University: Mr. F. L. Celauro, Mrs. Ruth E.

Goodman; Vassar College: Miss Janet C. Durand; Whitman College: Mrs. Pearl C. Miller.

Mr. J. K. Gore, president of the Prudential Insurance Company since 1934, died June 22, 1943, at the age of 79 years.

President C. C. Jones of the University of New Brunswick died August 19, 1943, at the age of 71 years.

Mr. L. L. Locke who retired in 1942 died August 28, 1943. He had been a member of the Society since 1900.

Mr. T. C. Rafferty, consulting actuary, Missouri State Life Insurance Company, died September 10, 1943. He had been a member of the Society since 1925.

Mr. H. M. Showman, lecturer and registrar of the University of California at Los Angeles, died June 24, 1943.

Professor Emeritus E. B. Van Vleck of the University of Wisconsin died on June 2, 1943. He had been a member of the Society since 1893.

Mr. A. W. Whitney died on July 27, 1943. He had been a member of the Society since 1902.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

## ALGEBRA AND THEORY OF NUMBERS

### 243. Angeline J. Brandt: *The free Lie ring and Lie representations of the full linear group.*

The present paper is a continuation and amplification of a paper of Thrall (Amer. J. Math. vol. 64 (1942) pp. 371–388). In the latter paper a recursion formula was developed from which the irreducible constituents of the $m$th Lie representation for $m \leqq 10$ were obtained. The main result of the present paper is a direct formula for the character of the $m$th Lie representation, namely $[m]=[\sum \mu(d)s_d^{d'}]/m$ where $dd'=m$, the sum is over all divisors $d$ of $m$, $\mu(d)$ is the familiar Mobius function and $s_d$ is the trace of $A^d$, $A$ being an arbitrary element of the full linear group. The nature of the irreducible invariant subspaces of $L^m$ (all forms of degree $m$ in the free Lie ring) is determined by this formula for $m \leqq 14$. Certain necessary and certain sufficient conditions are determined in order that a given ideal $I$ be characteristic. Since the intersection of $I$ with the module $L^m$ is an invariant subspace of $L^m$ relative to the representation of the full linear group afforded by $L^m$, it becomes important to know what elements constitute such a subspace and a method for determining this is developed. (Received October 1, 1943.)

### 244. R. H. Bruck: *Some results in the theory of quasigroups.*

This paper is primarily intended as an illustration of the usefulness of isotopy in quasigroup theory and as ground work for a later paper on linear non-associative algebras. It is largely devoted to the theory and construction of quasigroups with the inverse property. A quasigroup $Q$, finite or infinite, has the inverse property if there exist two one-to-one reversible mappings $L$, $R$, not necessarily distinct, of $Q$ on itself, such that $b^L \cdot ba = ab \cdot b^R = a$ for all $a$, $b$ of $Q$. Also contained in the paper are several new theorems on Moufang quasigroups, an explicit construction of all (Murdoch) abelian quasigroups, and necessary and sufficient conditions that the direct product of two finite quasigroups should contain no sub-quasigroup except itself. (Received August 13, 1943.)

### 245. Nathan Jacobson: *An extension of Galois theory to non-separable and non-normal fields.*

The extension of Galois theory obtained in this paper is based on two important concepts: self-representation of a field P, that is, a representation of P by matrices with elements in P, and composite (ring) of P with itself. The latter is a generalization of the usual concept of a composite field. With any two self-representations, associate a product self-representation obtained by substituting for the elements of the matrices of the first representation the matrices that represent these elements in the second representation. This leads to a definition of the product of composites. The latter concept is used to define Galois composites. The fundamental theorem then establishes a (1-1) correspondence between the Galois composites of P and the subfields $\Phi$ of P over which P is finite. This result gives a theorem recently announced by Kalujnine and specializes still further to the classical theorem for subfields $\Phi$ over which P is finite, separable and normal. In addition to the main result a general study of self-representations of fields is given in this paper. (Received August 9, 1943.)

### 246. R. E. Johnson: *On the equation* $\chi\alpha=\gamma\chi+\beta$ *over an algebraic division ring.*

The main purpose of this paper is to give necessary and sufficient conditions in order that the equation $\chi\alpha=\gamma\chi+\beta$ have a solution over an algebraic division ring. If $\alpha$ and $\gamma$ are not transforms of each other, this equation always has a unique solution. In case $\gamma$ equals $\alpha$, this equation has a solution if and only if $\beta\alpha\beta^{-1}$ is a unilateral solution of a particular equation over the division ring. For the special case in which the division ring is a quaternion algebra, $\chi\alpha=\alpha\chi+\beta$ has a solution if and only if $\beta\alpha=\bar{\alpha}\beta$, where $\bar{\alpha}$ is the quaternion conjugate to $\alpha$. (Received September 20, 1943.)

### 247. R. E. O'Connor and Gordon Pall: *On integral quadratic forms in many variables.*

A difficult problem proposed by Ko and Erdos, to find an integral quadratic form of determinant 1 and minimum greater than 2, is solved by the construction of a form in 24 variables of minimum 3, and a similar form in 40 variables. Improvements are found on the limits within which quadratic forms in genera of one class may exist. (Received August 6, 1943.)

### 248. R. R. Stoll: *Representations of completely simple semigroups.*

Every finite semigroup (that is, a finite system closed under an associative multiplication) can be represented by a set of correspondences of a suitable finite set to itself. This suggests the problem of the determination of all such representations of a given semigroup. Towards this end semigroups of correspondences are first studied. It is shown that if $P$ is a semigroup on the set $N$ then $N$ can be written as the class sum of mutually disjoint subsets which form "weakly transitive" systems for $P$. Next, each weakly transitive system has a unique decomposition as the union of "transitive" and "quasi-transitive" systems. This reduces the representation problem to the determination of all transitive and quasi-transitive representations and methods for composing such representations. The above results are used to determine all representations of a type of finite semigroup called a "Kerngruppe" by Suschkewitsch (Communications de la Société Mathématique de Kharkow vol. 6 (1933) pp. 27–38) and a "completely simple semigroup without zero" by Rees (Proc. Cambridge Philos. Soc. vol. 36 (1940) pp. 387–400). (Received October 1, 1943.)

### 249. Dorothy M. Stone: *An algebraic characterization of measure algebras.* Preliminary report.

Let $E$ be a non-atomic Boolean $\sigma$-algebra. This paper gives purely algebraic conditions on $E$ which are necessary and sufficient for the existence of a countably additive, real-valued, non-negative, finite measure function on $E$, vanishing only for $\phi$, the zero of $E$. $E$ is given the star-topology of G. Birkhoff, and it is shown that $E$ will have such a measure if and only if it is metrizable in a certain way in this topology. It is next shown that $E$ is a topological space if and only if it satisfies a certain distributive law. A basis for the neighborhoods of $\phi$ can then be characterized algebraically, making it possible to state simple algebraic equivalents for the various separation axioms. If $E$ satisfies the countable chain condition, $T_3$ implies metrizability, which gives an outer measure on $E$. A measure is obtained by means of an additional algebraic requirement. Thus, if $E$ satisfies the distributive law referred to, the countable chain condition, the algebraic equivalent of $T_3$, and the additional requirement, there exists a measure-function on $E$. These conditions are easily seen to be necessary. It is not known whether they are independent. (Received October 2, 1943.)

### 250. R. M. Thrall: *On the decomposition of modular tensors.* II.

Let $G$ be the $n$-rowed full linear group over a field $k$ of characteristic $p$. A representation of $G$ is called a tensor representation if its space is a direct sum of subspaces and factor spaces of tensor spaces. A main result of the present paper is that for a finite field $k$, the $k$-group ring of $G$ has a faithful tensor representation. In paper I the representations afforded by all tensors of rank $m<2p$ were determined subject to the condition that $k$ has more than $p$ elements. In this paper the same is done for the field $k$ with $p$ elements. A main tool in this investigation is the construction of a representaton of $G$ from each irreducible representation of the non-modular full linear group, and a corresponding extension of the Brauer-Nesbitt modular character theory to this case. The presence of zero divisors in the ring of polynomial functions over a finite field enters into the treatment of the case of tensors of rank $2p-1$ over a two-dimensional vector space, and the situation in that case should help point the way to the general decomposition theory. (Received October 1, 1943.)

## ANALYSIS

### 251. Stefan Bergman: *Fundamental solutions of partial differential equations of the second order.*

As was previously shown, for every differential equation $L(U)=U_{Z\bar{Z}}+H(Z, \bar{Z})U=0$, $Z=X+iY$, $\bar{Z}=X-iY$, there exists a function $\mathbf{E}(Z, \bar{Z}; t)=1+Z\bar{Z}t^2\mathbf{E}^*(Z, \bar{Z}, t)$ such that $U=\mathrm{P}(f)=\int_{-1}^{+1}\mathbf{E}(Z, \bar{Z}, t)f(Z(1-t^2)/2)dt/(1-t^2)^{1/2}$, where $f$ is an arbitrary analytic function, is a solution of $L(U)=0$ (see Duke Math. J. vol. 6 (1940) p. 537). The author shows that a fundamental solution $\Gamma(z, \bar{z}, \zeta, \bar{\zeta})$ of the equation $S(v)=v_{z\bar{z}}+F(z, \bar{z})v=0$ is given by $\mathrm{P}(1/2\pi)\log|Z|+G(Z, \bar{Z})$. Here $Z=z-\zeta$, $\bar{Z}=\bar{z}-\bar{\zeta}$, and $\mathrm{P}$ is the operator introduced above for the equation $L(U)=0$ with $H(Z, \bar{Z})=F(Z+\zeta, \bar{Z}+\bar{\zeta})$. $G(Z, \bar{Z})=-\int_0^Z\int_0^{\bar{Z}}D(Z, \bar{Z})dZd\bar{Z}+\int_0^Z\int_0^{\bar{Z}}H(Z, \bar{Z})(\int_0^Z\int_0^{\bar{Z}}D(Z, \bar{Z})dZd\bar{Z})dZd\bar{Z}+\cdots$ where $D(Z, \bar{Z})=(1/2)\int_{-1}^{+1}t^2[2\mathbf{E}^*+\bar{Z}\mathbf{E}^*_{\bar{Z}}+Z\mathbf{E}^*_Z]dt/(1-t^2)^{1/2}$. Using the representations of functions $v(z, \bar{z})$, $S(v)=0$, in the form of a line integral over a closed curve in terms of $\Gamma$, $\partial\Gamma/\partial n$, $v$ and $\partial v/\partial n$ the author studies the growth of $v$ and $\partial v/\partial n$ along circles $|z|=r$, $r\to\infty$. The existence of an analogous function $\mathbf{E}(X, Y, t)$ for every equation $H(U)=U_{XY}+H(X, Y)U=0$ (of hyperbolic type) has been established (see above reference). $(1/2\pi)\int_{-1}^{+1}\mathbf{E}(X, Y, t)dt/(1-t^2)^{1/2}$ is now shown to be the Riemann function of the equation $v_{xy}+F(x, y)v=0$, where $X=x-\xi$, $Y=y-\eta$ and $H(X, Y)=F(X+\xi, Y+\eta)$. Analogous relations hold for more general equations. (Received September 11, 1943.)

### 252. B. H. Bissinger: *Generalizations of continued fractions.*

The simple continued fraction for a real number $x$, $0<x<1$, may be written in the form $f(a_1+f(a_2+\cdots$, where the $a$'s are the partial denominators and $f(t)=1/t$. The present paper generalizes the simple continued fractions by using functions $f(t)$, other than $1/t$. When $f(t)$ belongs to an appropriate class $F$ of real functions, defined for $t\geqq 1$, and $a_1, a_2, \cdots$ is a sequence of positive integers, the "$f$-expansion" $f(a_1+f(a_2+\cdots$ converges to a number $x$, $0<x<1$. The "$f$-expansion of $x$," $0<x<1$, is given by an algorithm equivalent to the euclidean algorithm when $f(t)=1/t$. The main result of the paper is the identity of the $f$-expansion of $x$ with the $f$-expansion which converges to $x$. The remainder of the paper is concerned with generalizations of certain results in the theory of simple continued fractions, established by Borel and F. Bernstein. One of these results is that, for almost all $x$, $0<x<1$, the $a$'s form an unbounded sequence. This and theorems of a similar nature are proved for generalized continued fractions, provided that $f(t)$ belongs to a certain subclass of $F$. (Received August 5, 1943.)

### 253. Fritz Herzog and B. H. Bissinger: *Generalization of Borel's and F. Bernstein's theorems on continued fractions.*

Let $f(a_1+f(a_2+\cdots$ be the $f$-expansion of $x$, $0<x<1$, as defined by Bissinger in his paper *Generalizations of continued fractions* (abstract 49-11-252). For the case $f(t)=1/t$ (simple continued fractions) Borel and F. Bernstein have proved: (a) the set of $x$, $0<x<1$, for which the $a$'s are bounded has measure zero; (b) the set of $x$, $0<x<1$, for which all $a$'s are greater than unity has measure zero. In the paper referred to above the statements (a) and (b) were proved to hold when $f(t)$ belongs to a certain class of polygonal functions. The present paper is concerned with the problem of characterizing analytically a class of functions which does include $f(t)=1/t$ and such that statements (a) and (b) hold when $f(t)$ belongs to that class. This investigation at the same time reveals by which analytic properties of the function $f(t)=1/t$ the original statements of Borel and F. Bernstein may be proved. (August 5, 1943.)

### 254. R. C. Buck: *A note on subsequences.*

Starting from a result of H. Steinhaus (Prace Matematyczno-Fizyczne vol. 22 (1911) p. 129), it is shown that a sequence $s_n$, limitable by a regular matrix method $T$, is convergent if and only if each of its subsequences is also limitable $T$. This general tauberian theorem gives rise to a similar result for series. If there exists a regular matrix method $T$ which sums every series obtained from $\sum a_n$ by bracketing blocks of terms, then $\sum a_n$ is itself convergent. (Received September 11, 1943.)

### 255. R. C. Buck: *Multiple sequences.*

An $r$-multiple sequence of points of a limit space $L$ is a single value function $p(j)$ from $\mathfrak{J}$ to $L$, where $\mathfrak{J}$ is the $r$-fold cartesian product of the integers with themselves. Under an inclusion ordering, $\mathfrak{J}$ is a directed system; a subsequence may then be defined in terms of a monotonic function $\lambda(j)$, defined on $\mathfrak{J}$ to $\mathfrak{J}$. In terms of a product measure (C. Visser, Studia Mathematica vol. 7, p. 143) it is shown that the set of all convergent subsequences of $p(j)$ form a measurable set, with measure 0 or 1. By means of a lemma on measure preserving translations and sets of measure 1, it is shown that if $p(j)$ is divergent, so are almost all of its subsequences. This is a generalization of a previous result of H. Pollard and the author. (Received September 29, 1943.)

### 256. R. C. Buck and Harry Pollard: *Convergence and summability properties of subsequences.*

The authors discuss the relation of the convergence or summability of a sequence to that of its subsequences. The set of subsequences of a sequence can be mapped in a 1:1 fashion on the interval (0, 1). In terms of Lebesgue measure on this set, it is shown that a sequence is convergent if and only if almost all of its subsequences are convergent. Similarly, for bounded sequences, a sequence is $(C, 1)$ summable if and only if almost all of its subsequences are $(C, 1)$ summable. For unbounded sequences, $(C, 1)$ summability of $s_n$ follows from that of almost all of its subsequences, but not conversely. The principal tools are the properties of homogeneous sets, and the Rademacher functions. These results are intimately connected with probability; in particular, a theorem of Birnbaum and Zuckerman (Amer. J. Math. vol. 62 (1940) pp. 787–791) follows easily. (Received September 11, 1943.)

### 257. Nelson Dunford and D. S. Miller: *On the ergodic theorem.*

Let $S$ be a space with measure $|e|$ in a $\sigma$-field $F$, and $|S|$ finite. Define $\phi$ to be a 1-1 map of $S$ into all of itself with $\phi^{-1}e \in F$ when $e \in F$. For every real function $f$ on $S$ let $Tf = g$ where $g(t) = f(\phi t)$. In this paper it is shown that a necessary and sufficient condition for the mean ergodic theorem to hold for such an operator $T$ is that $n^{-1}\sum_{i=0}^{n-1}|\phi^{-i}e| \leqq M|e|$, $e \in F$, $n = 1, 2, \cdots$. It is also proved that in spite of the fact that $\phi$ may not be a measure preserving transformation it is nevertheless still true that the mean ergodic theorem implies the point ergodic theorem. The two above results are also obtained for the $n$-parameter continuous case. (Received August 4, 1943.)

### 258. Paul Erdös and Hans Fried: *On the connection between gaps in power series and the roots of their partial sums.*

Let $f(z) = 1 + a_1 z + \cdots + a_n z^n + \cdots$ be a power series with the radius of convergence 1. It has Ostrowski gaps if there exists a $\rho < 1$ and a pair of infinite sequences $m_k$ and $n_k$ with $\lim_{k\to\infty} m_k/n_k > 1$, such that $|a_n| < \rho^n$ if $m_k \leqq n \leqq n_k$. It has infinite Ostrowski gaps if $\lim_{k\to\infty} m_k/n_k = \infty$. The number of roots of $f_n(z) = 1 + a_1 z + \cdots + a_n z^n$ in the circle of radius $1+r$ is denoted by $A_n^{1+r}$. The following theorems are proved. If $f(z)$ has Ostrowski gaps, then there exists an $r > 0$ such that $\liminf_{n\to\infty} A_n^{1+r}/n < 1$. This converse is also true. If there exists an $r > 0$ such that $\liminf_{n\to\infty} A_n^{1+r}/n < 1$, then $f(z)$ has Ostrowski gaps. This theorem is not new. Mr. Bourion (*L'Ultraconvergence dans les Séries de Taylor*, Actualités Scientifiques et Industrielles, no. 472) proved it but his proof is quite different from that given here. Further, the following theorem is proved. If $f(z)$ has infinite Ostrowski gaps, then there exists an $r > 0$, such that $\liminf_{n\to\infty} A_n^{1+r}/n = 0$. The converse is also true. One of the main tools is the following lemma which may be of interest in itself. Let $F(\rho)$, where $\rho < 1$, be the family of all polynomials of the form $1 + a_1 z + \cdots + a_m z^m + \cdots + a_n z^n$ where $|a_k| < \rho^k$ if $m \leqq k \leqq n$, and if $r < 1/\rho$, then there exists a $c_r$ such that the number of roots of every polynomial belonging to $F(\rho)$ outside the circle with the radius $r$ is greater than $c_r(n-m)$. (Received August 4, 1943.)

### 259. Herbert Federer: *Surface area.* I.

Suppose $m \leqq n$ are positive integers For each subset $S$ of $n$-space let $\gamma(S)$ be the superior of the $m$-dimensional Lebesgue measures of the perpendicular projections of

$S$ into all $m$-dimensional subspaces of $n$-space. $\gamma_r(S)$ is the inferior of numbers of the form $\sum_{i=1}^{\infty}\gamma(A_i)$ where $A_1, A_2, A_3, \cdots$ are open connected subsets of $n$-space of diameter less than $r$ whose union contains $S$. $\Phi(S)=\lim_{r\to 0+}\gamma_r(S)$ $\Phi$ is an $m$-dimensional measure over $n$-space which agrees with Carathéodory linear measure in case $m=1$ and with Lebesgue measure if $m=n$ The author is interested in the relation of $\Phi$ to $m$-dimensional surfaces in $n$-space He proves: If $m=2$, $n=3$, $R$ is a rectangle and $f$ is a continuous numerically valued function on $R$, then the Lebesgue surface area of $f$ equals the $\Phi$ measure of the set of all points of the form $(u, v, f(u, v))$ with $(u, v)$ in $R$ (Received September 23, 1943 )

## 260. Herbert Federer: *Surface area.* II.

Retaining the conventions stated in abstract 49-11-259 the author investigates the validity of the formula $\int N(f, T, y)d\Phi y=\int_T Jf(x)dx$ where $f$ is a Lebesgue measurable function on $m$-space to $n$-space and $T$ is a Lebesgue measurable subset of $m$-space Here $N(f, T, y)$ is the number (possibly $\infty$) of points $x$ in $T$ for which $f(x)=y$; and $Jf(x)=(\det(\bar{L}L))^{1/2}$ where $L$ is the approximate differential of $f$ at $x$ and $\bar{L}$ its conjugate The formula holds for general $m$ whenever $\limsup\text{approx}_{z\to x}|f(z)-f(x)|/|z-x|<\infty$ for every $x$ in $T$. In the special case $m=2$ (here $Jf=(EG-F^2)^{1/2}$) the formula holds if corresponding to each point $x$ in $T$ there are three distinct unit vectors $x^1, x^2, x^3$ such that $\limsup_{t\to 0+}|f(x+tx^j)-f(x)/t|<\infty$ for $j=1, 2, 3$. Finally if $m=n=1$ and if the upper right-hand derivative of $f$ is finite at each point of $T$, then the formula is likewise true and $Jf$ may be replaced in it by the absolute value of the Dini derivative. (Received September 23, 1943.)

## 261. M. H. Heins: *On a problem of Walsh concerning the Hadamard three circles theorem.*

The present paper consists of a contribution to the solution of the following problem proposed to the author by J. L. Walsh: Let $\mathfrak{A}$ consist of the class of functions $f(z)$ which satisfy the following requirements: (a) $f(z)$ is analytic for $z<R(>0)$, (b) $|f(z)|<M(>0)$ for $|z|<R$, (c) $|f(z)|\leqq m(<M)$ for $|z|\leqq r(<R)$. Under these circumstances it is required to determine l.u.b. $M(f, \rho)$ with $f\in\mathfrak{A}$, where $r<\rho<R$ and $M(f, \rho)=\max_{|z|=\rho}|f(z)|$, and the associated extremal functions. The main concern of the paper is with the descriptive properties of the extremal functions. A method highly suggestive of the typical Tchebycheff argument is used to show that the extremal functions define $(1, k)$ directly conformal maps of $|z|<R$ onto $|w|<M$ and that they are unique when suitably normalized. Under the assumption that $r$ is sufficiently small, the degree $k$ is determined and the associated extremal functions are calculated for the special case where $r^2\leqq m/M$. Related questions are treated for the class of functions satisfying (a) and (b) and (c) modified by replacing "$|z|\leqq r$" with "$z\in E$" where $E$ belongs to and is closed relative to $-1<x<+1$, and l.u.b. $E<1$. (Received August 6, 1943.)

## 262. M. H. Heins: *On the problem of Milloux for functions analytic throughout the interior of the unit circle.*

Let $E$ denote a point set lying in and closed relative to $|z|<1$ and having the property that it has a non-void intersection with $|z|=r$ for $0\leqq r<1$. Let $m$ denote a positive number less than one. Finally let $f(z)$ denote a function analytic and of modulus less than one for $|z|<1$ which satisfies the further condition that there exists an $E$-set $E_f$ such that $|f(z)|\leq m$ for $z\in E_f$. It is desired to determine l.u.b. $\{\max_{|z|=\rho}|f(z)|\}$ $(0<\rho<1)$ and the corresponding extremal functions. This problem is an extremal problem associated with the original Milloux problem. The extremal value and the corresponding extremal functions are explicitly determined in terms of the Jacobi $sn$-function. Two other methods are given for treating the problem. One employs the theory of Blaschke products, the other gives an algorithm in terms of a process involving the composition of (1, 2) directly conformal maps of the interior of the unit circle onto itself. (Received August 14, 1943.)

### 263. M. R. Hestenes: *The isoperimetric problem of Bolza in parametric form.*

The purpose of the present paper is to establish a conjecture due to McShane that for a strong relative minimum it is sufficient that for each admissible variation $y$ there is a set of multipliers with which the arc under consideration satisfies the Euler-Lagrange equation, the transversality condition, the Weierstrass condition, nonsingularity and such that the second variation is a modification of those used by McShane (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 344–379) and Myers (Duke Math. J. vol. 10 (1943) pp. 73–97). An interesting feature of the method is that one obtains the sufficiency theorem for the isoperimetric problem without transformation of the problem to a non-isoperimetric problem. Moreover, one also obtains simultaneously the analogues of the theorems of Osgood. (Received August 4, 1943.)

### 264. Norman Levinson: *On a nonlinear differential equation of the second order.*

In the equation $\ddot{x}+f(x)\dot{x}+x=e(t)$, $e(t)$ is continuous and periodic of period $L$; $f(x)$ is continuous and greater than 0 except possibly at a finite number of points. If the integral of $f(x)$ over $(x_0, \infty)$ diverges then the above differential equation has a periodic solution of period $L$ and *all* other solutions tend to this solution as $t\to\infty$. (Received September 25, 1943.)

### 265. Brockway McMillan: *Random point patterns.* Preliminary report.

$X$ is a space, $X^k$ the $k$-fold Cartesian product of $X$ by itself. $W$ is a space whose elements are countable subsets $w=\{x_i\}\subseteq X$. N. Weiner (Amer. J. Math. vol. 60 (1938) pp. 925, 928) exhibited a probability measure $P$ in $W$ such that if $F(w)=f_0+\sum_k\sum' f_k(x_1, \cdots, x_k)$, where the $\sum'$ is over all $k$-tuples of distinct points $x_i\in w$, then (i) $\int F(w)dP=f_0+\sum_k\int f_k(x_1, \cdots, x_k)dm_k=L(F)$, where $m_k$ is a suitable measure in $X^k$. Such $P$'s are characterized by moment conditions on the number of points in $Aw$, $A\subseteq X$. An inverse problem has been treated by N. Wiener and A. Wintner (Abstract 46-11-480): Given $L(F)$ defined by (i), does a probability measure $P$ exist with $L(F)=\int F(w)dP$? The present paper solves this inverse problem by applying previous results about absolutely monotone functions of sets (Abstract 47-11-471). Any completely monotone ($CM$) function of sets $A\subseteq X$ defines a measure in $W$. $P$ exists whenever $1+\sum_k(-1)^k(k!)^{-1}m_k(A^k)$ converges absolutely to a $CM$ function of $A$. (Received August 3, 1943.)

### 266. A. P. Morse: *A theory of covering and differentiation.*

A covering theory is developed which embraces a variety of theorems of the Vitali type. By substituting a new and frequently useful concept of regularity for the classical notion the author arrives at a general theory of differentiation. This includes and supplements the Lebesgue theory and, at the same time, encompasses results in the differential theory of nets. Use is made of the closed subset theorems alluded to in the abstract which follows. (Received September 11, 1943.)

### 267. A. P. Morse and J. F. Randolph: *The $\phi$ rectifiable subsets of the plane.*

Suppose $A$ is a plane set, $\phi$ is a measure with $\phi(A)<\infty$, and suppose that, at $\phi$ almost all points of $A$, the upper linear $\phi$ density of $A$ is less than 1.01 times the lower linear $\phi$ density of $A$. Under these circumstances it is shown that all except a $\phi$ small part of $A$ lies on a rectifiable arc. A systematic study of the relations between density and $\phi$ rectifiability is made. In fact by setting $\phi$ equal to Carathéodory linear measure, one of our theorems becomes in essence a compendium of the results obtained by Besicovitch in his paper *On the fundamental geometrical properties of linearly measurable plane sets of points*. II, Math. Ann. vol. 115 (1938) pp. 296–329. Some closed subset theorems are obtained. These involve measure and are apparently heretofore unknown. The paper under discussion together with the paper whose abstract precedes will appear in an early issue of the Transactions. (Received September 11, 1943.)

### 268. F. J. Murray: *On the existence of quasi-complements in Banach spaces.*

Let $B$ be a reflexive Banach space. Let $M$ and $N$ be closed additive subsets of $B$. $N$ is a quasi-complement of $M$ if $M\cdot N=(0)$ and $M\cdot+\cdot N$ is dense in $B$. It is proved that for every closed additive $M$, such a quasi-complement exists. (Received September 23, 1943.)

### 269. H. E. Newell: *On the asymptotic forms of the solutions of a linear matric differential equation in the complex domain.*

In an earlier paper (*The asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain*, Duke Math. J. vol. 9 (1942)) the author discussed the existence of solutions to the matric differential equation $(d/dx)Y(x,\ \lambda)=\{\lambda(\delta_{ij}r_j(x))+(q_{ij}(x,\ \lambda))\}Y(x,\ \lambda)$, in which $x$ and $\lambda$ are complex variables. Under certain conditions of suitability imposed upon the coefficient functions and the domains of variation of $x$ and $\lambda$, it was possible to show that regions of existence can be constructed in which the differential equation possesses solutions of the form $P(x,\ \lambda)E(x,\ \lambda)$, where $E(x,\ \lambda)=(\delta_{ij}\exp\{\lambda\int^x r_j(x)dx\})$, and $P(x,\ \lambda)$, analytic in $x$, reduces uniformly in $x$ to the identity matrix when $\lambda$ becomes infinite. The author applied the elegant and basic concepts of associated and fundamental regions originated by R. E. Langer (*The boundary problem of an ordinary linear differential system in the complex domain*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 151–162, and correction, p. 467) to some of the cases in which the functions $r_j(x)$ have poles and the differences $r_i(x)-r_j(x)$, $i\neq j$, have poles or zeros on the boundary of the $x$ region in question. In adapting the concept of associated regions to the cases considered, the condition was imposed upon the differences $r_i(x)-r_j(x)$, $i\neq j$, that those having simple poles be of the form $a^{ij}(x-x_0)^{-1}$, where $a^{ij}(+0)$ is a constant. The present paper removes that restriction. (Received August 4, 1943.)

### 270. C. E. Rickart: *An abstract Radon-Nikodym theorem.*

Let $\mathfrak{M}$ be a $\sigma$-field of subsets of an abstract set $M$ and let $m(e)$ be a non-negative measure function defined on $\mathfrak{M}$. This paper contains a proof of the following generalization of the Radon-Nikodym theorem: *If $M$ is the union of a countable number of sets of finite measure and $X(e)$ is any completely additive function defined on $\mathfrak{M}$ to an arbitrary Banach space, then a necessary and sufficient condition that $X(e)$ be representable as an integral, with respect to $m(e)$, is that $X(e)$ be absolutely continuous relative to* $m(e)$ (that is, $m(e)=0$ implies $X(e)=0$). The integral used is the Pettis integral (B. J. Pettis, Trans. Amer. Math. Soc. vol. 44) extended (R. S. Phillips, Trans. Amer. Math. Soc. vol. 47) so that it is defined for certain "contractive" functions (that is, multivalued set functions such that $e_1 \subseteq e_2$ implies $f(e_1) \subseteq f(e_2)$). (Received August 6, 1943.)

### 271. H. M. Schwartz: *Riemann integration with respect to additive interval functions.* Preliminary report.

It is shown first that in order to have finite upper Darboux integrals for continuous integrands, the integrator $g(i)$ (defined and additive over the closed intervals $i$ lying in a fixed finite many-dimensional interval) must be of bounded variation. It is then shown that when $g(i)$ is continuous, all the essential properties of ordinary Riemann integration remain valid. Since every $g(i)$ can be decomposed into a continuous function and a function of jumps (in an extended sense), the essentially new properties are obtained by studying the integration with respect to the latter function. It is also shown that the class of Riemann integrable functions (taken with the Moore-Smith limit) can be generated from the class of simple step functions by employing a natural notion of limit of functions, which is closely related to the one employed by Arzelà and Hobson (cf. E. W. Hobson, *Theory of functions of a real variable*, vol. 2, 3d edition, pp. 312–314). (Received August 7, 1943.)

### 272. Max Shiffman: *Isoperimetric inequalities and continuity of area for classes of surfaces.*

The isoperimetric inequality $A \leqq L^2/4\pi$ has been extended to minimal surfaces by Carleman, and to sufficiently regular surfaces of nonpositive curvature by Beckenbach and Radó. Advanced methods were used, involving analytic functions or conformal mapping and subharmonic functions. Here it is shown by purely elementary methods that the isoperimetric inequality applies to polyhedra all of whose interior vertices have the sum of face angles greater than or equal to $2\pi$. The result can then be extended to surfaces of nonpositive curvature through approximation by such polyhedra. The isoperimetric inequality can be used as a decisive analytical tool in the estimation of areas of boundary strips. For example, one can establish general theorems involving the continuity of area of surfaces of nonpositive curvature. See, for example, Shiffman, *Unstable minimal surfaces with any rectifiable boundary*, Proc. Nat. Acad Sci U.S.A. vol. 28 (1942) pp. 103–108, where minimal surfaces are considered. (Received October 1, 1943.)

### 273. Dorothy M. Stone: *The representation of abstract measure functions.*

In this paper a standard form is obtained for general abstract measure algebras in which the measures have values in abelian semi-groups. Incidentally it is shown that the theory of such abstract measures can be reduced to the theory of ordinary numerical measures. Let $E$ be a non-atomic Boolean $\sigma$-algebra satisfying the countable chain condition. Let $\lambda$ be an abstract measure-function on $E$; $\lambda$ is assumed to be countably additive and non-atomic. Writing $a\sim b$ if $\lambda(a)=\lambda(b)$, an equivalence relation between certain pairs of elements of $E$ is defined, which has four simple properties. Conversely, any equivalence relation with these properties determines such a measure $\lambda$, uniquely to within isomorphism. Working only with the equivalence relation it is shown that there exists a finite isomorphism $\phi$ of $E$ onto a certain subalgebra of a direct sum of measure algebras $E_\alpha$, each having a numerical-valued measure $\mu_\alpha$, in such a way that $a\sim b$ if and only if $\mu_\alpha[\phi_\alpha(a)]=\mu_\alpha[\phi_\alpha(b)]$ (for all $\alpha$), where $\phi_\alpha(x)$ denotes the projection of $\phi(x)$ onto $E_\alpha$. A $\sigma$-representation can be deduced from this by reducing modulo a certain $\sigma$-ideal in the direct sum. (Received October 2, 1943.)

### 274. J. V. Wehausen: *A remark concerning a set of completely continuous transformations.*

Let $E$ be a Banach space, $K$ a closed convex subset of $E$, $\mathfrak{T}$ the set of all completely continuous transformations defined in $E$ with $T(K)\subset K$, $\mathfrak{T}'$ the subset of $\mathfrak{T}$ consisting of transformations with unique fixed points in $K$, and $\mathfrak{T}''$ the subset consisting of those with more than one fixed point in $K$. From the fixed point theorem of Schauder $\mathfrak{T}=\mathfrak{T}'+\mathfrak{T}''$. Let $\mathfrak{T}$ be metrized by $\|T_1-T_2\|=\sup_x\|T_1x-T_2x\|$. Then $\mathfrak{T}$ is complete and the following theorem holds. If $\mathfrak{T}'$ is dense in $\mathfrak{T}$, then $\mathfrak{T}'$ is of the second category and $\mathfrak{T}''$ of the first category of $\mathfrak{T}$. This generalizes a result of Orlicz (Bulletin international de l'Académie des sciences de Cracovie A, Nr. 8/9 (1932) pp. 221–228) concerning the differential equation $y'=f(x, y)$. (Received September 24, 1943.)

### 275. J. E. Wilkins: *Multiple integral problems in parametric form in the calculus of variations.*

This paper discusses the problem of minimizing a multiple integral in a class of varieties defined parametrically. In order that the multiple integral be independent of the parametric representation, it is necessary and sufficient that the integrand function satisfy a certain homogeneity condition. The implications of this homogeneity condition for the derivatives of the integrand of arbitrary order are investigated. The usual necessary conditions known in the non-parametric case for multiple integrals are extended to the parametric case. A discussion of the form of the Weierstrass condition is given. Finally, as a first step in obtaining a field theory, necessary and sufficient conditions that an integral be invariant are given. (Received August 5, 1943.)

### 276. Antoni Zygmund: *On certain integrals.*

(a) Given a function $\phi(z)$ regular for $|z|<1$ and of the class $H^r$, $r>1$, let $g^*(\theta)=\{\int_0^1(1-\rho)\chi^2(\rho, t)d\rho\}^{1/2}$, where $\chi^2(\rho, \theta)=\pi^{-1}\int_0^{2\pi}|\phi'(\rho e^{i(\theta+t)})|^2P(\rho, t)dt$, $P(\rho, t)$ denoting the Poisson kernel. Then the ratio $M_r(g^*(\theta))/M_r(\phi(e^{i\theta}))$ is contained between two positive numbers depending on $r$ only. (b) Let $f(\theta)$ be of period $2\pi$ and of the class $L^r$, $r>1$. Let $F(\theta)$ be the indefinite integral of $f$, and $\mu(\theta)=\{\int_0^\pi t^{-3}[F(\theta+t)+F(\theta-t)-2F(\theta)]dt\}^{1/2}$. The ratio $M_r(\mu)/M_r(f)$ is again comprised between two positive numbers depending on $r$ only. (c) Every lacunary series $\sum c_k$ which is absolutely summable $A$ must be absolutely convergent. (Received September 10, 1943).

## Applied Mathematics

### 277. Garrett Birkhoff: *The reversibility paradox and camber.*

A comparison is made between the predictions of the (reversible) Kutta-Joukowsky lift theory and the experimental findings of Eiffel, for circular airfoils. It is found that the predicted angle of the chord to the airfoil with the airstream in the position of zero lift ("first axis") varies from two to seven times the observed angle. Whereas with camber 1/7, the observed increase in lift/effective angle of attack is about 83 per cent; the predicted increase 4 per cent. A possible irreversible explanation of this is pointed out. (Received August 14, 1943.)

### 278. H. B. Curry: *The method of steepest descent for nonlinear minimization problems.*

A practical method for calculating approximately a stationary value of a function $G(x_1, x_2, \cdots, x_n)$ is desirable in connection with certain nonlinear least square problems (abstract 49-11-286). Such a method may be exhibited as follows. Let $x_i^0$ be an initial approximation. The direction in which $g(t) = G(x_i^0 + t\xi_i)$ decreases most rapidly is obtained by putting $\xi_i = -\partial G/\partial x_i$. By trial $t_0$ can be determined so that $g(t_0) < g(0)$. Then the point $x_i^0 + t_0\xi_i$ can be used as a new approximation and the process repeated. It is shown here that if $t_0$ is chosen so that $g'(t_0) = 0$, $g'(t) < 0$ for $0 \leq t < t_0$, then the successive approximations converge to a point where $\partial G/\partial x_i = 0$ provided that $G$ takes a smaller value at $x_i^0$ than at any point on the boundary of that region $S$ within which differentiability and other usual conditions hold. Although the process is well known in analysis, it does not appear to have been noticed recently in this connection. It is applicable when the initial approximation $x_i^0$ is too rough for the standard least square procedure. The problem includes that of solving $n$ simultaneous nonlinear equations in $n$ unknowns which was handled by Cauchy. (Received October 1, 1943.)

### 279. D. W. Dudley and Hillel Poritsky: *The geometry of cutting and hobbing of worms and gears.*

The exact shape of the teeth of a worm $W$ or a helical gear $G$ which is produced by a milling cutter $C$ or a hob $H$ of a given tooth profile is investigated. This problem and its converse are of great interest in the manufacture of gears and worms. Assuming numerous cutter teeth one may replace the cutter $C$ by a surface of revolution $S$. In its motion relative to $W$, $S$ occupies a one-parameter family of positions whose envelope is $W$. The curve of contact of $S_0$, any position of $S$, with the envelope represents $C$, the curve of deepest cutting. $C$ may be determined from the condition that along it the velocity of the motion of $S$ relative to $W$ is tangent to $S$. In the converse problem where $W$ is given and $S$ is sought, the motion of $W$ relative to $S$ is utilized in a similar way; this motion can now be simplified to a rotation about the cutter axis. A similar treatment applies to the hob-gear problem except that here a two-parameter family of motions is involved and two kinematic conditions are now required to determine the deepest cutting. (Received August 12, 1943.)

### 280. Bernard Friedman: *A method of approximating the complex roots of polynomial equations.*

By using successive divisions, an iterative process is set up to approximate the quadratic factors of a polynomial. In this way, the complex roots of largest and smallest absolute value can be determined. Conditions for convergence and for the rapidity of convergence are also investigated. (Received August 31, 1943.)

### 281. R. E. Gaskell: *On moment balancing in structural dynamics.*

The moment balancing process (H. Cross, Transactions of the American Society of Civil Engineers vol. 96 p. 1) is developed for the dynamics of structures with the help of the dynamic form of the four-moment equation (W. Prager, Ingenieur-Archiv, vol. 1, p. 527). This combination gives a method of finding stresses in plane, rigid-frame structures, each of whose bars is of uniform cross section and whose joints are not subject to translation, when an oscillating load is applied. The method converges at least for oscillations whose frequency is less than the lowest natural frequency of the structure. (Received August 5, 1943.)

### 282. G. H. Handelman: *On a principle of M. A. Sadowsky.* Preliminary report.

Sadowsky (Journal of Applied Mechanics vol. 10 (1943) pp. A-65) has applied a heuristic principle of maximum plastic resistance to several types of combined states of stress. The principle has been proved by Prager for a beam under combined tension and torsion. This paper is concerned with the additional case of a symmetric beam subject to bending and torsion. The basic differential equation, derived from the stress-strain relations and the yield condition, is shown to be the Euler-Lagrange equation for the following variational problem: to find that stress distribution for a given applied torque for which the value of the bending couple is stationary. (Received August 9, 1943.)

### 283. Carl Holtom: *Permanent configurations in the n-body problem.*

Let $n$ masses in a plane be projected with initial velocities and thereafter be allowed to move only under the force of attraction according to the Newtonian law. If the potential function of the system is $U(x_i, y_i)$, then the masses $m_i$ at points $(x_i, y_i)$ form a permanent configuration if there exists a real solution of the equations $\partial U/\partial x_i + m_i\omega^2 x_i = 0$ and $\partial U/\partial y_i + m_i\omega^2 y_i = 0$, where $\omega^2$ is the angular velocity of the system. It is shown that for the case of $n-1$ masses there exists at least one real point of liberation at which $m_n = 0$ may be placed, and that the solutions vary continuously and remain real as $m_n$ increases to a positive value. For the case $n > 4$ a restriction on the masses or mass ratios is sufficient, though it may not be necessary, to insure the existence of a permanent configuration. The equations may be solved for the coordinates as functions of the mass ratios, and several terms in the series solutions are given. (Received August 6, 1943.)

### 284. L. C. Hutchinson: *Free vibrations in a rectangular rod.* Preliminary report.

In this paper the author studies, directly from the general equations of motion, the possible free vibrations in a rectangular rod of isotropic homogeneous material, and their relationship to the corresponding acoustic and electromagnetic waves. (Received August 6, 1943.)

### 285. Arthur Korn: *Compressibility with repect to vibrations of high frequency.*

When the velocities and accelerations remain under a certain limit, the compressibility of the air can be quite well handled by means of the well known state equation $p=R\rho T$, in which $p$ is the pressure, $\rho$ the density, $T$ the absolute temperature and $R$ a constant. If one takes account of friction the formula is slightly modified. For high values of velocities and accelerations, especially when one has to do with vibrations of high frequency, new approximations have to be found. The state equation of compressible matter not only contains according to the author's ideas all the secrets of quantum mechanics, but also has important relations to the theory of the electromagnetic field. For vibrations $p=p_0+p_1 \cos \nu t+p_2 \sin \nu t$ with frequencies, as they are considered in quantum mechanics, for example, in the theory of the spectra, the state equation may have the form $p=R\rho T+R'\nu\rho(T)^{1/2}$, where $R$ and $R'$ are constants and $\nu$ the frequency of the vibration. For high frequencies the pressure will become more and more proportional to the frequency. When the frequency still becomes higher and higher, one may make the assumption that for such vibrations approximate incompressibility takes place. (Received August 12, 1943.)

### 286. Kenneth Levenberg: *A method for the solution of certain non-linear problems in least squares.*

The standard method of least squares for problems involving a set of residual functions $f_i(\alpha, \beta, \gamma, \cdots)$, $i=1, 2, \cdots, n$, which are nonlinear in the parameters $\alpha, \beta, \gamma, \cdots$, consists of reducing the $f_i(\alpha, \beta, \gamma, \cdots)$ to linear form by first order Taylor approximations $F_i(\alpha, \beta, \gamma, \cdots)$, taken about an initial trial solution for the parameters $\alpha_0, \beta_0, \gamma_0, \cdots$, and minimizing $S=\sum_1^n F_i^2(\alpha, \beta, \gamma, \cdots)$ by solving the corresponding linear normal equations (Whittaker and Robinson, Calculus of Observations, 1937, p. 214). If $\alpha_1, \beta_1, \gamma_1, \cdots$ are the solution of these normal equations, it may happen, especially in certain engineering applications, that $\sum_1^n f_i^2(\alpha_1, \beta_1, \gamma_1, \cdots) \geqq \sum_1^n f_i^2(\alpha_0, \beta_0, \gamma_0, \cdots)$ in which case the process has failed to improve the initial solution. This paper offers a method of solving the problem in this case. It is shown that the initial solution can be improved by the minimization of $wS+Q$, where $Q$ is a positive definite quadratic form in $\alpha-\alpha_0, \beta-\beta_0, \gamma-\gamma_0, \cdots$, and $w$ is a sufficiently small positive quantity for which an approximate expression, which may be improved by trial, is given. For $Q=(\alpha-\alpha_0)^2+(\beta-\beta_0)^2+(\gamma-\gamma_0)^2+\cdots$, the method consists of revising the normal equations by the addition of a constant $1/w$ to the coefficients of the principal diagonal. (Received October 1, 1943.)

### 287. L. L. Merrill: *The mathematical determination of cooling rates during arc welding.*

This is an extension of work done by Dr. Daniel Rosenthal who determined the temperature distribution in very thick or very thin plates during arc welding under the simplifying assumptions that the heat source is a point source and the plate metal in the neighborhood of the source undergoes no change. The present investigation considers plates of various finite thickness and, by introducing an "input factor," takes into account the facts that the heat source is not a point source and that a pool of molten metal trails the moving electrode. To accomplish this two temperature-time records for each plate thickness must be determined experimentally at a point at the edge of the molten metal, one with the plate initially at room temperature and the other with the plate at some convenient pre-heat temperature. It is then possible to predict the cooling rate at any temperature during the cooling, for any initial plate temperature, for any travel speed, and for any heat input within the range of interest. Results of the investigation make it possible to determine welding conditions necessary to produce any desired cooling rate at any temperature level during cooling and thus to make a weld with desired characteristics. (Received August 4, 1943.)

### 288. A. D. Michal: *An analogue of the Maupertuis-Jacobi "least" action principle for dynamical systems of infinite degrees of freedom.*

More than a century ago, Jacobi gave his famous least action principle for the dynamical trajectories of dynamical systems with finite degrees of freedom. Although Hamilton's principle has been used widely in the derivation of the equations of motion for infinite degree of freedom problems in physics and engineering, analogues of the Jacobi least action principle do not seem to be in the literature. In this paper, analogues of the Jacobi least action principle are given for various infinite degree of freedom problems. The functionals that are made stationary are more general than those considered in the usual calculus of variations. (Received August 6, 1943.)

### 289. A. D. Michal: *Physical models of some curved differential-geometric metric spaces of infinite dimensions.* I. *Wave motion as a study in geodesics.*

In this paper, the second order partial differential equations of wave motion are shown to define dynamical states that can be considered as generating geodesics in some of the author's general "Riemannian" spaces with linear topological coordinates (Bull. Amer. Math. Soc. vol. 45 (1939) pp. 529–563, especially pp. 551–559). The author's earlier geometries (1927–1931) in infinitely dimensional spaces are not suitable for these particular physical models partly because some of the functionals have a higher order continuity than zero. (Received August 6, 1943.)

### 290. Isaac Opatowski: *Cantilever beam of circular cross section and constant strength under the action of its own weight.*

The determination of the cross section consists in the solution of a nonlinear integral equation, which gives the profile of the beam in the shape of two parabolas tangent at their common vertex. A calculation of deflections, based on the usual approximation concerning the curvature, gives an infinite deflection at the free end. The use of an exact expression of the curvature gives an imaginary deflection curve, at least within a certain range containing the free end. These impossible conclusions are due to the use of the theory of prismatic beams for a beam of variable cross section. An extension is made to the case in which, besides the beam's own weight, a known force $F$ acts at its free end. This leads to a hyperelliptic integral which gives a known formula when the beam's own weight is negligible with respect to $F$ (see, for example, Hütte, *Des Ingenieurs Taschenbuch*, 25th edition, Berlin, 1925, vol. 1, p. 627). (Received August 6, 1943.)

### 291. M. O. Peach: *Simplified technique for constructing orthonormal functions.*

To orthonormalize the functions $f_1, f_2, \cdots$ over a given region $R$ evaluate the quantities $d_{p,q} = \int_R f_p \bar{f}_q dR$, $0 < p$, $q \leqq n$, where $n$ orthonormal functions are desired.

A computer, whose mathematical attainments need extend no further than a knowledge of the algebraic rules of sign, may complete the calculations. The quantities $d_{p,q}$ are formed into a matrix, the unit matrix is adjoined, and a process $P$ is applied to the combined matrix which reduces its order by one. Repeated application of $P$ yields directly the coefficients of all $n$ orthogonal functions. A root extraction is required to obtain the normalizing factor for each function. The process $P$ consists of the repeated evaluation of certain second-order determinants. Proof depends upon certain properties of $P$ and is obtained largely from the elementary properties of determinants. The method is particularly adapted to the use of modern computing machines. (Received August 9, 1943.)

### 292. A. E. Ross: *A note on forced oscillations of a conducting sphere.*

J. A. Stratton and L. J. Chu (Journal of Applied Physics vol. 12 (1941) pp. 236–240) studied the problem of forced oscillations in a conducting sphere. Taking the equatorial line as the effective region of an applied oscillatory field $E'$, they obtained a series expansion of the current $I_\theta$ crossing the equatorial plane and used it to compute and plot the curves for the radiation conductance, and susceptance, and input admittance of the first few modes of the sphere, and also for the total radiation conductance. It was noted by Stratton and Chu that the above series expansions were not convergent. In the present paper, at the suggestion of Professor Brillouin, the author takes an equatorial strip of finite fixed width $\Delta\theta$ as the effective region of the applied field $E'$, shows that the resulting expansion for the current $I_\theta$ through the surface $\theta=$const. is convergent, and employs it to plot the graphs of the above mentioned quantities. It is of interest to observe that to different values of $\Delta\theta$ correspond different selections of dominant modes in the expansions of the input impedance $Y_i$. (Received August 6, 1943.)

### 293. J. A. Shohat: *Parseval formula in its application to Van der Pol's and generalized equations.*

First, an error is indicated in the paper by the author: *A new analytical method* · · · , Journal of Applied Physics vol. 14 (1943) pp. 40–48. It is further shown, by a simple application of the Parseval formula for Fourier series, that the main result of Van der Pol—approximating the solution by a fundamental oscillation—is mathematically correct, and the error of such an approximation is estimated. Its amplitude is derived anew, with the same numerical result as previously obtained. The present method is shown to apply to the more general equation $d^2u/dt^2-\epsilon F(u)du/dt+u=0$. (Received September 15, 1943.)

### 294. C. A. Truesdell: *The differential equations of the membrane theory of shells of revolution.*

Previously the "membrane equations" have been derived from a figure. Here they are shown to be consequences of the general equations of elasticity. First a generalization of spherical polar coordinates is introduced in which one family of coordinate surfaces is that obtained by revolving curves parallel to some fixed curve about an axis. Then the assumption is made that there is no force normal to the shell across any cut, and no force component in any direction across the surface of the shell or across any parallel surface. There result equations similar to those usually given for the membrane theory and not more difficult to solve. If the thickness of the shell is small compared with its radii of curvature, the usual equations can be deduced, in which the stress does not vary along a normal. The significance of the assumptions is discussed. Finally it is shown that solutions of the equations of the membrane theory lead to a state of strain which does not satisfy the conditions of compatibility, so that displacements calculated in the membrane theory violate geometry. (Received August 6, 1943.)

### 295. D. V. Widder: *Positive temperatures on an infinite rod.*

In this paper it is proved that any solution $u(x, t)$ of the heat equation $\partial^2u/\partial x^2 = \partial u/\partial t$ which is non-negative for positive $t$ and which vanishes for $t=0$ is identically zero. By use of this result it is shown that any solution which is non-negative for $t>0$ has the Poisson-Stieltjes representation $u(x, t)=\int_{-\infty}^{\infty}k(x-y, t)d\alpha(y)$. Here $\alpha(y)$ is nondecreasing and $k(x, t)$ is the familiar source solution $(4\pi t)^{-1/2} \exp(-x^2/4t)$. As a consequence any such solution is analytic in $x$ and in $t$. (Received August 24, 1943.)

## GEOMETRY

### 296. R. C. Buck: *Partition of space.*

By an application of elementary topology, it is shown that $n$ hyperplanes, with general intersection, partition Euclidean $r$-space into $M_r(p, n)$ $p$-dimensional regions, where $M_r(p, n)=\sum_{r-p}^{r} C_{n,k}C_{k,r-p}$, of which $C_{r,p}C_{n,r+1}(r+1)/(n+p-r)$ are bounded. The problem is also solved for projective $r$-space, yielding $\sum_{0}^{[(r-p)/2]} C_{r-2k,r-p}C_{n,r-2k}$ as the number of $p$-dimensional regions. This completely solves the well known "cheese slicing" problem. (Received September 11, 1943.)

### 297. John DeCicco: *Dynamical and curvature trajectories in space.*

Kasner has studied the geometry of dynamical trajectories in the plane and in space in the Princeton Colloquium (Amer. Math. Soc. Colloquium Publications, vol. 3). This paper considers the problem of determining all quintuply-infinite systems of curves in space which are at once dynamical and curvature trajectories. In the plane, Kasner has shown that the appropriate families are the trajectories of all central or parallel fields of force. It is shown that the systems of $\infty^5$ curves which are simultaneously dynamical and curvature trajectories are the dynamical trajectories of the following three distinct types of fields of force: (I) Those whose lines of force all lie in a pencil of planes. (II) Those whose lines of force are orthogonal to a family of $\infty^2$ circular helices, all of which possess the same axis and the same period. (III) Those of the central or parallel type. Each of these types is projectively invariant. (Received August 11, 1943.)

### 298. Edward Kasner and John DeCicco: *Union-preserving transformation of space.*

Sophus Lie showed that the only lineal-element transformations of the contact type are the extended point transformations. This result is extended by studying transformations from differential curve-elements of order $n$: $(x, y, z, y', z', \cdots, y^{(n)}, z^{(n)})$, where $n$ is 2 or more, into lineal-elements $(X, Y, Z, Y', Z')$. The entire class of the union-preserving transformations is determined. Any general union-preserving transformation from curve-elements of order $n$ into lineal-elements is completely determined by a new directrix equation $\Omega(X, Y, Z, x, y, z, y', z', \cdots, y^{(n-1)}, z^{(n-1)})=0$. The only available union-preserving transformations (in the whole domain of curve-elements) are the group of point transformations and the union-preserving transformations from curve-elements of order $n$ into lineal-elements, or the extensions of these two types. An additional theorem is that any transformation from curve-elements of the $(x, y, z)$-space into lineal-elements of the $(X, Y, Z)$-space, by which any union of the $(X, Y, Z)$-space corresponds to exactly $\infty^{2(n-1)}$ curves of the $(x, y, z)$-space, is union-preserving. (Received August 11, 1943).

### 299. L. A. MacColl: *Geometrical characterizations of some families of dynamical trajectories.*

This paper deals with a certain five-parameter family of curves, which can be regarded as the family of trajectories of an electrified particle in an arbitrary static magnetic field. A set of geometrical properties is given which completely characterizes the family of curves. Certain other related families of curves, including the four-parameter family of trajectories of the particle moving with an arbitrarily prescribed value of the energy, are also discussed and characterized by sets of geometrical properties. (Received October 1, 1943.)

### 300. Alice T. Schafer: *Two singularities of space curves.*

This paper uses the methods of projective differential geometry to study an analytic space curve in the neighborhood of an inflexion point and, second, a planar point. Canonical power-series expansions representing the curve in the neighborhood of each singular point are deduced by suitably choosing the projective coordinate system. These canonical expansions are then used to study properties of the curve in this neighborhood. Particular emphasis is placed on the surfaces osculating the curve, sections of the tangent developable of the curve made by the faces of the tetrahedron of reference, and projections of the curve onto the faces of the tetrahedron of reference. (Received October 1, 1943.)

## STATISTICS AND PROBABILITY

### 301. W. K. Feller: *On a general class of "contagious" distributions.*

This paper is concerned with some properties of a class of contagious distributions which contains, among others, some distributions studied by Greenwood and Yule, Polya, and Neyman, respectively. (Received August 3, 1943.)

### 302. H. B. Mann and Abraham Wald: *On the statistical treatment of linear stochastic difference equations.*

For any integer $t$ let $x_{1t}, \cdots, x_{rt}$ be a set of $r$ random variables which satisfy the system of linear stochastic difference equations $\sum_{j=1}^{r}\sum_{k=0}^{p_{ij}}\alpha_{ijk}x_{j,t-k}+\alpha_i=\epsilon_{it}(i=1, \cdots, r)$. The coefficients $\alpha_{ijk}$ and $\alpha_i$ are (known or unknown) constants and the vectors $\epsilon_t=(\epsilon_{1t}, \cdots, \epsilon_{rt})$ $(t=1, 2, \cdots)$ are independently distributed random vectors each having the same distribution. It is assumed that $E(\epsilon_{it})=0$. The problem dealt with in this paper is to estimate the unknown coefficients $\alpha_{ijk}$ and $\alpha_i$ on the basis of $Nr$ observations $x_{it}(i=1, \cdots, r; t=1, \cdots, N)$. The statistics used as estimates of the unknown coefficients are identical with the maximum likelihood estimates if $\epsilon_t$ is normally distributed. The joint limiting distribution of these estimates is obtained without assuming normality of the distribution of $\epsilon_t$. (Received August 7, 1943.)

### 303. Abraham Wald and Jacob Wolfowitz: *An exact test for randomness in the non-parametric case based on serial correlation.*

Let $X_1, \cdots, X_n$ be $n$ chance variables, about the distribution of which nothing is known. Let the problem be to test the (null) hypothesis that $X_1, \cdots, X_n$ are independently distributed with the same distribution function. It is shown that an exact test of this hypothesis based on the serial correlation coefficient can be made. For this purpose the distribution of the serial correlation coefficient in the sub-population consisting of all possible permutations of the observed values is employed. Under the null hypothesis this distribution is independent of the distribution function of $X_i (i=1, \cdots, n)$. Several exact moments are obtained and asymptotic normality is proved. (Received August 7, 1943.)

## TOPOLOGY

### 304. L. M. Blumenthal: *Metric study of generalized elliptic spaces.* Preliminary report.

Let $\Sigma$ be semimetric with diameter $d$, $\phi(x/\rho)$ a real single-valued monotonic decreasing function ($\rho$, positive parameter) defined over the distance set of $\Sigma$, with $\phi(0)=1$, $\phi(d/\rho)=0$. The space is called generalized elliptic $E^{\phi}_{n,\rho}$ provided: I. For each positive integer $k$ and each $k+1$ points $p_1, p_2, \cdots, p_{k+1}$ there corresponds an *allowable matrix* $(\epsilon_{ij})$; $\epsilon_{ii}=1$, $\epsilon_{ij}=\epsilon_{ji}=\pm 1$ $(i, j=1, 2, \cdots, k+1)$ with every nonvanishing principal minor of the determinant $|\epsilon_{ij}\phi(p_ip_j/\rho)|$ positive. II. The integer $n$ is the smallest for which there exist $n+1$ points $p_1, p_2, \cdots, p_{n+1}$ such that $|\epsilon_{ij}\phi(p_ip_j/\rho)|$ does not vanish for any allowable matrix $(\epsilon_{ij})$. For the ordinary elliptic space, $\Sigma$ is the surface of the sphere $S_{n,\rho}$ with opposite points identified and "shorter arc" metric, while $\phi(x/\rho)=\cos(x/\rho)$. An interesting feature of these spaces is that, in contrast to others (that is, euclidean, hyperbolic, spherical) the mutual distances of a set of points does not suffice to determine the dimension of the subspace which contains them. Thus a given set of three numbers may be distances of three points on an $E^{\phi}_{1,\rho}$, and also distances of three points not on any $E^{\phi}_{1,\rho}$, but on an $E^{\phi}_{2,\rho}$. It is found that certain pseudo-$E^{\phi}_{n,\rho}$ $(n+3)$-tuples are contained in an $E^{\phi}_{n+1,\rho}$. New theorems concerning determinants are a by-product of the study. (Received August 3, 1943.)

### 305. L. M. Blumenthal: *New formulations of some imbedding theorems.*

The theorems deal with congruent imbedding of metric spaces in Hilbert space, and center about the two following results: I. A complete connected ptolemaic metric space in which every point is contained in a closed convex neighborhood is convex. II. A complete, convex, externally convex metric space in which the Theorem of Pythagoras is valid is congruently contained in Hilbert space. An application of a well known theorem of Menger-Schoenberg yields the first result when it is shown that each two points of the space are joined by an arc with everywhere vanishing metric curvature. To establish the second theorem one notes that every pair of lines that intersect at "right angles" is congruently imbeddable in the plane. It follows that the space has the *weak euclidean four-point property* and the conclusion follows from a result due to the writer. (Received August 3, 1943.)

### 306. R. H. Fox: *The complete homotopy group.*

The fundamental group $\tau_\omega(Y)=\pi_1(Y^T)$ of the space of continuous mappings of the infinite dimensional torus $T$ into a topological space $Y$ has the homotopy groups $\pi_n(Y)$, $n=1, 2, \cdots$, as subgroups. The infinite symmetric group $\{P\}$ is a group of operators on $\tau_\omega$. The complete homotopy group $\tau$ is the minimal subgroup of $\tau_\omega$ containing all the subgroups $\pi_n^P$. The group $\tau$ can be given a direct geometric definition and can be built up from the homotopy groups by a countable number of splitting extensions. The classical automorphisms of $\pi_n$ induced by elements of $\pi_1$ are induced by inner automorphisms of $\tau$. In fact the Abe group $\kappa_n$, $n\geqq 2$, (Jap. J. Math. vol. 16 (1940) p. 169) is the minimal subgroup containing $\pi_1$ and $\pi_n$. Similarly the multiplication $\alpha\cdot\beta\in\pi_{m+n-1}$, $\alpha\in\pi_m$, $\beta\in\pi_n$ (Whitehead, Ann. of Math. vol. 42 (1941) p. 411) is represented as a commutator $\alpha\beta^P\alpha^{-1}\beta^{-P}$ where $P$ is a certain type of permutation. The commutators $\alpha^Q\beta^R\alpha^{-Q}\beta^{-R}$ which are not $\{P\}$-automorphs of commutators of this type are all $=1$. Generalization of the theory to groups modulo a subset $B$ of $Y$ is automatic. (Received October 1, 1943.)

### 307. D. W. Hall: *A note on primitive skew curves.*

It is shown that a locally connected continuum $M$ not separated by the removal of any pair of its points can contain no primitive skew curve of type II unless it contains a primitive skew curve of type I. As a corollary there is the theorem of F. B. Jones (Abstract 48-11-340) to the effect that a locally connected continuum $M$ separated by no pair of its points but by every one of its simple closed curves must be homeomorphic with a sphere provided $M$ contains no primitive skew curve of type I. This follows immediately in view of a theorem of S. Claytor (Ann. of Math. vol. 35 (1934) pp. 809–835). (Received August 3, 1943.)

### 308. M. E. Shanks: *The space of metrics on a compact metrizable space.* II.

Denote by $\boldsymbol{M}_0(X)$ the set of all metrics on the compact metrizable space $X$ compatible with its topology, and by $\boldsymbol{M}(X)$ the complete extension of $\boldsymbol{M}_0(X)$. Then $\boldsymbol{M}_0(X)$ and $\boldsymbol{M}(X)$ are semi-linear normed spaces. In a previous abstract 47-5-284 the author stated that $X$ and $Y$ are homeomorphic if and only if $\boldsymbol{M}(X)$ and $\boldsymbol{M}(Y)$ are congruent. In this paper the same result is obtained from the stronger statement that the homeomorphism of $X$ and $Y$ implies, and is implied by, the linear isomorphism $\boldsymbol{M}_0(X)$ and $\boldsymbol{M}_0(Y)$. The methods are new and make essential use of the lattice of upper semi-continuous decompositions of $X$. (Received October 1, 1943.)

### 309. J. W. T. Youngs: *On surfaces of class* $K_1$. Preliminary report.

In a paper entitled *On the semi-continuity of double integrals in parametric form* (Trans. Amer. Math. Soc. vol. 51 (1942) pp. 336–361) Radó has proved a variety of important theorems concerning a class of surfaces designated by the symbol $K_1$. The purpose of this paper is to show that every surface is in the class $K_1$. A new functional $I^*(S)=\sup I(T, B, \|X\|)$ is discussed, the supremum being taken with respect to all representations $(T, B)$ of the surface $S$ for which $I(T, B, \|X\|)$ has meaning. (For the terminology see the paper of Radó.) (Received September 22, 1943.)

# NEW PUBLICATIONS

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KELLOGG, O. D. Foundations of potential theory. New York, Murray Printing Co., 1943. Photographic reproduction of 1929 edition. 9+384 pp. $6.00

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WEYL, F. J. See WEYL, H.

WEYL, H., and WEYL, F. J. Meromorphic functions and analytic curves. (Annals of Mathematics Studies, no. 12.) Princeton University Press; London, Humphrey Milford and Oxford University Press, 1943. 9+269 pp. $3.50.

# THE TERNARY OPERATION $(abc)=ab^{-1}c$ OF A GROUP

JEREMIAH CERTAINE

**1. Introduction.** This note is the result of some investigations into the ternary operation $ab^{-1}c$ in a group. We shall assume familiarity on the part of the reader with the notions of a group, a one-one transformation (we shall use the shorter term permutation) of an arbitrary set of elements, an automorphism, and a coset.[1] We shall use the multiplicative notation for a group $G$ with elements $a, b, c, \cdots$. We shall also use the following convention for multiplication of permutations. Given two permutations $T_i: x\to xT_i$ $(i=1, 2)$, then $T_1T_2$ is $x\to(xT_1)T_2$. Finally, we denote automorphisms by small Greek letters.

In §2 we shall review certain properties of the ternary operation in a given group, determining all subsets closed with respect to this operation and the group of permutations of $G$ which preserve this operation. These results had been previously obtained by Reinhold Baer.[2]

In §§3 and 4 we give postulates for this operation with proofs of their independence and consistency. Thus, if a ternary operation satisfies these postulates in an arbitrary set of elements, then the set may be made into a group (unique within isomorphism) in which $(abc)=ab^{-1}c$. The first set of postulates appears as a weakened form of a set given by Baer in his paper,[3] in which he mentions the group property. This and an equivalent set completely determine the ternary function as $ab^{-1}c$. However, by further weakening one of these postulates, it is possible to get a system which no longer has this last property. That is, the group property still holds but the ternary operation is not determined by the group operation.

In the remaining sections we get a geometric interpretation of the ternary operation and derive therefrom simple conditions on pairs of elements (vectors) under which they form a group. In the case where an abelian group is desired, the conditions are even simpler, reducing essentially to a single law.

I wish to express my gratitude to Garrett Birkhoff for his kind

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[1] Cf. H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Leipzig and Berlin, 1937, pp. 1, 5, 41 and 10.

[2] *Zur Einführung des Scharbegriffs*, J. Reine Angew. Math. vol. 160 (1929) pp. 199–206.

[3] Cf. Baer, op. cit. p. 202, footnote.

assistance and encouragement, without which this note would probably not have been written.

## 2. The ternary operation in a group.

THEOREM 1. *$S \subset G$ is closed under $(abc)$ if and only if $S$ is a coset of some subgroup of $G$; indeed a right- (left-) coset of $S^{-1}S$ $(SS^{-1})$.*

PROOF. For the first part see Baer's paper.[4] As for the second observe that if $S = sT$ $(s \in S)$, then $T = s^{-1}S$ $(s \in S)$ and hence $T = S^{-1}S$. Similarly, $S = Ts$ $(s \in S)$ implies $T = SS^{-1}$.

DEFINITION 1. *The set of all permutations of $G$ of the form $\alpha T_e$: $x \rightarrow (x\alpha)a$, where $\alpha$ is an automorphism of $G$, is called the holomorph of $G$, or simply the holomorph.*[5]

THEOREM 2. *The group of all permutations which preserve the ternary operation is the holomorph.*[6]

*Explanation*. A permutation $T$ preserves the ternary operation, by definition, if and only if $(abc)T = (aT\ bT\ cT)$. The group property follows from the general theorem that the set of all automorphisms of any algebra form a group, and the set in question is exactly that of the automorphisms with respect to the ternary operation.[7]

**3. Postulates for the ternary operation.** The reader will observe, as stated in the introduction, that we may consider the postulates given below as postulates for a group under the ternary operation. The first set is interesting, considered as postulates for a group, because it does not (explicitly) require the existence of either the identity or the inverse.[8] The other sets require only the existence of an identity. An analogous situation is that of generalized groups defined by the use of an $n$-ary function.[9] However, the postulates given below seem to be the simplest for the general case.

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[4] Cf. Baer, op. cit. Satz 3 (part 3). As may be seen, Baer's "schar" is simply a coset studied under the operation $ab^{-1}c$. By Theorem 1, and also by the fact that $sT = sTs^{-1}s$, where $sTs^{-1}$ is a subgroup if and only if $T$ is (and indeed equals $T$ if and only if $T$ is normalized by $s$), we see that the property of being a coset is intrinsic.

[5] Cf. Zassenhaus, op. cit. p. 46.

[6] Cf. Baer, op. cit. Satz 11 (part 3). An alternative proof would be to consider $T$ as given and define $d = eT$, $e$ the identity of $G$, and $\alpha$: $x \rightarrow (xT)d^{-1}$. Using $ab = (aeb)$, it is easy to verify that $(ab)\alpha = (a\alpha)(b\alpha)$, and that $T = \alpha T_d$. The converse is straightforward.

[7] Cf. Garrett Birkhoff, Proc. Cambridge Philos. Soc. vol. 31 (1935) p. 434.

[8] Cf. Rainich, *Note on group postulates*, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 81–84.

[9] Cf. E. L. Post, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 208–350.

We shall assume, unless otherwise stated, that the systems defined below are closed with respect to $(abc)$ and that they contain all elements under discussion.

DEFINITION 2. *Let $G$ be a set of elements on which there is defined a ternary operation $(abc)$ satisfying the following postulates:*

$$(3.1) \qquad A_1\colon ((abc)de) = (ab(cde)), \qquad A_2\colon (abb) = a, \qquad A_3\colon (bba) = a.$$

*We shall call $G$ an abstract coset.*

We shall not use these postulates directly but use a weakened yet equivalent set given below. The equivalence is a corollary to Theorem 3 proved below.

$$(3.2) \qquad \begin{aligned} &A_1\colon ((abc)de) = (ab(cde)). \\ &B\colon \text{There exists } u \text{ in the set satisfying (a) } (auu) = a, \text{ (b) } (aau) = u. \end{aligned}$$

THEOREM 3. *If $G$ is a set satisfying* (3.2) *and we define $ab = (aub)$, then $G$ becomes a group and $(abc) = ab^{-1}c$.*

PROOF. Closure is obvious.

In $A_1$ take $b = d = u$, and we get the associative law $(ac)e = a(ce)$.

By definition and B(a), it follows that $u$ is a right identity.

If $a$ is given, choose $x = (uau)$. Then $ax = (au(uau)) = ((auu)au) = (aau) = u$. It follows that $G$ is a group under the binary operation and hence $u$ is a left identity also, that is, $(uua) = ua = a$ for all $a$.

Finally, $(abc) = ((auu)bc) = (au(ubc)) = a(ubc) = a(ub(uuc)) = a((ubu)uc) = ab^{-1}c$.

COROLLARY 1. (3.1) *and* (3.2) *are equivalent.*

PROOF. The proof is obvious.

Thus we see that if $G$ satisfies (3.1), we may choose any element $u$ in $G$ and define a group $G_u$ with $u$ as its identity, and $ab = (aub)$ as its law of composition. However, the following corollary shows that we get essentially the same group no matter which element we choose for the identity.

COROLLARY 2. *The groups $G_u$ are isomorphic for all $u$ in $G$, an abstract coset. Moreover, $(abc) = ab^{-1}c$.*

PROOF. Consider $G_u$ and $G_v$. Define $T\colon x \rightarrow (xuv)$. $T$ is in the holomorph of $G_u$ with $\alpha$ the identity permutation. By Theorem 2 it follows that $(aub)T = (aT\ uT\ bT)$. But $uT = v$, which completes the proof.

*Remark.* $G$ may thus be considered either as a group or as an abstract coset. We could define the holomorph of an abstract coset as the group of all permutations preserving the ternary operation $(abc)$. This evidently coincides with the holomorph of $G$ (considered as a group) given by Definition 1.

Now we shall examine the system we get from (3.2) by weakening postulate $A_1$.

DEFINITION 3. *Let $G$ be a set of elements on which there is defined a ternary operation satisfying, for some $u$ in $G$,*

$$(3.3)\qquad \begin{aligned} &A\colon ((auc)de) = (au(cde)),\\ &B\colon (a)\ (auu) = a, \quad (b)\ (aau) = u. \end{aligned}$$

THEOREM 4. *If $G$ is a set satisfying Definition 3, and we define $ab = (aub)$, then $G$ becomes a group and the following properties are equivalent:*

$$T\colon (abc) = ab^{-1}c, \qquad A_1\colon ((abc)de) = (ab(cde)), \qquad A_2\colon (bba) = a.$$

PROOF. The fact that $G$ is a group follows from the proof of Theorem 3. It is obvious that T implies $A_1$ and $A_2$. But $A_1$ implies T by Theorem 3. It suffices to prove that $A_2$ implies T. But, by the proof of Theorem 3 again, $(abc) = a(ubc)$. Now $b(ubc) = (bbc) = c$ or $(ubc) = b^{-1}c$, so the result follows.

The following example shows that T does not hold for all sets satisfying (3.3).

*Example.* Let $G$ be the set of two elements $u$ and $a$ combined according to the following rules:

$$\begin{aligned} (uuu) &= (aau) = (aaa) = (aua) = u,\\ (auu) &= (uau) = (uua) = (uaa) = a. \end{aligned}$$

Obviously B is satisfied. The reader may check A, noting that it is unnecessary to check A for the first element equal to $u$ since $(uux) = x$ for all $x$ in $G$. But, since $(aaa) = u \neq a$, we see that $(xyz)$ is not in general equal to $xy^{-1}z$. Also, since any group of order 2 (and there is only one) satisfies Definition 3 with $(abc) = ab^{-1}c$, we see that this set of postulates is insufficient to determine the ternary operation in terms of the group operation.

*Remark.* It is obvious that if we replace B(b) by $A_2$ in (3.3) we get a set equivalent to (3.1) and (3.2).

DEFINITION 4. *$G$ is said to be commutative if $(abc) = (cba)$.*

THEOREM 5. *A necessary and sufficient condition that an abstract* *coset be commutative is that* $(abc)=(cba)$ *where* $b$ *is some fixed element and* $a$ *and* $c$ *are arbitrary.*

PROOF. Necessity is obvious. For the sufficiency, let $b$ be given as in the hypothesis and $b'$ be arbitrary. Consider $G_b$ and $G_{b'}$. $G_b$ is commutative by hypothesis. By Theorem 3, Corollary 2, it follows that $G_{b'}$ is also commutative, whence $(ab'c)=(cb'a)$.

*Remark.* An analogous theorem for sets satisfying Definition 3 does not hold. For if it did then $(ubc)=(cbu)=c(ubu)=cb^{-1}$, whence it would follow that $(abc)=ab^{-1}c$. Yet, in the above example, we have $(xuy)=(yux)$ for all $x$, $y$ and $(xyz)\neq xy^{-1}z$ in general. As a corollary to this remark we have proved that every commutative system satisfying Definition 3 is a (commutative) abstract coset. We also have complete associativity, that is, $((abc)de)=(a(bcd)e)=(ab(cde))$.[10] Thus any system of this type is not only a Prüfer schar[11] but also a Dörnte 3-group.[12] In fact, the commutative law, $A_1$ and $A_2$ ($A_3$ is then also true) constitute exactly Prüfer's set of postulates. To sum up, we may assert that *in the commutative case the three sets of postulates are equivalent to those of Prüfer and Dörnte and to each other.*

4. **Consistency and independence of the sets of postulates.** The consistency of these postulates was really proved in the example given above. Actually, any group with $(abc)=ab^{-1}c$ (or $cb^{-1}a$) will satisfy these postulates, with the possible exception of the commutative law. The latter is obviously satisfied if $G$ is commutative.

For the independence, we observe first that the commutative law is obviously independent of the others. To prove the other laws independent we find that a single set of systems will suffice for all three sets of postulates.

Consider the two element systems defined as follows:

$$(abb)=(aaa)=(aba)=(bba)=a,\ (bbb)=(baa)=(aab)=(bab)=b.\ (G_1).$$

$$(abb)=(aaa)=(aba)=(aab)=a,\ (baa)=(bbb)=(bab)=(bba)=b.\ (G_2).$$

Finally, let $G_3$ be derived from $G_2$ by defining $[a'b'c']=(c'b'a')$.

THEOREM 6. (3.1) *is a set of independent postulates.*

PROOF. Note first that the closure postulate is independent. But $G_1$ satisfies $A_2$ and $A_3$ by inspection. Yet $(ab(bab))=(abb)=a\neq b$

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[10] Pointed out by the referee.

[11] Cf. H. Prüfer, *Theorie der Abelschen Gruppen*, Math. Zeit. vol. 20 (1924) pp. 166–187.

[12] Cf. W. Dörnte, *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Zeit. vol. 29 (1928) pp. 1–19.

$=(aab)=((abb)ab)$ so that $A_1$ is independent of $A_2$ and $A_3$.

Now consider $G_3$ and observe that $(xyz)=x$ for all $x$, $y$, and $z$. This gives $A_1$ and $A_2$. However $A_3$ is not satisfied since $(aab)=a\neq b$.

We observe finally that the definition of $G_3$ from $G_2$ carries $A_1$ into itself and permutes $A_2$ and $A_3$. For instance $[[a'b'c']\,d'e']=(e'd'(c'b'a'))=((e'd'c')b'a')=[a'b'[c'd'e']]$. This proves $A_2$ is independent of $A_1$ and $A_3$.

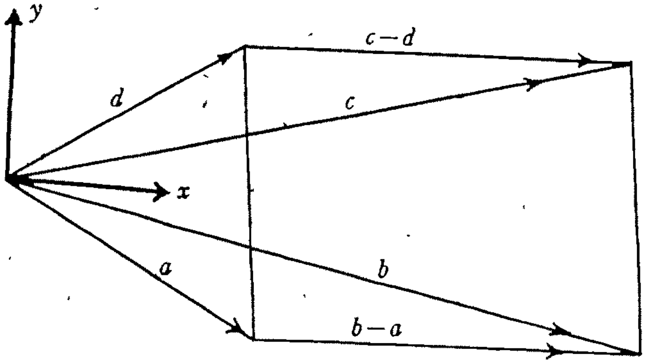
Corollary 1. (3.2) *is an independent set of postulates.*

Proof. We observe first that the systems are symmetric in $a$ and $b$. The corollary then follows immediately.

Corollary 2. (3.3) *is a set of independent postulates.*

Proof. By symmetry, and proof of Theorem 6, $G_1$ gives the independence of A. The remainder of the proof follows from the above corollary.

5. **Axioms for free vectors.** By a system of free vectors is usually meant a set of vectors which may be translated without changing their value. For example, the system of forces in physical space constitute such a set. Let us investigate the (commutative) abstract coset derived from the group of all points in the plane under addition, or even a subset which is a coset. We shall then have $(abc)=a-b+c=d$.



By means of the accompanying diagram we see that $d$ is simply the fourth vertex of a parallelogram whose vertices are described cyclicly as $a$, $b$, $c$, and $d$. For obviously $c-d=b-a$ or $d=a-b+c$. We may thus describe the closure of $G$ as the property of containing with any three points the fourth vertex of the cyclicly ordered parallelogram

$(a, b, c, d)$. This concept generalizes to any group by defining a parallelogram as the ordered quadruple $(a, b, c, ab^{-1}c)$. But we may also consider the difference $b-a$ as a vector (the vector from $a$ to $b$, say) and interpret the equality $c-d=b-a$ as the statement that parallel vectors of the same length and direction are equal. This is the same as saying that these vectors are free. We shall now define this concept in the abstract.

DEFINITION 5. *By a set of free vectors we mean all pairs of elements (points) of an arbitrary set, the pairs* $(a, b)$ *being connected by an equivalence relation (reflexive, symmetric and transitive relation) denoted by* $\sim$ *and satisfying*:

$V_1$: $(a, b)\sim(a', b')$ *implies* $(b, a)\sim(b', a')$.

$V_2$: $(a, c)\sim(a', c')$, $(b, c)\sim(b', c')$ *imply* $(a, b)\sim(a', b')$.

$V_3$: *Given* $(a, b)$ *and* $c$, *there exists a unique* $d$ *such that* $(a, b)\sim(d, c)$.

These postulates have a very simple interpretation in the light of the preceding paragraph. $V_1$ simply says that opposite sides of parallelograms are equal when their sense is taken into consideration. $V_2$ may be interpreted as a statement on congruent triangles and $V_3$ as a guarantee of the existence of a unique vector through a given point parallel to a given vector.

THEOREM 7. *Given any system of free vectors it is possible to define from them a group.*[13]

*Explanation.* We shall define as elements of the group the couples $ab$ which are respectively the classes of all pairs equivalent to $(a, b)$. We shall define $ab+cd=ah$, where $(d, c)\sim(h, b)$ and $h$ is given by $V_3$.

PROOF. We shall divide the proof into several parts.

(a) Given $(a, b)$ and $c$ there exists a unique $d$ such that $(a, b)\sim(c, d)$. By $V_3$, there exists a $d$(unique) such that $(b, a)\sim(d, c)$. Apply $V_1$.

(b) Addition is unique. To prove this we must show that if $(a, b)\sim(a', b')$ and $(c, d)\sim(c', d')$, then $ab+cd=a'b'+c'd'$. But consider $h, h'$ where $(d, c)\sim(h, b)$, $(d', c')\sim(h', b')$. We get by $V_1$ and hypothesis $(h, b)\sim(h', b')$. $V_2$ and hypothesis give $(a, h)\sim(a', h')$.

(c) $ab+bc=ac$. By definition, this sum is $ah$ where $(c, b)\sim(h, b)$. The reflexive law and $V_3$ imply $h=c$, the desired result.

(d) Now let $u$ be fixed. Then $uu$ is a right zero, for by $V_3$ an arbitrary vector $ab=cu$ for suitable $c$. Hence $ab+uu=cu+uu=cu=ab$. Again, by (a) $ab=ud$ for suitable $d$, whence by $V_1$, $ba=du$. Thus

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[13] In connection with this result, which shows that Definition 5 may be considered as a definition of a group, cf. B. A. Bernstein, Trans. Amer. Math. Soc. vol. 43 (1938) pp. 1–6 and H. Boggs and G. Y. Rainich, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 81–84.

$ab+ba=uu$, so $ba$ is a right negative. Since the associative law obviously follows from (c) and (a), the proof is complete.

In the commutative case the postulates may be reduced to a very simple form. We shall prove the following theorem.

THEOREM 8. *Any set of pairs of elements for which there is defined an equivalence relation satisfying the following postulates constitutes a commutative group of free vectors (under the definitions of Theorem* 7).

$V_0$: $(a, b)\sim(a', b')$ *implies* $(a, a')\sim(b, b')$.

$V_3$: *Given* $(a, b)$ *and* $c$ *there exists a unique* $d$ *such that* $(a, b)\sim(d, c)$.

PROOF. We first show that these form a system of free vectors. To prove $V_1$, let $(a, b)\sim(a', b')$. By $V_0$, we get $(a, a')\sim(b, b')$ or $(b, b')\sim(a, a')$. By $V_0$ again we have $(b, a)\sim(b', a')$. The hypotheses of $V_2$ give $(a, a')\sim(b, b')\sim(c, c')$, using $V_0$. Hence $(a, b)\sim(a', b')$. It remains only to prove that the group which we get is commutative. To this end consider $ab$ and $cd$ and choose $e$ such that $(b, e)\sim(c, d)$ and $f$ such that $(a, b)\sim(e, f)$. But $ab+cd=ae$ while $cd+ab=be+ef=bf$. Since $(a, e)\sim(b, f)$, the result follows.

The question which naturally arises is whether all systems of free vectors (as defined in Definition 5) are necessarily commutative, in which case we should surely use the conditions of Theorem 8. In fact *a necessary and sufficient condition that the group of free vectors be commutative is that* $V_0$ *hold.* Sufficiency being obvious, suppose $(a, b)\sim(a', b')$. Then $aa'=ab+ba'=ba'+ab=bb'$ since $(a, b)\sim(a', b')$. Hence we get $(a, a')\sim(b, b')$. To settle this question completely and to prove the consistency of these postulates, we now show how any group may be made into a system of free vectors.

THEOREM 9. *Any group $G$ with elements $a, b, \cdots$ may be converted into a group of free vectors $V$, and conversely. Moreover, $G$ and $V$ are isomorphic.*

PROOF. *Necessity.* Let $G$ be given and define $(a, b)\sim(c, d)$ if and only if $ab^{-1}=cd^{-1}$. This is obviously an equivalence relation. $V_3$ is also obvious. But $ab^{-1}=a'b'^{-1}$ implies $ba^{-1}=b'a'^{-1}$ ($V_1$); and $ac^{-1}=a'c'^{-1}$, $bc^{-1}=b'c'^{-1}$ imply $ab^{-1}=ac^{-1}(bc^{-1})^{-1}=a'c'^{-1}(b'c'^{-1})^{-1}=a'b'^{-1}$, which completes the proof.

*Sufficiency.* Let $V$ be given and let $u$ be any fixed element. Then the correspondence $a\rightleftarrows au$ is one-one between the elements of the proposed $G$ and the elements of $V$. We define a binary operation in $G$ by $ab\rightleftarrows au+bu$, thus making $G$ into a group isomorphic to $V$. Now, by Theorem 7, $a^{-1}\rightleftarrows ua$, and $ab^{-1}\rightleftarrows au+ub=ab$. Thus $ab^{-1}=cd^{-1}$ if and only if $(a, b)\sim(c, d)$.

As for the isomorphism, we need only consider the case where $V$ is defined from $G$. Obviously, the correspondence $ab^{-1}\rightleftarrows ab$ is one-one from $G$ onto $V$. Moreover, the correspondent of $ab+cd$ is $ah^{-1}$ where $dc^{-1}=hb^{-1}$ or $h^{-1}=b^{-1}cd^{-1}$. This establishes the isomorphism.

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# AN INVARIANT OF INTERSECTION OF TWO SURFACES

CHUAN-CHIH HSIUNG

1. **Introduction.** Projective invariants of several pairs of surfaces have been deduced and characterized geometrically by various authors.[1] In this paper we shall supplement their investigations by studying in ordinary space two surfaces intersecting at an ordinary point with distinct tangent planes.

In §2 we show by analysis the existence of a projective invariant determined by the neighborhood of the second order of the two surfaces at the point of intersection.

The final two sections are devoted to the presentation of projectively, as well as metrically, geometric characterizations of this invariant.

2. **Derivation.** Suppose that two surfaces $S_1$, $S_2$ in ordinary space intersect at an ordinary point $O$ with distinct tangent planes $\tau_1$, $\tau_2$, and let the common tangent $t$ be distinct from the asymptotic tangents. Let $t_1$, $t_2$ be the harmonic conjugate lines of $t$ with respect to the asymptotic tangents of the surfaces $S_1$, $S_2$, respectively, at the point $O$. If we choose the point $O$ to be the origin, the lines $t$, $t_2$, $t_1$ to be, respectively, the axes $x$, $y$, $z$ of a general nonhomogeneous projective coordinate system, then the power series expansions of the surfaces $S_1$, $S_2$ in the neighborhood of the point $O$ may be written in the form

$$S_1:\quad y = l_1x^2 + m_1z^2 + \cdots, \tag{1}$$

$$S_2:\quad z = l_2x^2 + m_2y^2 + \cdots. \tag{2}$$

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[1] See the bibliography at the end of the paper.

Let us now consider the most general projective transformation of coordinates which leaves the lines $t$, $t_1$, $t_2$ unchanged:

$$(3) \qquad \begin{cases} x = a_{22}x^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*), \\ y = a_{33}y^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*), \\ z = a_{44}z^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*), \end{cases}$$

where $a_{ik}$ are arbitrary constants. The effect of this transformation on equations (1), (2) is to produce two other equations of the same form whose coefficients, indicated by stars, are given by the formulas

$$(4) \qquad \begin{cases} a_{33}l_1^* = a_{22}^2 l_1, & a_{33}m_1^* = a_{44}^2 m_1, \\ a_{44}l_2^* = a_{22}^2 l_2, & a_{44}m_2^* = a_{33}^2 m_2, \end{cases}$$

from which it follows immediately that

$$(5) \qquad I = l_1^2 m_2/l_2^2 m_1$$

is an invariant determined by the neighborhoods of the second order of the two surfaces $S_1$, $S_2$ at the point $O$.

**3. A projectively geometric characterization.** Let us consider two pairs of planes, one of which is determined by the line $t_2$ and the asymptotic tangents of the surface $S_1$ at the point $O$, and the other by the line $t_1$ and the asymptotic tangents of the surface $S_2$ at the point $O$. The lines of intersection of these four planes lie on two planes passing through the line $t$:

$$(6) \qquad (l_1m_2)^{1/2}y \pm (l_2m_1)^{1/2}z = 0.$$

On the other hand, we observe that the plane sections of the surfaces $S_1$, $S_2$ made by any plane through the line $t$ have, in general, an ordinary contact at the point $O$. In particular, we can uniquely determine a plane through the line $t$ such that the plane sections have contact of the third order at the point $O$. The equation of this plane is easily found to be

$$(7) \qquad l_2y - l_1z = 0.$$

From equations (6), (7) it follows at once that the double ratio of the four planes $\tau_1$, $\tau_2$, (7), (6) is equal to

$$(8) \qquad (0, \infty, -l_1/l_2, \pm (l_2m_1/l_1m_2)^{1/2}) = \pm I^{1/2}.$$

Hence we obtain the following characterization of the invariant $I$:

*Suppose that two surfaces $S_1$, $S_2$ in ordinary space intersect at an ordinary point $O$ with distinct tangent planes $\tau_1$, $\tau_2$, and let the common* *tangent $t$ be distinct from the asymptotic tangents. Let $t_1$, $t_2$ be the harmonic conjugate lines of $t$ with respect to the asymptotic tangents of the surfaces $S_1$, $S_2$, respectively, at the point $O$; and let $\pi_1$ be one of the two planes through $t$ on which lie the lines of intersection of two pairs of planes, one of which is determined by $t_2$ and the asymptotic tangents of the surface $S_1$ at the point $O$, and the other by $t_1$ and the asymptotic tangents of the surface $S_2$ at the point $O$. If $\pi_2$ be the unique plane through the line $t$ such that the plane sections of the surfaces $S_1$, $S_2$ made by $\pi_2$ have contact of the third order at the point $O$, then the projective invariant $I$ associated with the surfaces $S_1$, $S_2$ at the point $O$ is equal to the square of the double ratio $(\tau_1\, \tau_2,\ \pi_2\, \pi_1)$.*

**4. A metrically geometric characterization.** It is deemed worthwhile to find a simple metrical characterization of the projective invariant $I$. For this purpose let us make a projective transformation which leaves the point $O$ unchanged and carries the lines $t$, $t_1$, $t_2$ into three mutually perpendicular lines $\bar t$, $\bar t_1$, $\bar t_2$. Let $\bar S_1$, $\bar S_2$ be the transformed surfaces of $S_1$, $S_2$, and $\bar x$, $\bar y$, $\bar z$ the nonhomogeneous Cartesian coordinates of a point in space referred to the orthogonal coordinate system determined by the lines $\bar t$, $\bar t_1$, $\bar t_2$; then the power series expansions of the surfaces $\bar S_1$, $\bar S_2$ in the neighborhood of the point $O$ may be written in the form

$$\bar S_1:\quad \bar y = \bar l_1 \bar x^2 + \bar m_1 \bar z^2 + \cdots, \tag{9}$$

$$\bar S_2:\quad \bar z = \bar l_2 \bar x^2 + \bar m_2 \bar y^2 + \cdots, \tag{10}$$

and the projective invariant $I$ takes the form

$$I = \bar l_1^3 \bar m_2 / \bar l_2^3 \bar m_1. \tag{11}$$

Let $\bar K_1$, $\bar K_2$ be the total curvatures of the surfaces $\bar S_1$, $\bar S_2$ at the point $O$; and $\bar R_1$, $\bar R_2$ the radii of curvature at the point $O$ of the curves in which the tangent planes $\bar z = 0$, $\bar y = 0$ intersect with the surfaces $\bar S_1$, $\bar S_2$ respectively. Then it can be demonstrated that

$$\begin{cases} \bar K_1 = 4\bar l_1 \bar m_1, & \bar K_2 = 4\bar l_2 \bar m_2, \\ \bar R_1 = 1/2\bar l_1, & \bar R_2 = 1/2\bar l_2, \end{cases} \tag{12}$$

and therefore that

$$I = \bar R_2^4 \bar K_2 / \bar R_1^4 \bar K_1. \tag{13}$$

Thus we obtain the following metrical characterization of the projective invariant $I$.

*Let $K_1$, $K_2$ be the total curvatures of the surfaces $S_1$, $S_2$ at the point $O$,* *and* $R_1$, $R_2$ *the radii of curvature at* $O$ *of the curves in which the tangent planes* $\tau_2$, $\tau_1$ *intersect the surfaces* $S_1$, $S_2$, *respectively. Then* $R_2^4K_2/R_1^4K_1$ *is the projective invariant* $I$ *of the surfaces* $S_1$, $S_2$ *above indicated.*

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# ON THE CONVERGENCE OF CERTAIN PARTIAL SUMS OF A TAYLOR SERIES WITH GAPS

GEORGE PIRANIAN

We consider the function $f(z)$ determined by the power series

$$f(z) = \sum_{1}^{\infty} c_n z^{\lambda_n} \tag{1}$$

and its direct analytic continuation. For simplicity, it is supposed that $\limsup |c_n|^{1/\lambda_n} = 1$.

We write

$$S_n(z) = \sum_{1}^{n} c_\nu z^{\lambda_\nu},$$

$$M(r) = \max_{|z|=r} |f(z)| \qquad (0 < r < 1),$$

$$M(r) = 1 \qquad (r \leqq 0),$$

$$\theta_n = \lambda_{n+1}/\lambda_n - 1.$$

Ostrowski has proved[1] that if $\{\theta_{n_i}\}$ is a sequence extracted from the sequence $\{\theta_n\}$ such that $\liminf \theta_{n_i} > 0$, then every regular point of $f(z)$ on the circle $|z| = 1$ is the center of a circle in which the sequence $\{S_{n_i}(z)\}$ converges uniformly to $f(z)$. Restricting ourselves to the question of convergence at the regular points themselves, we shall prove the following theorem:

*If*

$$\limsup_{i\to\infty} \frac{\log \left(M(1-\theta_{n_i}^2)/\theta_{n_i}\right)}{\lambda_{n_i}\theta_{n_i}^2} < \infty, \tag{2}$$

*then* $\lim S_{n_i}(z) = f(z)$ *at all regular points of* (1) *on the circle* $|z| = 1$.

For the proof, we shall assume that $\lim \theta_{n_i} = 0$; afterwards, we shall remove this restriction, with the aid of Ostrowski's theorem.

Let $z_1$ be a regular point for (1) on the circle $|z| = 1$, and let $z_0$ be a point on the segment joining $z_1$ to the origin. We write $|z_1 - z_0| = a$, and for every positive integer $i$ we define the three circles

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[1] A. Ostrowski, *Über eine Eigenschaft gewisser Potenzreihen mit unendlich vielen verschwindenden Koefficienten*. Preuss. Akad. Wiss. Sitzungsber. vol. 34 (1921) pp. 557–565. Essentially the same proof is to be found in P. Montel's *Leçons sur les familles normales de fonctions analytiques et leurs applications*, pp. 204–207.

$$\Gamma_i' : \left| z - z_0 \right| = \rho_i' \qquad (0 < \rho_i' < a),$$

$$\Gamma(a) : \left| z - z_0 \right| = a, \quad \Gamma_i'' : \left| z - z_0 \right| = \rho_i'' \qquad (\rho_i'' > a),$$

where $\rho_i''$ shall be chosen so that the function $f(z)$ is holomorphic in the closed region bounded by $\Gamma_i''$. By $M_i'$, $M_i(a)$, and $M_i''$ we denote the maximum value of $|f(z) - S_{n_i}(z)|$ on $\Gamma_i'$, $\Gamma(a)$, and $\Gamma_i''$, respectively. We write

$$R_i' = 1 + \rho_i' - a, \qquad R_i'' = 1 + \rho_i'' - a,$$

and we choose $r_i$ so that $R_i' < r_i < 1$.

By Cauchy's formula we have, on the circle $|z| = R_i'$,

$$\left| f(z) - S_{n_i}(z) \right| \leqq \sum_{p=\lambda_{n_i}+1}^{\infty} \frac{M(r_i)}{r_i^p} R_i'^p = \frac{M(r_i) r_i}{r_i - R_i'} \left( \frac{R_i'}{r_i} \right)^{\lambda_{n_i}(1+\theta_{n_i})},$$

and by the principle of the maximum

$$\log M_i' \leqq \log \frac{M(r_i) r_i}{r_i - R_i'} + \lambda_{n_i}(1 + \theta_{n_i}) \log \frac{R_i'}{r_i} \cdot$$

On the circle $|z| = R_i''$, and therefore also on $\Gamma_i''$, we have

$$\left| S_{n_i}(z) \right| \leqq M(r_i) \sum_{0}^{\lambda_{n_i}} \left( \frac{R_i''}{r_i} \right)^p < \frac{M(r_i) R_i''}{R_i'' - r_i} \left( \frac{R_i''}{r_i} \right)^{\lambda_{n_i}}.$$

Since $f(z)$ is holomorphic in the closed region bounded by $\Gamma_i''$, and since at least one of the expressions $R_i''/(R_i'' - r_i)$, $(R_i''/r_i)^{\lambda_{n_i}}$ tends to $\infty$ as $i \to \infty$, we may write, for any positive $\eta$ and for $i$ sufficiently large,

$$\log M_i'' < \log \frac{M(r_i)(1 + \eta) R_i''}{R_i'' - r_i} + \lambda_{n_i} \log \frac{R_i''}{r_i} \cdot$$

Applying Hadamard's three-circle theorem to the function $S_{n_i}(z) - f(z)$ on the circles $\Gamma_i'$, $\Gamma(a)$, $\Gamma_i''$, we have now

$$\begin{aligned} \log \frac{\rho_i''}{\rho_i'} \log M_i(a) &\leqq \log \left( \frac{\rho_i''}{a} \right) \log M_i' + \log \left( \frac{a}{\rho_i'} \right) \log M_i'' \\ &< \log \left( \frac{\rho_i''}{a} \right) \left\{ \log \frac{M(r_i) r_i}{r_i - R_i'} + \lambda_{n_i}(1 + \theta_{n_i}) \log \frac{R_i'}{r_i} \right\} \qquad (3) \\ &\quad + \log \left( \frac{a}{\rho_i'} \right) \left\{ \log \frac{M(r_i)(1 + \eta) R_i''}{R_i'' - r_i} + \lambda_{n_i} \log \frac{R_i''}{r_i} \right\}. \end{aligned}$$

In the proof of his theorem Ostrowski now takes the point $z_0$ near to the point $z_1$ and chooses for $\Gamma'_i$, $\Gamma(a)$, and $\Gamma''_i$ three fixed circles with radii sufficiently near to $|z_1-z_0|$ (with $\Gamma(a)$ including the point $z_1$ instead of passing through it). In our case it is necessary to take $z_0$ near to the origin and to let $\rho'_i$ and $\rho''_i$ tend to $a$ as $i$ becomes large. We choose $\rho'_i=(1-b_i)a$, $\rho''_i=(1+b_i)a$, $r_i=1-k_i$, where $0<b_i<1$ and $0<k_i<ab_i$; substituting these values in (3), expanding terms such as $\log(1+ab_i)$ in power series, and dividing both sides of the inequality by $b_i$, we get

$$\begin{aligned}
&2(1+b_i^2/3+\cdots)\log M_i(a) < (1-b_i/2+b_i^2/3-\cdots)\\
&\qquad\cdot\left\{\log\frac{M(1-k_i)(1-k_i)}{(1-k_i/ab_i)ab_i}+\lambda_{n_i}(1+\theta_{n_i})\log\frac{1-ab_i}{1-k_i}\right\}\\
&\quad+(1+b_i/2+b_i^2/3+\cdots)\cdot\\
&\qquad\cdot\left\{\log\frac{M(1-k_i)(1-k_i)(1+\eta)}{(1+k_i/ab_i)ab_i}+\lambda_{n_i}\log\frac{1+ab_i}{1-k_i}\right\}\\
&< 2\eta+3k_i+3k_i/ab_i-3\log a\\
&\quad+2(1+b_i^2/3+\cdots)\log\frac{M(1-k_i)}{b_i}\\
&\quad+\lambda_{n_i}\{\theta_{n_i}[1-b_i/2+b_i^2/3-\cdots]\\
&\qquad\cdot[-ab_i-a^2b_i^2/2-\cdots+k_i+k_i^2/2+\cdots]\\
&\quad+[1-b_i/2+b_i^2/3-\cdots][-ab_i-a^2b_i^2/2-\cdots]\\
&\quad+[1+b_i/2+b_i^2/3+\cdots][ab_i-a^2b_i^2/2+\cdots]\\
&\quad+2[1+b_i^2/3+\cdots][k_i+k_i^2/2+\cdots]\}\\
&< 2\eta+3k_i+3k_i/ab_i-3\log a+3\log\frac{M(1-k_i)}{b_i}\\
&\quad-(\lambda_{n_i}/2)\{ab_i[\theta_{n_i}-b_i(1-a)]-3k_i\}
\end{aligned}\tag{4}$$

provided $\theta_{n_i}$ and $b_i$ are sufficiently small and the sum of the terms in the braces of the last term is positive.

Now suppose that

$$\limsup_{i\to\infty}\frac{\log(M(1-\theta_{n_i}^2)/\theta_{n_i})}{\lambda_{n_i}\theta_{n_i}^2}<H\qquad(1<H<\infty).$$

Choose $b_i=b\theta_{n_i}$, $k_i=b_i\theta_{n_i}/b=\theta_{n_i}^2$, where $b>14H$. The last member of (4) becomes

$$-(\lambda_{n_i}\theta_{n_i}^2b/2)\{a[1-b(1-a)]-3/b\}.$$

If $a$ is chosen sufficiently near to 1, $b\{a[1-b(1-a)]-3/b\}>10H$, and for sufficiently large values of $i$ (4) becomes

$$(2+\epsilon)\log M_i(a) < K(a,b) + 3\log M(1-\theta_{n_i}^2)/\theta_{n_i} - 5\lambda_{n_i}\theta_{n_i}^2 H,$$

where $K(a, b)$ depends on $a$ and $b$ only. But $\lim_{i\to\infty}\theta_{n_i}=0$ implies, together with the validity of (2), that $\lim_{i\to\infty}\lambda_{n_i}\theta_{n_i}^2=\infty$. For sufficiently large $i$ we have

$$(2+\epsilon)\log M_i(a) < -\lambda_{n_i}\theta_{n_i}^2 H,$$

that is,

$$\lim_{i\to\infty}\log M_i(a) = -\infty,$$

and, in particular,

$$\lim_{i\to\infty} S_{n_i}(z_1) = f(z_1).$$

Now let $\{\theta_{n_i}\}$ be any sequence of values $\theta_{n_i}$ for which (2) is satisfied, and let $z_1$ be a regular point for (1). From every subsequence of $\{\theta_{n_i}\}$ we can extract a further subsequence $\{\theta_{m_j}\}$ such that either $\lim\theta_{m_j}=0$ or $\liminf\theta_{m_j}>0$. In the first case, $\lim S_{m_j}(z_1)=f(z_1)$ by what we have just proved; in the second case, by Ostrowski's theorem. From every subsequence of $\{n_i\}$ we can therefore extract a further subsequence $\{m_j\}$ such that $\lim S_{m_j}(z_1)=f(z_1)$. It follows that the sequence $\{S_{n_i}(z_1)\}$ itself tends to $f(z_1)$, and our theorem is proved.

Condition (2) may be replaced by one that is somewhat less general, but can be expressed more immediately in terms of the behavior of $M(r)$:

If $\liminf\theta_{n_i}=\theta$, we define, for $h>\theta$,

$$\lambda(h) = \operatorname*{g.l.b.}_{\theta_{n_i}\leqq h}\lambda_{n_i};$$

for $0\leqq h\leqq\theta$, we write $\lambda(h)=\infty$. It follows from our theorem that $\lim S_{n_i}(z_1)=f(z_1)$ whenever $z_1$ is a regular point for (1) on the circle $|z|=1$, provided

$$\limsup_{h\to 0+}\frac{\log(M(1-h^2)/h)}{h^2\lambda(h)} < \infty. \tag{5}$$

If we write $r=1-h^2$, (5) becomes

$$\limsup_{r\to 1-0}\frac{\log[M(r)/(1-r)^{1/2}]}{(1-r)\lambda(1-r)^{1/2}} < \infty.$$

The author is indebted to Professor S. Mandelbrojt, who suggested the existence of the present theorem and gave valuable assistance in its development.

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# ON ABEL AND LEBESGUE SUMMABILITY

OTTO SZÁSZ

1. **Introduction.** A series $\sum_1^\infty a_n$ is called Abel summable to the value $s$ if the power series $\sum a_n r^n$ converges for $0<r<1$, and if $\sum a_n r^n \to s$ as $r \uparrow 1$; it is called Lebesgue summable if the sine series

$$\sum_1^\infty a_n \frac{\sin nt}{n} = F(t) \tag{1.1}$$

converges in some interval $0<t<\tau$, and if

$$t^{-1}F(t) \to s \quad \text{as} \quad t \downarrow 0. \tag{1.2}$$

We write in the first case $A\sum a_n = s$, and in the latter case $L\sum a_n = s$ (summability $A$ or $L$ respectively). It is known that convergence does not imply $L$-summability and conversely $L$-summability does not imply convergence of $\sum a_n$. Tauberian type problems which arise out of this situation have been discussed.[1] It is also known that either convergence or $L$-summability imply $A$-summability. As to the converse (restricting ourselves to real $a_n$) we have proved the following theorems:

THEOREM 1. [8, pp. 582–583]. *If*

$$\sum_n^{2n} (|a_\nu| - a_\nu) = O(1) \quad as \quad n \to \infty, \tag{1.3}$$

*and if*

$$\sum_1^\infty a_n r^n = O(1) \quad as \quad r \uparrow 1, \tag{1.4}$$

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[1] See [8], where further references are given; numbers in brackets refer to the bibliography at the end of this paper.

*then*

$$(1.5)\qquad t^{-1}F(t) = O(1) \quad \text{as} \quad t\downarrow 0.$$

THEOREM 2. [8, p. 585]. *If* (1.3) *holds and if*

$$(1.6)\qquad \lim_{\lambda\downarrow 1}\ \liminf_{n\to\infty}\ \min_{n\leq k\leq \lambda n}\ \sum_{n}^{k} a_\nu \geq 0,$$

*then A-summability implies L-summability.*

Note that $A$-summability and (1.6) (without (1.3), which need not be satisfied) imply convergence (by a theorem of R. Schmidt) and are also necessary for convergence, while the series need not be $L$-summable.

We remark also that, in the assumption and in the conclusion of Theorem 1, $O(1)$ can be replaced by $o(1)$; for if

$$(1.7)\qquad \sum_{n}^{2n} (|a_\nu| - a_\nu) = o(1) \quad \text{as} \quad n\to\infty,$$

then (1.6) holds. Moreover by the previous remark the series $\sum a_n$ converges (to zero).

We shall complete and generalize these results by proving the following theorems:

THEOREM 3. *If* (1.3) *holds then each of the statements* (1.4), (1.5) *and*

$$(1.8)\qquad \sum_{1}^{n} a_\nu = O(1) \quad \text{as} \quad n\to\infty$$

*implies the two others.*

THEOREM 4. *If* (1.3) *holds then A-summability implies L-summability, but not necessarily convergence.*

THEOREM 5. *If* (1.3) *holds and if* $\sum a_n$ *converges, then* $\sum a_n \sin nt/nt$ *converges uniformly in* $0<t<\pi$.

This generalizes Theorem 6′ of my paper [8].

2. **Proof of Theorem 3.** We prove the following lemma.

*Lemma* 1. *If* (1.3) *and* (1.4) *hold, then*

$$(2.1)\qquad s_n = \sum_{1}^{n} a_\nu = O(1), \qquad \sum_{n}^{2n} |a_\nu| = O(1), \qquad \sum_{1}^{n} \nu |a_\nu| = O(n),$$

$$(2.2)\qquad \sum_{1}^{\infty} \nu^{-1} |a_\nu| < \infty, \qquad \sum_{n}^{\infty} \nu^{-1} |a_\nu| = O(n^{-1}) \quad \text{as} \quad n\to\infty.$$

The statement $s_n = O(1)$ is an immediate corollary of a previous result [6, Lemma 2]. Combining it with (1.3) we get

$$\sum_n^{2n} |a_\nu| = \sum_n^{2n} (|a_\nu| - a_\nu) + s_{2n} - s_{n-1} = O(1) \quad \text{as} \quad n \to \infty.$$

Furthermore, where $\sum_\alpha^\beta$ means summation over the range $\alpha < \nu \leqq \beta$,

$$\sum_1^n \nu |a_\nu| = \sum_{k=0}^n \sum_{n/2^{k+1}}^{n/2^k} \nu |a_\nu| \leqq \sum_{k=0}^n \left( \frac{n}{2^k} \sum_{n/2^{k+1}}^{n/2^k} |a_\nu| \right)$$
$$= O\left( n \sum_0^\infty 2^{-k} \right) = O(n).$$

(2.1) is now proved. We have thus $\sum_n^{2n} |a_\nu| < c$, a positive constant, and $\sum_n^{2n} \nu^{-1} |a_\nu| < c/n$, hence

$$\sum_1^n \nu^{-1} |a_\nu| \leqq \sum_{k=1}^n \sum_{2^{k-1}}^{2^k} \nu^{-1} |a_\nu| < c \sum_{k=1}^\infty 2^{1-k} = 2c.$$

This proves the first part of (2.2). Finally

$$\sum_n^\infty \nu^{-1} |a_\nu| \leqq \sum_{k=1}^\infty \sum_{n \cdot 2^{k-1}}^{n \cdot 2^k} \nu^{-1} |a_\nu| < \frac{c}{n} \sum_1^\infty 2^{1-k} = \frac{2c}{n},$$

which proves the lemma.

We now prove Theorem 3. If (1.3) holds, then (1.8) implies (1.5) by Theorem 5 of my paper [8], and (1.4) follows from the remark to the same theorem. By the same remark (1.4) implies (1.8), hence also (1.5). Finally, assuming (1.5), to prove (1.8) we write

$$t^{-1}F(t) - s_n = \sum_1^n a_\nu \left( \frac{\sin \nu t}{\nu t} - 1 \right) + \sum_{n+1}^\infty a_\nu \frac{\sin \nu t}{\nu t} = S_1 + S_2.$$

From $0 < 1 - \sin \nu t / \nu t < \nu^2 t^2$ we get

$$|S_1| < t^2 \sum_1^n \nu^2 |a_\nu| < n t^2 \sum_1^n \nu |a_\nu| = t^2 O(n^2);$$

furthermore, by Lemma 1,

$$|S_2| < t^{-1} \sum_n^\infty \nu^{-1} |a_\nu| = O(n^{-1} t^{-1}).$$

On putting now $t = n^{-1}$ we get

$$n F(n^{-1}) - s_n = O(1) \quad \text{as} \quad n \to \infty;$$

this proves (1.8) and a fortiori (1.4), which completes the proof of Theorem 3.

3. **Proof of Theorem 4.** We first prove the following lemmas.

LEMMA 2. *Let*

$$\Delta_n = \sin nt/nt - \sin (n+1)t/(n+1)t,$$
$$\Delta_n^2 = \Delta(\Delta_n) = \sin nt/nt - 2 \sin (n+1)t/(n+1)t + \sin (n+2)t/(n+2)t;$$

*then*

(3.1) $$0 < \Delta_n^2 < t^2 \quad \textit{for} \quad (n+2)t < \pi/2,$$

(3.2) $$|\Delta_n| < 2/n \quad \textit{for} \quad nt > 1.$$

Applying the mean value theorem to $\Delta^2$ we get easily (see [8, Lemma 4])

$$0 < \Delta_n^2 < t^2 \quad \text{for} \quad (n+2)t < \pi/2.$$

Furthermore

$$\Delta_n = \frac{\sin (n+1)t}{n(n+1)t} - 2\,\frac{\sin (t/2) \cos ((2n+1)t/2)}{nt},$$

which yields

$$|\Delta_n| < 1/n(n+1)t + 1/n < 2/n \qquad \text{for } nt > 1.$$

LEMMA 3. *If $\sum a_n$ is Abel summable and if* (1.3) *holds, then $\sum a_n$ is Cesàro summable of any order $\alpha > 0$.*

By Lemma 1, $s_n = O(1)$; this and $A$-summability imply $(C, 1)$ summability, as was proved first by Littlewood in 1910. For a short proof (with a more general assumption) cf. [5]. That Abel summability and $s_n = O(1)$ imply $(C, \alpha)$ summability for any $\alpha > 0$ has been proved by Andersen [1, p. 80]. We shall apply only the case $\alpha = 1$.

Let now $\sum_1^n s_\nu = s_n'$, then $n^{-1}s_n'$ tends to a limit $s$; we can assume without loss of generality that $s = 0$ (otherwise replace $a_1$ by $a_1 - s$). To a given positive $\epsilon < 1/2$ we now choose $n_0(\epsilon)$ so that

(3.3) $$|s_n'| < \epsilon^2 n \quad \text{for} \quad n > n_0(\epsilon) > 3.$$

By (2.2) $\sum \nu^{-1}a_\nu \sin \nu t$ converges absolutely; we write

$$t^{-1}F(t) = \sum_1^\infty a_\nu \frac{\sin \nu t}{\nu t} = \sum_1^n + \sum_{n+1}^\infty = T_1 + T_2.$$

We restrict ourselves to $0<t<n_0^{-1}$, and choose $n=1+[\epsilon^{-1}t^{-1}]>\epsilon^{-1}t^{-1}>\epsilon^{-1}n_0>2n_0$; Abel's summation by parts yields

$$T_1 = s_n \frac{\sin nt}{nt} + s'_{n-1}\Delta_{n-1} + \sum_1^{n-2} s'_\nu \Delta_\nu^2.$$

Now $nt>\epsilon^{-1}$. Hence

$$(3.4)\qquad |s_n \sin nt/nt| < |s_n|/nt < \epsilon|s_n| = \epsilon O(1) \quad \text{as} \quad t\downarrow 0,$$

and, from (3.2) and (3.3),

$$(3.5)\qquad |s'_{n-1}\Delta_{n-1}| < 2\epsilon^3;$$

furthermore

$$(3.6)\qquad |T_2| < t^{-1}\sum_n^\infty \nu^{-1}|a_\nu| = O(n^{-1}t^{-1}) = O(\epsilon) \quad \text{as} \quad t\downarrow 0.$$

Finally, write

$$\sum_1^{n-2} s'_\nu\Delta_\nu^2 = \left(\sum_1^{k-1} + \sum_k^{n-2}\right) s'_\nu\Delta_\nu^2, \qquad 2 \leqq k \leqq n-2,$$

and choose

$$k = 1 + [t^{-1}] > t^{-1} > n_0(\epsilon) > 3.$$

By (3.1), as $(k+1)t<(2+t^{-1})t<3/2<\pi/2$,

$$(3.7)\qquad \left|\sum_1^{k-1} s'_\nu\Delta_\nu^2\right| < t^2 \sum_1^k |s'_\nu| = o(t^2k^2) = o(1).$$

It remains to estimate $\sum_k^{n-2} s'_\nu\Delta_\nu^2$. We decompose this sum according to the changes of sign of the factors $\Delta_\nu^2$, and write

$$\sum_k^{n-2} s'_\nu\Delta_\nu^2 = \sum_1 + \sum_2 + \cdots + \sum_\rho.$$

To estimate $\rho$ we note that there are not more changes of sign in the sequence $\Delta_\nu^2$ than there are zeros $x_1, x_2, \cdots$ of $D_2(x^{-1}\sin x)$ in the interval $0<x<(n-1)t$. A simple calculation yields for $x_\nu$ the estimate

$$x_\nu = (\nu+1)\pi - \psi_\nu, \quad 0<\psi_\nu<\pi/3, \quad \nu = 1, 2, 3, \cdots;$$

hence,

$$\rho\pi < x_\rho < (n-1)t < \epsilon^{-1}.$$

But each $\sum$ is in absolute value less than $4\epsilon^2nk^{-1}$ (from (3.2) and (3.3)), and

$$\epsilon^2 n k^{-1} < \epsilon^2 n t < 2\epsilon^2;$$

thus

$$\left| \sum_{k}^{n-2} s_\nu' \Delta_\nu^2 \right| < 2\rho\epsilon^2 < \epsilon. \tag{3.8}$$

Collecting the estimates (3.4) to (3.8) we find

$$\left| t^{-1}F(t) \right| < \epsilon O(1) + o(1) \quad \text{as} \quad t \downarrow 0;$$

$\epsilon$ being arbitrarily small the positive part of Theorem 4 follows. For the negative part we refer to the examples in §5.

4. **Proof of Theorem** 5. We write, for $\lambda > 1$,

$$\sum_{n+1}^{\infty} a_\nu \frac{\sin \nu t}{\nu t} = \sum_{n+1}^{\lambda n} + \sum_{\nu > \lambda n} = R_1 + R_2,$$

say; then by (2.2)

$$\left| R_2 \right| < t^{-1} \sum_{\nu > \lambda n} \nu^{-1} \left| a_\nu \right| = \frac{1}{\lambda n t} O(1).$$

Abel's summation by parts yields

$$\sum_{1}^{n} a_\nu \frac{\sin \nu t}{\nu t} = s_n \frac{\sin nt}{nt} + \sum_{1}^{n-1} s_\nu \Delta_\nu,$$

whence

$$\sum_{n+1}^{n+k} a_\nu \frac{\sin \nu t}{\nu t} = s_{n+k} \frac{\sin (n+k)t}{(n+k)t} - s_n \frac{\sin nt}{nt} + \sum_{n}^{n+k-1} s_\nu \Delta_\nu.$$

We may assume that the limit of $s_n$ is zero; given $\epsilon > 0$, we choose $n_0(\epsilon)$ so that $|s_n| < \epsilon^3$ for $n > n_0$; then

$$\left| s_{n+k} \frac{\sin (n+k)t}{(n+k)t} - s_n \frac{\sin nt}{nt} \right| < 2\epsilon^3 \quad \text{for} \quad n > n_0(\epsilon).$$

We define $k$ by $n+k = [\lambda n]$, thus $k = [\lambda n] - n \leqq (\lambda - 1)n$. We subdivide the range $n \leqq \nu < \lambda n$ into consecutive parts in each of which $\Delta_\nu$ has constant sign; denote the number of subdivisions by $\sigma$. Denoting the positive zeros of $u^{-1} \sin u$ by $u_1 < u_2 < \cdots$, we find easily $u_\nu = \nu\pi + \alpha_\nu$, where $0 < \alpha_\nu < \pi/2$; the number of zeros in the interval $nt < u < \lambda nt$ is therefore less than $2\lambda nt/\pi$, and

$$\sigma \leqq \lambda nt + 2.$$

In each section $\left|\sum s_\nu \Delta_\nu\right| < 2\epsilon^3$, hence

$$\left|\sum_{n}^{n+k-1} s_\nu \Delta_\nu\right| < 2\epsilon^3(2 + \lambda n t),$$

and

$$|R_1| < 2\epsilon^3(3 + \lambda n t).$$

We now choose $\lambda = 1/\epsilon^2 n t$, for whatever $n > n_0(\epsilon)$ and any $0 < t < \pi$, if $\epsilon^2 n t < 1$, and put $\lambda = 1$ (that is $R_1 = 0$) otherwise. In the latter case $\left|\sum_{n+1}^{\infty} a_\nu \sin(\nu t)/\nu t\right| < (nt)^{-1} O(1) < \epsilon^2 O(1)$, while in the first case

$$\left|\sum_{n+1}^{\infty} a_\nu \frac{\sin \nu t}{\nu t}\right| < \epsilon^2 O(1) + 2\epsilon^3\left(3 + \frac{1}{\epsilon^2}\right) < \epsilon O(1)$$

for $n > n_0(\epsilon)$ and $0 < t < \pi$. This proves our theorem.

Note that convergence of $\sum a_n$ is a necessary condition for the uniform convergence of $\sum a_n \sin(nt)/nt$. For if, for any $\epsilon > 0$,

$$\left|\sum_{n+1}^{n+k} a_\nu \frac{\sin \nu t}{\nu t}\right| < \epsilon \quad \text{for} \quad n > n_0(\epsilon), \quad k = 1, 2, 3, \cdots, 0 < t < \pi,$$

then, letting $t \downarrow 0$ we get $\left|\sum_{n+1}^{n+k} a_\nu\right| \leqq \epsilon$. Moreover we have uniform convergence in the closed interval.

It is shown easily that the assumption (1.3) is equivalent to either of the following conditions: There exists a constant $\lambda > 1$ such that

$$\sum_{n}^{\lambda n} (|a_\nu| - a_\nu) = O(1); \tag{4.1}$$

$$\sum_{1}^{n} \nu(|a_\nu| - a_\nu) = O(n), \quad \text{as} \quad n \to \infty. \tag{4.2}$$

For a more general statement see [7, p. 129].

A consequence of our results is the following theorem:

THEOREM 6. *If*

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_{n}^{\lambda n} (|a_\nu| - a_\nu) = 0, \tag{4.3}$$

*then A-summability of* $\sum a_n$ *implies uniform convergence of the series* $\sum a_n \sin(nt)/nt$ *in* $0 < t < \pi$.

Clearly (4.3) implies (4.1), whence (1.3). Now, by Theorem 4, $\sum a_n$ is $L$-summable; furthermore by Theorem 4 of our paper [8] $L$-summability and (4.3) imply convergence of $\sum a_n$. Theorem 6 now follows from Theorem 5.

5. **Negative results.** We quote the following lemma.

LEMMA 4. *Let* $n \geqq 1$ *and*

$$P_n(z) = \frac{1}{n} + \frac{z}{n-1} + \cdots + \frac{z^{n-1}}{1} - \frac{z^n}{1} - \cdots - \frac{z^{2n-1}}{n};$$

*then, when* $|z| \leqq 1$,

$$|P_n(z)| < 6.$$

For the proof see Fejér [2, pp. 36–37].

Consider the polynomial series $\sum_1^\infty n^{-2} z^{\lambda_n} P_{k_n}(z)$, where $\lambda_1 = 1$, $k_1 = 3$, $2\lambda_n = 2^{n^2}$, $2k_n = \lambda_{n+1} - \lambda_n$, $n \geqq 2$. In view of the above lemma the series converges uniformly in $|z| \leqq 1$, so that the function

$$F(z) = \sum_1^\infty n^{-2} z^{\lambda_n} P_{k_n}(z)$$

is regular in $|z| < 1$ and continuous in $|z| \leqq 1$. The degree of the $n$th term is $2k_n + \lambda_n - 1 < \lambda_{n+1}$, hence writing out the polynomials explicitly we get a power series, convergent for $|z| < 1$,

$$F(z) = \sum a_n z^n. \tag{5.1}$$

For $|z| = 1$ we get a Fourier power series of a *continuous* function $F(e^{it})$. The structure of $P_n$ and the inequality $(n+1)^{-2} \log k_n < \log 2$ easily yield

$$\sum_n^{2n} |a_\nu| = O(1) \quad as \quad n \to \infty.$$

But $\sum a_n$ diverges, as there are sections $\sum a_\nu = n^{-2} \sum_1^{k_n} 1/\nu$ which do not tend to zero. On the other hand the series (5.1) is evidently $L$-summable at every point on $|z| = 1$.

Next we define a series $\sum a_n$ by putting $s_n = 1$ for $n = 2^k$, $k = 0, 1, 2, \cdots$, and $s_n = 0$ otherwise. Now $n^{-1} \sum_1^n s_\nu \to 0$, moreover $\sum_n^{2n} |a_\nu| \leqq 3$, hence the series is summable $L$. But $\sum a_n$ diverges, in fact $\limsup |a_n| = 1$, and $\sum a_n \cos nt$ is not a Fourier series.

Another example of this kind is due to Neder [4].

In contrast Menchoff [3] tried to prove that $A$-summability and (1.3) imply convergence of $\sum a_n$; the error lies in his Lemma 4 which is false. It is based on a false interpretation of an argument used by Landau.

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# ON SUBHARMONIC FUNCTIONS

MAXWELL READE

If $p(x, y)$ is continuous in a domain (non-null connected open set) $\mathfrak{D}$, then $p(x, y)$ is subharmonic in $\mathfrak{D}$ if and only if the inequality

$$p(x, y) \leqq A(p; x, y; r) \equiv \frac{1}{\pi r^2} \int\int_{D(x,y;r)} p(x + \xi, y + \eta) d\xi d\eta$$

holds for all circular discs $D(x_0, y_0; r): (x - x_0)^2 + (y - y_0)^2 = \xi^2 + \eta^2 \leqq r^2$ in $\mathfrak{D}$. If $p(x, y)$ is continuous along with its partial derivatives of the second order in $\mathfrak{D}$, then $p(x, y)$ is subharmonic there if and only if $\Delta p(x, y) \geqq 0$ in $\mathfrak{D}$, where $\Delta$ is the Laplace operator [2, 3].[1]

If $p(x, y)$ is continuous in $\mathfrak{D}$, then $p(x, y)$ is said to be of class $PL$ [2] in $\mathfrak{D}$ provided (i) $p(x, y) \geqq 0$ and (ii) $\log p(x, y)$ is subharmonic wherever $p(x, y) \neq 0$. If $p(x, y) \geqq 0$ and is continuous along with its partial derivatives of the second order, then $p(x, y)$ is of class $PL$ if and only if $p\Delta p - p_x^2 - p_y^2 \geqq 0$ wherever $p(x, y) \neq 0$.

Beckenbach [1] has proved the following theorem characterizing functions of class $PL$.

THEOREM A. *If $p(x, y) \geqq 0$ in $\mathfrak{D}$, then $p(x, y)$ is of class $PL$ in $\mathfrak{D}$ if and only if $[(x-\alpha)^2+(y-\beta)^2]p(x, y)$ is subharmonic in $\mathfrak{D}$ for every choice of the real constants $\alpha$, $\beta$.*

The Beckenbach theorem is comparable to the classic Montel-Radó theorem, which was later generalized by Kierst and Saks [3, 4] for functions $p(x, y)$ with continuous partial derivatives of the second order; this generalization is the following theorem.

THEOREM B. *Let $f(t)$ have a continuous second derivative, with $f'(t) > 0$, for $-\infty < t < \infty$. If $v(x, y)$ has continuous partial derivatives of the second order in $\mathfrak{D}$, and if the function $f(\alpha x + \beta y + v(x, y))$ is subharmonic in $\mathfrak{D}$ for every choice of the real constants $\alpha$, $\beta$, then $v(x, y)$ is subharmonic in $\mathfrak{D}$.*

The question arises as to the possibility of exhibiting a Kierst-Saks type of generalization for Beckenbach's Theorem A. Our result is the following.

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[1] The numbers in square brackets refer to references listed in bibliography at end of paper.

THEOREM C. *Let* $f(t)$ *have a continuous second derivative, with* $f'(t) > 0$, *for* $-\infty < t < \infty$. *If* $v(x, y)$ *has continuous partial derivatives of the second order in* $\mathfrak{D}$, *and if the function*

$$\Phi(v; \alpha, \beta; x, y) \equiv f\{\log [(x-\alpha)^2 + (y-\beta)^2] + v(x, y)\} \tag{1}$$

*is subharmonic in* $\mathfrak{D}$ *for every choice of the real constants* $\alpha$, $\beta$, *then* $v(x, y)$ *is subharmonic in* $\mathfrak{D}$.

PROOF. We shall show $\Delta v(x, y) \geqq 0$. From the hypothesis, it follows that $\Phi(v; \alpha, \beta; x, y)$ has a non-negative Laplacian, that is,

$$\begin{aligned} f'(t)\Delta v(x, y) + f''(t)\Bigg\{&\left[\frac{2(x-\alpha)}{(x-\alpha)^2 + (y-\beta)^2} + v_x(x, y)\right]^2 \\ &+ \left[\frac{2(y-\beta)}{(x-\alpha)^2 + (y-\beta)^2} + v_y(x, y)\right]^2\Bigg\} \geqq 0, \end{aligned} \tag{2}$$

where $t \equiv \log [(x-\alpha)^2 + (y-\beta)^2] + v(x, y)$, for all real $\alpha$, $\beta$.

Let $(x_0, y_0)$ be a point of $\mathfrak{D}$. We distinguish two cases. (a) $v_x^2 + v_y^2 \neq 0$ at $(x_0, y_0)$. Then it is sufficient to choose

$$\alpha = x_0 + 2v_x/(v_x^2 + v_y^2), \qquad \beta = y_0 + 2v_y/(v_x^2 + v_y^2),$$

where the partial derivatives are evaluated at $(x_0, y_0)$, and then to make use of the positiveness of $f'(t)$ to obtain $\Delta v(x_0, y_0) \geqq 0$ from (2). (b) $v_x^2 + v_y^2 = 0$ at $(x_0, y_0)$. If $(x_0, y_0)$ can be approached by a sequence of points $(x_n, y_n)$, $n = 1, 2, \cdots$, at each of which $v_x^2 + v_y^2 \neq 0$, then it follows from (a) and the continuity of $\Delta v(x, y)$ that $\Delta v(x_n, y_n) \to \Delta v(x_0, y_0) \geqq 0$, as $n \to \infty$. If no such sequence exists, then $v_x^2 + v_y^2 \equiv 0$ in a neighborhood of $(x_0, y_0)$, so that $\Delta v(x_0, y_0) = 0$. This completes the proof.

Beckenbach has also given an isoperimetric characterization of functions of class $PL$ [2]; if we use the abbreviation

$$L(p; x, y; r) \equiv \frac{1}{2\pi r}\int_{C(x,y;r)} p(x+\xi, y+\eta)\,ds$$

where $C(x, y; r)$ is the boundary of $D(x, y; r)$, then the theorem is the following.

THEOREM D. *If* $p(x, y)$ *is continuous and if* $p(x, y) \geqq 0$ *in* $\mathfrak{D}$, *then* $p(x, y)$ *is of class PL in* $\mathfrak{D}$ *if and only if, for every continuous function* $q(x, y)$ *of class PL in* $\mathfrak{D}$, *the inequality*

$$A(pq; x, y; r) \leqq L(p; x, y; r) \cdot L(q; x, y; r),$$

*holds for every circular disc* $D(x, y; r)$ *in* $\mathfrak{D}$.

This latter theorem may be extended as follows, for functions that have continuous second derivatives in $\mathfrak{D}$.

THEOREM E. *If $p(x, y)$ has continuous partial derivatives of the second order in $\mathfrak{D}$, if $p(x, y) \geqq 0$ in $\mathfrak{D}$, and if $a$, $b$ is a pair of real constants such that $(1/a)+(1/b)=2$, then $p(x, y)$ is of class PL in $\mathfrak{D}$ if and only if, for every continuous function $q(x, y)$ of class PL in $\mathfrak{D}$, the inequality*

$$A(pq;\, x, y;\, r) \leqq [L(p^a;\, x, y, r)]^{1/a} \cdot [L(q^b;\, x, y;\, r)]^{1/b} \tag{3}$$

*holds for every circular disc $D(x, y; r)$ in $\mathfrak{D}$.*

*Necessity.* A proof of the necessity part may be given that parallels Beckenbach's proof of his theorem; indeed, the only change would be to use the more general Hölder inequality where Beckenbach uses the Schwarz inequality. It is to be noted that the hypothesis on the existence of any derivative of $p(x, y)$ is superfluous here.

*Sufficiency.* Let $(x_0, y_0)$ be a point of $\mathfrak{D}$ such that $p(x_0, y_0) \neq 0$; if there were no such point, then $p(x, y) \equiv 0$ and hence of class $PL$ in $\mathfrak{D}$. We shall consider only analytic $q(x, y)$, so that we may expand both $p(x, y)$ and $q(x, y)$ in finite Taylor expansions about $(x_0, y_0)$, and then substitute in (3) to obtain

$$\begin{aligned} &pq + (r^2/8)\Delta(pq) \\ &\quad \leqq [p^a + (r^2/4)\Delta(p^a) + o(r^2)]^{1/a}[q^b + (r^2/4)\Delta(q^b) + o(r^2)]^{1/b} + o(r^2), \end{aligned}$$

where $o(r^2)$ is a quantity (not always the same quantity) such that $[o(r^2)/r^2] \to 0$, as $r \to 0$. We use binomial expansions, for small $r$, and *then* let $r \to 0$, to obtain the further inequality

$$\Delta(pq)/8 \leqq (pq/4ap^a)\Delta(p^a) + (pq/4bq^b)\Delta(q^b), \tag{4}$$

where the derivatives are evaluated at $(x_0, y_0)$. If we express the derivatives of $pq$, $p^a$ and $q^b$ in terms of the derivatives of $p$ and $q$, and if we set $q = e^{\alpha x+\beta y}$, we obtain

$$\begin{aligned} 0 \leqq\ &(2b-1)(\alpha^2+\beta^2)p^2 - 2\alpha pp_x \\ &- 2\beta pp_y + p\Delta p + (2a-2)(p_x^2 + p_y^2), \end{aligned} \tag{5}$$

which must hold for all real constants $\alpha$, $\beta$. Hence the discriminant of the right-hand member of (5) must satisfy

$$0 \leqq (2b-1)^2[p\Delta p + (2a-2)(p_x^2 + p_y^2)] - (2b-1)(p_x^2 + p_y^2),$$

which yields, since $2b-1>0$ and $(1/a)+(1/b)=2$,

$$0 \leqq p\Delta p - p_x^2 - p_y^2.$$

Hence $p(x, y)$ is of class $PL$ in $\mathfrak{D}$.

The usual averaging process does not appear to be adaptable to the lessening of differentiability conditions in Theorems B, C and E.

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# A NOTE ON SUBSEQUENCES

R. CREIGHTON BUCK

It is well known that a sequence is convergent if all of its subsequences are convergent. The purpose of this note is to show that this may be extended to a result of great generality.

THEOREM. *A sequence is convergent if there exists a regular matrix summability method $T$, which sums every subsequence of the given sequence.*

Thus, for example, if every subsequence of a given sequence is Cesàro summable of any order, then the sequence is convergent. The general theorem depends upon the following well known result of H. Steinhaus.[1]

LEMMA. *For any regular summability method $T$, given by the matrix $(A_{nm})$, there exists a sequence of* 0*'s and* 1*'s not summable* $T$.

To establish the theorem, we suppose that the given sequence $\{S_n\}$ is divergent; we then show that, for any $T$, we can choose a subsequence $\{S_{r_n}\}$ not summable $T$. We may obviously restrict ourselves to bounded sequences. Set $S_n' = (S_n - \liminf S_n)/(\limsup S_n - \liminf S_n)$. We observe that $\liminf S_n' = 0$, $\limsup S_n' = 1$, so that we may choose two disjoint subsequences

$$\{S'_{\alpha_n}\} \quad \text{and} \quad \{S'_{\beta_n}\}$$

which converge to 1 and 0 respectively.

We next construct a "replica" $\{S_n^*\}$ of the sequence $\{S_n'\}$ by defining

$$S_n^* = \begin{cases} 1 & \text{if} \quad n = \alpha_k \quad \text{for some } k. \\ 0 & \text{if} \quad n = \beta_k \quad \text{for some } k. \\ S_n' & \text{otherwise.} \end{cases}$$

To justify the use of the term replica, we observe that $\{S_n^*\}$ and $\{S_n'\}$ have the same limit points, that $\lim (S_n' - S_n^*) = 0$, and that in general the two sequences behave alike.

Among the terms of $\{S_n^*\}$, there are an infinite number of 1's and of 0's; by the Steinhaus Lemma, for any $T$, we can choose a subse-

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[1] Hugo Steinhaus, *Kilka słów o uogólnieniu pojęcia granicy* (Some remarks on the generalization of limit), Prace Matematyczno-Fizyczne vol. 22 (1911) pp. 121–134 (p. 129).

quence $\{S_{r_n}^*\}$ which is not summable by the regular method $T$. Since $S_n' = S_n^* + \theta_n$ where $\theta_n$ is a null sequence, $\{S_{r_n}'\}$ is not summable $T$; thus, $\{S_{r_n}\}$ is a subsequence of the original sequence, and is not summable by the method $T$.

The theorem just proved may be stated in the form of a general Tauberian theorem—admittedly difficult to apply!

THEOREM. *A sequence $\{S_n\}$, summable by a regular matrix method $T$, is convergent if and only if each of its subsequences is also summable $T$.*

As a corollary, we may obtain the following result for series:

COROLLARY. *The series $\sum a_n$ is convergent if there exists a regular matrix summability method $T$, which sums every series formed from the given one by bracketing blocks of terms.*

For, the partial sums of a bracketed series form a subsequence of the partial sums of the original series.

In closing, we note that the restriction to matrix methods is not essential, for the theorem is also true for Abel summability. However, it is apparent that it is *not* true for the Banach-Steinhaus generalized limit, which sums all bounded sequences.[3]

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[3] S. Banach, *Théorie des opérations linéaires*, p. 34.

# ON THE SURFACES OF COINCIDENCE

SU-CHENG CHANG

The purpose of this note is to give a characteristic property of the surfaces of coincidence. We show first that there are six and only six asymptotic osculating quadrics $Q_j$ ($j=1, \cdots, 6$) associated with a point $P$ of a non-ruled analytic surface $(M)$ each of which contains the four consecutive asymptotic tangents $t_{i(j)}$ ($i=1, 2, 3, 4$) of one system along a curve $T$ through $P$ and on the surface $(M)$. As the curve $T$ must be tangent to a direction of Darboux at $P$, it is further proved that each quadric $Q_j$ is an osculating quadric of an asymptotic ruled surface $R$ along a curve of Darboux when and only when the surface $(M)$ is a surface of coincidence.

Let the surface $(M)$ be referred to its asymptotic parameters $u$, $v$, and let $\{M, M_1, M_2, M_3\}$ be the normal tetrahedron of Cartan, $M_1M$ and $M_2M$ being the two asymptotic tangents and $MM_3$, $M_1M_2$ the directrices of Wilczynski. Then the surface $(M)$ is, except for a projective transformation, determined by the following system of equations:

$$(1)\begin{cases} \dfrac{\partial M}{\partial u} = \dfrac{1}{2}\,\dfrac{\partial \log \gamma}{\partial u} M + M_1, \quad \dfrac{\partial M_1}{\partial u} = B^2 M - \dfrac{1}{2}\,\dfrac{\partial \log \gamma}{\partial u} M_1 + \beta M_2, \\ \dfrac{\partial M_2}{\partial u} = \dfrac{k_1}{2} M + \dfrac{1}{2}\,\dfrac{\partial \log \gamma}{\partial u} M_2 + M_3, \\ \dfrac{\partial M_3}{\partial u} = A^2\beta M + \dfrac{1}{2} k_1 M_1 + B^2 M_2 + \dfrac{-1}{2}\,\dfrac{\partial \log \gamma}{\partial u} M_3; \\ \dfrac{\partial M}{\partial v} = \dfrac{1}{2}\,\dfrac{\partial \log \beta}{\partial v} M + M_2, \quad \dfrac{\partial M_2}{\partial v} = A^2 M - \dfrac{1}{2}\,\dfrac{\partial \log \beta}{\partial v} M_2 + \gamma M_1, \\ \dfrac{\partial M_1}{\partial v} = \dfrac{k_1}{2} M + \dfrac{1}{2}\,\dfrac{\partial \log \beta}{\partial v} M_1 + M_3, \\ \dfrac{\partial M_3}{\partial v} = B^2\gamma M + A^2 M_1 + \dfrac{1}{2} k_1 M_2 - \dfrac{1}{2}\,\dfrac{\partial \log \beta}{\partial v} M_3; \end{cases}$$

where $M$ denotes a point of the surface with coordinates $M^i$ ($i=1, 2, 3, 4$) and

$$k_1 = \beta\gamma - (\log \beta)'', \qquad k_1 = \beta\gamma - (\log \gamma)''.$$

The integrability conditions of the system (1) are

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$$
(2)\qquad \begin{gathered}
\frac{\partial A^2}{\partial u} = \frac{k_1}{2}\,\frac{\partial(\log \beta k_1)}{\partial v}, \qquad \frac{\partial B^2}{\partial v} = \frac{k_1}{2}\,\frac{\partial(\log \gamma k_1)}{\partial u}, \\
A\,\frac{\partial(A\beta)}{\partial v} = B\,\frac{\partial(B\gamma)}{\partial u}.
\end{gathered}
$$

Following Godeaux,[1] we denote by $\cdots, U_n, \cdots, U_1, U, V, V_1, \cdots, V_n, \cdots$ the self-polar Laplace sequence in $S_5$, where $U = |M, M_1|$ and $V = |M, M_2|$ are the images in $S_5$ of the two asymptotic tangents of the surface $(M)$ at $M$ respectively. It is well known that[2]

$$
(3)\qquad U^{10} = \beta V, \qquad V^{01} = \gamma U;
$$

$$
(4)\qquad \begin{cases}
U_n^{01} = U_{n+1} + U_n(\log \beta k_1 \cdots k_n)^{01}, \\
U_n^{10} = k_n U_{n-1}, \\
k_n = k_{n-1} - (\log \beta k_1 \cdots k_n)^{11};
\end{cases}
$$

and

$$
(5)\qquad \begin{cases}
V_n^{10} = V_{n+1} + V_n(\log \gamma k_1 \cdots k_n)^{10}, \\
V_n^{01} = k_n V_{n-1}, \\
k_n = k_{n-1} - (\log \gamma k_1 \cdots k_n)^{11},
\end{cases}
$$

where $(\ )^{ij}$ denotes the partial derivative of order $i+j$ formed by differentiating $(\ )$ $i$ times with respect to $u$ and $j$ times with respect to $v$.

Since seven points in $S_5$ are always dependent, we find that

$$
(6)\qquad \begin{aligned}
&U_3 + (\log \beta^3 k_1^2 k_2)^{01} U_2 + \big[(\log \beta k_1)^{02} - 4A^2 \\
&\quad + (\log \beta k_1)^{01}(\log \beta^2 k_1)^{01}\big] U_1 \\
&\quad - 4A^2(\log A\beta)^{01} U + 4\gamma B^2 V - \gamma(\log \gamma k_1)^{10} V_1 - \gamma V_2 = 0
\end{aligned}
$$

and

$$
(7)\qquad \begin{aligned}
&V_3 + (\log \gamma^3 k_1^2 k_2)^{10} V_2 + \big[(\log \gamma k_1)^{20} - 4B^2 \\
&\quad + (\log \gamma k_1)^{10}(\log \gamma^2 k_1)^{10}\big] V_1 \\
&\quad - 4B^2(\log B\gamma)^{10} V + 4\beta A^2 U - \beta(\log \beta k_1)^{01} U_1 - \beta U_2 = 0.
\end{aligned}
$$

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[1] See Godeaux, *La theorie des surfaces et l'espace réglé*, 1934, Paris. A direct definition of the sequence of Godeaux quadrics has been given by the present author. See S. C. Chang, *Some theorems on ruled surfaces*, in the press of Science Records, Academia Sinica.

[2] L. Godeaux, loc. cit.

If four consecutive asymptotic $u$-tangents ($v=\text{const.}$) belong to a regulus, then there is a plane in $S_3$ which intersects the surface $(U)$ at four consecutive points. In other words: $U$, $dU$, $d^2U$, $d^3U$ should be coplanar.

By means of (3), (4), (5) and (6) it is clear that the three points

$$\beta V du + U_1 dv, \tag{8}$$

$$\begin{aligned} &V[(\beta^{10} + \beta(\log\gamma)^{10})(du)^2 + \beta^{01} du dv + \beta d^2u] \\ &\qquad + \beta V_1(du)^2 + U_1 d^2v + [U_2 + U_1(\log \beta k_1)^{01}](dv)^2 \end{aligned} \tag{9}$$

and

$$\begin{aligned} &[U_3 - 4A^2(\log A\beta)^{01}U](dv)^3 + MU_1 + \beta V_2(du)^3 + NV \\ &\quad + V_1[\beta(\log \gamma^2 k_1)^{10}(du)^3 + 2\beta^{10}(du)^3 + 2\beta^{01}dv(du)^2 + 3\beta du d^2u] \\ &\quad + U_2[(\log \beta^2 k_1^2 k_2)^{01}(dv)^3 + 3dv d^2v] \end{aligned} \tag{10}$$

are collinear, where

$$\begin{aligned} M = {} & [(\log \beta k_1)^{01}]^2(dv)^3 + d^3v + k_1 du(dv)^2 + (\log \beta k_1)^{02}(dv)^3 \\ & + 3(\log \beta k_1)^{01} d^2v dv, \\ N = {} & 3\beta^{10} du d^2u + (du)^3[\beta^{20} + \beta((\log\gamma)^{10})^2 + \beta(\log\gamma)^{20} + 2\beta^{10}(\log\gamma)^{10}] \\ & + (du)^2 dv[2\beta^{11} + \beta(\log\gamma)^{11} + 2\beta^{01}(\log\gamma)^{10} + \beta k_1] + \beta^{02} du(dv)^2 \\ & + 2\beta^{01} d^2u dv + \beta^{01} du d^2v + \beta d^3u + 3\beta(\log\gamma)^{10} d^2u du. \end{aligned}$$

From (6), (8), (9) and (10) it follows that the coefficient of $V_2$ in (10) must vanish, so that

$$\beta(du)^3 + \gamma(dv)^3 = 0. \tag{11}$$

Moreover, we demand that the coefficients of $U_2$, $V_1$ in (9) should be proportional to those in (10), and therefore that

$$du d^2v - dv d^2u - (1/3) du dv[(\log \beta^2\gamma)^{10} du + (\log \beta^2)^{01} dv] = 0. \tag{12}$$

Differentiation of (11) shows that

$$3(du d^2v - dv d^2u) + (du)^2 dv(\log(\gamma/\beta))^{10} + du(dv)^2(\log(\gamma/\beta))^{01} = 0. \tag{13}$$

If (12) and (13) coincide with each other for any direction of Darboux, then

$$(\log \beta\gamma^2)^{10} = 0, \qquad (\log \gamma\beta^2)^{01} = 0. \tag{14}$$

Thus we may take

$$\beta = 1, \qquad \gamma = 1, \tag{15}$$

and accordingly,

$$N = d^2u + (du)^2dv, \qquad M = d^2v + du(dv)^2.$$

Hence the surface in consideration must be a surface of coincidence.

We inquire now whether the asymptotic tangents of some one system along a curve on a surface belong to a regulus. It is seen from (11) that the curve is necessarily a Darboux curve. In virtue of (11) and (15) we obtain a point

$$V[d^2u + (du)^2dv + (-4B^2)(dv)^3 - 3d^2vd^2u/dv] \\ + U_1[d^2v + du(dv)^2 + 4A^2(du)^3 - 3(d^2v)^2/dv].$$

This is collinear with the points (9) and (10) and coincides with the point (8) when

$$\begin{aligned} dv[d^2u + (du)^2dv - 4B^2(dv)^3 - 3d^2vd^2u/dv] \\ - du[d^2v + du(dv)^2 + 4A^2(dv)^3 - 3(d^2v)^2/dv] = 0. \end{aligned} \tag{16}$$

From (13), (15) and (16) we have

$$A^2 = 0, \qquad B^2 = 0. \tag{17}$$

A reference to (15) and (17) shows that *the surface must be $xyz=1$ or one of its projective transforms.*

NATIONAL UNIVERSITY OF CHEKIANG

# THE CHARACTERISTICS OF ASYMPTOTIC OSCULATING QUADRICS OF A CURVE ON A SURFACE

BUCHIN SU[1]

**1. Introduction.** Lane[2] has derived the equations of asymptotic osculating quadrics of a curve $C$ on a surface $S$, employing Wilczynski's notation and deducing some of their fundamental properties. We shall now investigate the characteristics of these quadrics along $C$. For the sake of convenience the normal tetrahedron of Cartan[3] is utilized throughout this note. In terms of the local coordinates with respect to this tetrahedron we have expressed simply the equations of asymptotic osculating quadrics and their characteristics and therefrom obtained new configurations projectively connected with a surface.

**2. The normal tetrahedron of Cartan.** Let us consider the directrices of Wilczynski at a generic point $M$ of a non-ruled surface $S$; the first of them intersects the quadric of Lie at another point $M_3$ and the second meets the asymptotic tangents at the points $M_1$ and $M_2$. The tetrahedron $\{MM_1M_2M_3\}$ is known as the normal tetrahedron of Cartan. If we denote, for simplicity, the corresponding projective coordinates of these points $M$, $M_1$, $M_2$, $M_3$ by the same notations, then they are solutions of the following system of differential equations:[4]

$$
\begin{aligned}
\frac{\partial M}{\partial u} &= \frac{1}{2} M \frac{\partial}{\partial u} \log \gamma + M_1, \\
\frac{\partial M_1}{\partial u} &= B^2 M - \frac{1}{2} M_1 \frac{\partial}{\partial u} \log \gamma + \beta M_2, \\
\frac{\partial M_2}{\partial u} &= KM + \frac{1}{2} M_2 \frac{\partial}{\partial u} \log \gamma + M_3, \\
\frac{\partial M_3}{\partial u} &= A^2 \beta M + KM_1 + B^2 M_2 - \frac{1}{2} M_3 \frac{\partial}{\partial u} \log \gamma;
\end{aligned} \tag{1}
$$

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[1] Research Professor in Mathematics, Academia Sinica.

[2] E. P. Lane, *The asymptotic osculating quadrics of a curve on a surface*, Bull. Amer. Math. Soc. vol. 33 (1927) pp. 195–200.

[3] Cf., for example, Fubini and Čech, *Introduction à la géométrie projective différentielle des surfaces*, Paris, 1931, p. 236.

[4] These were first used in this form by S. Finikoff, *Sur les couples de surfaces dont les asymptotiques se correspondent et qui, aux points correspondants, ont les mêmes directrices de Wilczynski*, C. R. Acad. Sci. Paris vol. 197 (1933) pp. 883–885.

$$
(1)\qquad
\begin{aligned}
\frac{\partial M}{\partial v} &= \frac{1}{2} M \frac{\partial}{\partial v} \log \beta + M_2, \\
\frac{\partial M_1}{\partial v} &= \bar{K} M + \frac{1}{2} M_1 \frac{\partial}{\partial v} \log \beta + M_3, \\
\frac{\partial M_2}{\partial v} &= A^2 M + \gamma M_1 - \frac{1}{2} M_2 \frac{\partial}{\partial v} \log \beta, \\
\frac{\partial M_3}{\partial v} &= B^2 \gamma M + A^2 M_1 + \bar{K} M_2 - \frac{1}{2} M_3 \frac{\partial}{\partial v} \log \beta,
\end{aligned}
$$

where the components of displacements of the tetrahedron $\beta$, $\gamma$, $A$, $B$, $K$, $\bar{K}$ satisfy the integrability conditions

$$
(2)\qquad
\begin{gathered}
2K = \beta\gamma - \frac{\partial^2}{\partial u \partial v} \log \beta, \qquad 2\bar{K} = \beta\gamma - \frac{\partial^2}{\partial u \partial v} \log \gamma, \\
\frac{\partial (A^2)}{\partial u} = K \frac{\partial}{\partial v} \log (K\beta), \qquad \frac{\partial (B^2)}{\partial v} = \bar{K} \frac{\partial}{\partial u} \log (\bar{K}\gamma), \\
A \frac{\partial (A\beta)}{\partial v} = B \frac{\partial (B\gamma)}{\partial u}.
\end{gathered}
$$

Any point $P$ in space can now be represented by the local coordinates $(y_1, y_2, y_3, y_4)$ with respect to the tetrahedron $\{MM_1M_2M_3\}$,

$$
(3)\qquad P = y_1 M + y_2 M_1 + y_3 M_2 + y_4 M_3.
$$

From (1) we easily show that the conditions of immovability for $P$ are

$$
(4)\qquad
\begin{array}{lllll}
\dfrac{\partial y_1}{\partial u} = & -\dfrac{1}{2}\left(\dfrac{\partial}{\partial u} \log \gamma\right) y_1 & -B^2 y_2 & -K y_3 & -A^2 \beta y_4, \\
\dfrac{\partial y_2}{\partial u} = & & -y_1 + \dfrac{1}{2}\left(\dfrac{\partial}{\partial u} \log \gamma\right) y_2 & * & -K y_4, \\
\dfrac{\partial y_3}{\partial u} = & * & -\beta y_2 & -\dfrac{1}{2}\left(\dfrac{\partial}{\partial u} \log \gamma\right) y_3 & -B^2 y_4, \\
\dfrac{\partial y_4}{\partial u} = & * & * & -y_3 & +\dfrac{1}{2}\left(\dfrac{\partial}{\partial u} \log \gamma\right) y_4; \\
\dfrac{\partial y_1}{\partial v} = & -\dfrac{1}{2}\left(\dfrac{\partial}{\partial v} \log \beta\right) y_1 & -\bar{K} y_2 & -A^2 y_3 & -B^2 \gamma y_4, \\
\dfrac{\partial y_2}{\partial v} = & * & -\dfrac{1}{2}\left(\dfrac{\partial}{\partial v} \log \beta\right) y_2 & -\gamma y_3 & -A^2 y_4, \\
\dfrac{\partial y_3}{\partial v} = & -y_1 & * & +\dfrac{1}{2}\left(\dfrac{\partial}{\partial v} \log \beta\right) y_3 & -\bar{K} y_4, \\
\dfrac{\partial y_4}{\partial v} = & * & -y_2 & * & +\dfrac{1}{2}\left(\dfrac{\partial}{\partial v} \log \beta\right) y_4.
\end{array}
$$

3. **Equations.** Suppose that a curve $C$ on $S$ is given by the equation

$$u = u(v). \tag{5}$$

The asymptotic ruled surface $R_u$ generated by the asymptotic $u$-tangents of $S$ along $C$ is given by

$$P = M + \rho M_1, \tag{6}$$

where $\rho$ and $v$ are independent parameters, the variable $u$ in $M$ and $M_1$ being given by (5).

In virtue of (1) there is no difficulty in calculating the partial derivatives of $P$ with respect to $\rho$ and $v$. Denoting $du/dv$, $d^2u/dv^2$, $d^3u/dv^3$ by $u'$, $u''$, $u'''$ respectively we have

$$\begin{aligned}
P_\rho &= M_1,\\
P_v &= \left\{\frac{1}{2}u'\frac{\partial}{\partial u}\log\gamma + \frac{1}{2}\frac{\partial}{\partial v}\log\beta + \rho(B^2u' + \overline{K})\right\}M\\
&\quad + \left\{u' + \frac{1}{2}\rho\left(-u'\frac{\partial}{\partial u}\log\gamma + \frac{\partial}{\partial v}\log\beta\right)\right\}M_1\\
&\quad + (1 + \rho\beta u')M_2 + \rho M_3,\\
P_{v\rho} &= (B^2u' + \overline{K})M + \frac{1}{2}\left(-u'\frac{\partial}{\partial u}\log\gamma + \frac{\partial}{\partial v}\log\beta\right)M_1\\
&\quad + \beta u'M_2 + M_3,\\
P_{vv} &= (^*)M + (^*)M_1\\
&\quad + \left\{u'\frac{\partial}{\partial u}\log\gamma + \beta u'^2 + \rho(2B^2u' + 2\overline{K}\right.\\
&\qquad\qquad\qquad \left. + \beta u'' + \beta_u'^2 + \beta_v u')\right\}M_2\\
&\quad + 2\left\{u' + \rho\left(-u'\frac{\partial}{\partial u}\log\gamma + \beta u'^2\right)\right\}M_3,
\end{aligned} \tag{7}$$

so that the differential equation of the curved asymptotics of[3] $R_u$,

$$2(PP_\rho P_v P_{v\rho})d\rho + (PP_\rho P_v P_{vv})dv = 0,$$

becomes

$$\frac{d\rho}{dv} = a\rho^2 + b\rho - u', \tag{8}$$

---

[3] The parenthesis denotes a determinant of the four points there inclosed.

where we have placed

$$
\begin{aligned}
a &= \bar{K} + B^2 u' + \frac{1}{2}\beta u'' + \frac{1}{2}\beta u'^2 \frac{\partial}{\partial u}\log(\beta\gamma) \\
&\quad + \frac{1}{2}\beta u' \frac{\partial}{\partial v}\log\beta - \frac{1}{2}\beta^2 u'^3, \qquad (9)\\
b &= u'\frac{\partial}{\partial u}\log\gamma - \beta u'^2.
\end{aligned}
$$

The osculating quadric $Q_u$ or $R_u$ along its generator $MM_1$ is the asymptotic osculating quadric of one family, and a second asymptotic osculating quadric $Q_v$ of $C$ at $M$ is obtained by using the other family of asymptotics of $S$. It is clear, therefore, that a generic point $Z$ on $Q_u$ must be represented by

$$
Z = P + \sigma\frac{dP}{dv} = M + \rho M_1 + \sigma\left(P_v + P_\rho \frac{d\rho}{dv}\right), \qquad (10)
$$

$\sigma$ being another parameter.

Substitution of (7) and (8) in (10) gives the local coordinates of the point $Z$, namely,

$$
\begin{aligned}
y_1 &= 1 + \sigma\left\{\frac{1}{2}u'\frac{\partial}{\partial u}\log\gamma + \frac{1}{2}\frac{\partial}{\partial v}\log\beta + \rho(B^2u' + \bar{K})\right\}, \\
y_2 &= \rho + \rho\sigma\left\{\frac{1}{2}u'\frac{\partial}{\partial v}\log\gamma + \frac{1}{2}\frac{\partial}{\partial v}\log\beta - \beta u'^2\right\} + a\rho^2\sigma, \qquad (11)\\
y_3 &= (1 + \rho\beta u')\sigma, \\
y_4 &= \rho\sigma.
\end{aligned}
$$

Thus we obtain *the equation of the first asymptotic osculating quadric $Q_u$ of $C$ at $M$*:

$$
\begin{aligned}
&y_1y_4 - y_2y_3 + \beta u' y_2 y_4 - \beta u'^2 y_3 y_4 \\
&\quad + \frac{1}{2}\beta\left\{u'' + \beta u'^3 + u'^2\frac{\partial}{\partial u}\log(\beta\gamma) + u'\frac{\partial}{\partial v}\log\beta\right\}y_4^2 = 0. \qquad (12)
\end{aligned}
$$

*The equation of the second asymptotic osculating quadric $Q_v$ of $C$ at $M$ is*

$$
\begin{aligned}
&u'^3(y_1y_4 - y_2y_3) - \gamma u' y_2 y_4 + \gamma u'^2 y_3 y_4 \\
&\quad + \frac{1}{2}\gamma\left\{- u'' + \gamma + u'\frac{\partial}{\partial v}\log(\beta\gamma) + u'^2\frac{\partial}{\partial u}\log\gamma\right\}y_4^2 = 0. \qquad (13)
\end{aligned}
$$

4. **Characteristics.** The asymptotic osculating quadrics $Q_u$ (or $Q_v$) at $M$ and its consecutive point $M'$ on the curve $C$ intersect each other at the asymptotic tangent $u$ (or $v$) and two other lines, which are called the characteristic of $Q_u$ (or $Q_v$) along $C$. In order to derive the equations of the characteristics of $Q_u$ we have to set the derivative along $C$ of the left-hand member of (12) equal to zero, remembering that the derivatives of $y$'s should be determined by (4). A simple calculation, which we shall omit here, suffices to demonstrate that the equations of the characteristic in consideration are (12) and

$$C_{33}y_3^2 - C_{34}y_3y_4 + C_{44}y_4^2 = 0, \tag{14}$$

where we have placed

$$C_{33} = \beta u'^3 + \gamma, \tag{15}$$

$$\begin{aligned} C_{34} = \beta u' \Big\{ & 3u'' + \beta u'^3 + \left(\frac{\partial}{\partial u} \log (\beta^2\gamma)\right) u'^2 \\ & + 3\left(\frac{\partial}{\partial v} \log \beta\right) u' + \gamma \Big\}, \end{aligned} \tag{16}$$

$$\begin{aligned} C_{44} = & - A^2\beta u' - B^2\gamma + \beta u'^2(B^2u + \bar{K}) - \beta u'(Ku' + A^1) \\ & + \frac{1}{2}\beta\Big\{u'' + \beta u'^3 + \left(\frac{\partial}{\partial u} \log (\beta\gamma)\right)u'^2 \\ & + \left(\frac{\partial}{\partial u} \log \beta\right)u'\Big\}\left\{u'\frac{\partial}{\partial u} \log \gamma + \frac{\partial}{\partial v} \log \beta\right\} \\ & + \frac{1}{2}\beta\Big\{u''' + \left(\frac{\partial}{\partial u} \log (\beta^2\gamma^2)\right)u'u'' \\ & + 2\left(\frac{\partial}{\partial v} \log \beta\right)u'' + 3\beta^2u'^2u'' \\ & + 2\left(\frac{\partial}{\partial u} \log \beta\right)u'^4 + \Big(2\frac{\partial}{\partial v} \log \beta + \frac{\partial^2}{\partial u^2} \log (\beta\gamma) \\ & + \frac{\partial}{\partial u} \log (\beta\gamma)\frac{\partial}{\partial u} \log \beta\Big)u'^3 \\ & + \left(\frac{\partial^2}{\partial u\partial v} \log (\beta^2\gamma) + 2\frac{\partial}{\partial u} \log (\beta\gamma)\frac{\partial}{\partial v} \log \beta\right)u'^2 \\ & + \left(\frac{\partial^2}{\partial v^2} \log \beta + \left(\frac{\partial}{\partial v} \log \beta\right)^2\right)u'\Big\}. \end{aligned} \tag{17}$$

From (14) it follows that the characteristic of $Q_u$ along $C$ consists of two straight lines intersecting the asymptotic $u$-tangent and the latter counted twice.

In a similar way we obtain another pair of lines intersecting the asymptotic tangent $MM_2$ as the second quadric moves along $C$. These four lines furnish obviously a generalization of the well known quadrilateral of Demoulin.

By means of (14) we can deduce some remarkable results.[6]

(I) *If the asymptotic tangent* $MM_2$ *is a part of the characteristic of the asymptotic osculating quadric* $Q_u$ *of* $C$ *at* $M$, *then* $C$ *must be tangent to a Darboux curve at* $M$, *and conversely.*

The necessary and sufficient condition for this is $C_{33}=0$, namely,

$$\beta u'^3 + \gamma = 0, \tag{18}$$

which represents the Darboux directions of the surface at $M$.

(II) *In order that the characteristic of the first and second asymptotic osculating quadrics along every Darboux curve of a surface* $S$ *should decompose into two asymptotic tangents, each being counted twice, the necessary and sufficient condition is that* $S$ *be a surface of coincidence.*

In fact, we have $C_{33}=0$, $C_{34}=0$, so that (14) becomes $y_4^2=0$. In consequence, the characteristic is given by the equations

$$y_4^2 = 0, \qquad y_2 y_3 = 0.$$

For a Darboux curve we have

$$\beta u'^3 + \gamma = 0, \quad u'' = -\frac{1}{3} u'^2 \frac{\partial}{\partial u} \log (\gamma/\beta) - \frac{1}{3} u' \frac{\partial}{\partial v} \log (\gamma/\beta). \tag{19}$$

Substituting these in the condition $C_{34}=0$, namely,

$$3u'' + \beta u'^3 + u'^2 \frac{\partial}{\partial u} \log (\beta^2 \gamma) + 3u' \frac{\partial}{\partial v} \log \beta + \gamma = 0, \tag{20}$$

and taking account of the fact that this equation must hold for the three Darboux curves, we arrive at

$$\frac{\partial}{\partial v} \log \beta = 0, \qquad \frac{\partial}{\partial v} \log \gamma = 0,$$

that is, both $\beta$ and $\gamma$ are functions of $u$ alone.

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[6] Cf. S. C. Chang, *On the surfaces of coincidence*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 900–903.

Since the same condition also holds for the second asymptotic osculating quadrics, we have that $\beta$ and $\gamma$ must be functions of $v$ alone, and therefore that they are constants. Hence the surface in question is a surface of coincidence.

(III) *If the characteristics of the quadrics $Q_u$ and $Q_v$ along every Darboux curve of a surface S are indeterminate, that is, the asymptotic osculating quadrics along every Darboux curve are stationary, then S must be projectively equivalent to the cubic surface $xyz=1$.*

For we have in this case

$$C_{33} = 0, \qquad C_{34} = 0, \qquad C_{44} = 0.$$

We may put $\beta=\gamma=1$, which gives

$$A^2 = 0, \qquad B^2 = 0.$$

The equation to the surface may easily be found by integrating the system (1).

5. **Associate directrices.** We come now to demonstrate that a certain pair of covariant lines can be constructed by means of the asymptotic osculating quadrics. Consider, for instance, the first quadric $Q_u$ given by (12). As was shown before, there exists an element of the second order, $E_2$, corresponding to each Darboux tangent, such that the characteristic of $Q_u$ at the point $M$ decomposes into the two asymptotic tangents $MM_1$ and $MM_2$, each being counted twice. We have therefore a plane containing this $E_2$. In order to find the equation of this plane we merely have to substitute the value of $u''$ given by (20) in the equation of the osculating plane of $C$ at $M$:

$$\begin{aligned} (21)\qquad & 2u'(x - u'y) \\ & - \left(u'' + u'^2 \frac{\partial}{\partial u} \log (\beta\gamma) + \gamma - u' \frac{\partial}{\partial v} \log (\beta\gamma) - \beta u'^3\right) z = 0, \end{aligned}$$

where $x$, $y$, $z$ denote the nonhomogeneneous coordinates of a point with respect to the tetrahedron of Fubini at $M$. The result of carrying out the computation is

$$(22)\qquad 2u'(x - u'y) - \left\{\phi u'^2/3 - \psi u' + 2\gamma\right\} z = 0,$$

where

$$(23)\qquad u' = -\epsilon(\gamma/\beta)^{1/3} \qquad (\epsilon \neq 1,\ \epsilon^3 = 1),$$

$$(24)\qquad \phi = \frac{\partial}{\partial u} \log (\beta\gamma^2), \qquad \psi = \frac{\partial}{\partial v} \log (\beta^2\gamma).$$

The three planes (22) corresponding to the Darboux tangents at $M$ form a trihedron; and the polar of the tangent plane of the surface at $M$ with respect to this is

$$x + \psi z/2 = 0, \qquad y + \phi z/6 = 0. \tag{25}$$

If the second quadric $Q_v$ is used instead of $Q_u$, a second covariant line is found to be

$$x + \psi z/6 = 0, \qquad y + \phi z/2 = 0. \tag{26}$$

These lines are analogous to the directrices of Sullivan[7] and will be called *the associate directrices* of the surface at $M$. The first of them, given by (25), is the intersection of the plane containing the asymptotic $v$-tangent and the first directrix of Wilczynski and the plane containing the asymptotic $u$-tangent and the first principal ray $c(-1/6)$ of Fubini. A similar construction is obtained for the second associate directrix. It shall be noted that the asymptotic $u$-tangent, the first directrix of Sullivan, the first associate directrix and the first directrix of Wilczynski are coplanar and the double ratio of them in this order is equal to $-3$.

There are two planes through each Darboux tangent at $M$, one being (22) and the other for $Q_v$:

$$x - u'y + (1/2)\{\psi - \phi u'/3 + 2\beta u'^2\} z = 0. \tag{27}$$

The harmonic conjugate of the tangent plane of the surface with respect to them osculates the corresponding Darboux curve at $M$, as we can easily show from the equations (19) and (21).

6. **Associate cones.** As was shown by Lane,[8] the directrices of Sullivan are related to a certain pair of cones of the third class enveloped by the osculating planes at $M$ of the extremals of certain integrals. We shall here prove that analogous cones also exist in the case of associate directrices.

Let us consider again the characteristic of the first quadric $Q_u$ along a curve $C$ on the surface $S$. For a given non-Darboux direction equation (14) shows that we can always adjust an element of the second order, $E_2$, to the curve $C$ of the given direction at $M$, such that the planes through the asymptotic $u$-tangent and the characteristic lines are harmonic conjugate to the tangent plane and the plane through the first directrix of Wilczynski. The plane containing this $E_2$ corresponding to every non-Darboux direction at $M$ envelopes a cone of the third class

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[7] C. T. Sullivan, *Scroll directrix curves*, Trans. Amer. Math. Soc. vol. 16 (1915) pp. 199–214.

[8] E. P. Lane, *Le direttrici di Sullivan*, Bollettino della Unione Matematica Italiana vol. 5 (1926) pp. 214–215.

$$\gamma u_1^3 + 2\beta u_2^3 + 3(\psi u_1/2 + \phi u_2/6 - u_3)u_1u_2 = 0, \tag{28}$$

where $u_1$, $u_2$, $u_3$ denote the coordinates of a plane through $M$. The three cuspidal planes of this cone evidently pass through the first associate directrix.

In a similar way a second cone of the third class

$$2\gamma u_1^3 + \beta u_2^3 + 3(\psi u_1/6 + \phi u_2/2 - u_3)u_1u_2 = 0 \tag{29}$$

is obtained.

In virtue of these cones we can further construct a third cone of the third class and a new canonical ray at the point $M$ of the surface.

For this purpose, we draw the tangent planes of the cones (28) and (29) through a non-Darboux tangent at $M$ and construct the harmonic conjugate of the tangent plane of the surface with respect to them. The equation of this plane is found to be

$$2u'(x - u'y) + (\beta u'^3 - \gamma + \phi u'^2/3 - \psi u'/3)z = 0. \tag{30}$$

It is easily seen that the three planes corresponding to the three Segre tangents at $M$ are concurrent in the canonical ray $c(1/6)$:

$$x - \psi z/6 = 0, \qquad y - \phi z/6 = 0. \tag{31}$$

This line and the first principal ray of Fubini harmonically separate the canonical tangent and the projective normal of the surface at $M$.

The plane (30) also envelopes a cone of the third class

$$\gamma u_1^3 + \beta u_2^3 - 2(\psi u_1/6 + \phi u_2/6 + u_3)u_1u_2 = 0 \tag{32}$$

and the three cuspidal planes are concurrent in the ray $c(1/6)$.

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# CONVERGENCE REGIONS FOR THE GENERAL CONTINUED FRACTION

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The purpose of this note is to prove convergence region theorems for continued fractions

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots, \tag{1}$$

where the $a_n$ and $b_n$ are complex numbers. As an application we obtain a new theorem for the associated continued fraction

$$\frac{a_1}{b_1 + z} + \frac{a_2}{b_2 + z} + \cdots. \tag{2}$$

This continued fraction plays a rôle in the moment problem and is also otherwise of interest as a means of representing analytic functions. For details see Perron [4, chapters 8 and 9].[1]

Recently a continued fraction very similar to (2) has been investigated by Hellinger and Wall [1]. They chose the name "$J$-fraction." A $J$-fraction is a continued fraction of the form

$$\frac{1}{b_1 + z} - \frac{c_1^2}{b_2 + z} - \frac{c_2^2}{b_3 + z} - \cdots. \tag{2$'$}$$

Our result for $J$-fractions is stated in Theorem C.

The results of this note are closely related to an as yet unpublished work of Wall and Wetzel on "positive definite $J$-fractions." In particular Theorem C seems to be contained in a theorem of theirs.

In what follows we shall denote by $H(b, \gamma)$ the half-plane (including the boundary) defined by the relation $z \in H(b, \gamma)$ if $\Re(ze^{-i\gamma}) \geqq b$. For the open half-plane we shall use the notation $H_0(b, \gamma)$. It is clear from the context that $b$ is a real number. Further for $a > 0$, $P(a, \gamma)$ shall be the parabolic region (including the boundary) bounded by the curve

$$\rho \leqq \frac{a^2/2}{1 - \cos(\theta - 2\gamma)}.$$

For $a = 0$, $P(a, \gamma)$ is to be the totality of points $r \cdot e^{i\gamma}$, $r \geqq 0$.

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[1] Numbers in brackets refer to the references listed at the end of the paper.

We are now in a position to state our theorems.

THEOREM A. *The continued fraction* (1) *converges if all the* $a_n$ *lie in a bounded part of the set* $P(a, \gamma)$, $a \geqq 0$, *and if all* $b_n$ *lie in the half-plane* $H(a+\epsilon, \gamma)$, *where* $\epsilon$ *is an arbitrary small positive number.*

Even though its statement is somewhat involved it seems worth while to state the following theorem too, as it is a generalization of the parabola theorems[1] for the continued fractions with $b_n = 1$.

THEOREM A′. *In the continued fraction* (1) *let all* $b_n$ *lie in the angular opening*

$$|\arg e^{-i\gamma}(b_n - 1)| < \pi/2 - \epsilon,$$

*where* $-\pi/2 < \gamma < \pi/2$ *and* $\epsilon$ *is an arbitrary small positive number. Further let all the* $a_n$ *lie in the parabolic region* $P(\cos\gamma, \gamma)$. *Then the continued fraction* (1) *converges if and only if* $\sum |b_n d_n| = \infty$, *where* $d_1 = 1/a_1$, $d_n = 1/d_{n-1}a_n$.

It is well known that the last condition of the theorem is satisfied if $\liminf a_n < \infty$. We note that neither of these two theorems is contained in the other. Both theorems could be used to obtain new "multiple" convergence regions for continued fractions $1 + K(a_n/1)$. For both of these theorems there are corresponding theorems for associated continued fractions and $J$-fractions. We only state the theorems corresponding to Theorem A.

THEOREM B. *If real numbers* $a \geqq 0$, $M > 0$, $b$ *and* $\gamma$ *exist such that for all* $n \geqq 1$, $a_n \in P(a, \gamma)$, $|a_n| < M$ *and* $b_n \in H(b, \gamma)$ *then the associated continued fraction* (2) *converges uniformly and hence to a holomorphic function of the complex variable* $z$ *in every closed region contained in the half-plane* $H_0(a - b, \gamma)$.

It is easily seen that if $a_n \in P(a, 0)$ then $c_n = a_n^{1/2}$ satisfies the relation $|\Im(c_n)| \leqq a/2$ and conversely. Hence the following theorem for $J$-fractions is an immediate consequence of Theorem B with $\gamma = 0$.

THEOREM C. *If real numbers* $a \geqq 0$, $M > 0$ *and* $b$ *exist such that for all* $n \geqq 1$, $|\Im(c_n)| \leqq a/2$, $|c_n| < M$ *and* $\Im(b_n) \geqq b$ *then the* $J$*-fraction* (2)′ *converges uniformly and hence to a holomorphic function of the complex variable* $z$ *in every closed region contained in the half-plane* $\Im(z) > a - b$.

The theorems are based on the following lemma.

LEMMA. *All the approximants of the continued fraction* (1) *lie in the half-plane* $H(-b/2, \gamma)$ *if for all* $n \geqq 1$, $a_n \in P(b, \gamma)$ *and* $b_n \in H(b, \gamma)$, *where* $b \geqq 0$.

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[1] See Leighton and Thron [2] and Paydon and Wall [3].

The proof of the lemma is by induction. To perform the induction we require the following two relations: for an arbitrary choice of $a_n \in P(b, \gamma)$, $b_n \in H(b, \gamma)$ and $v \in H(-b/2, \gamma)$ the two relations

$$a_n/b_n \in H(-b/2, \gamma) \tag{3}$$

and

$$\frac{a_n}{b_n + v} \in H(-b/2, \gamma) \tag{4}$$

must be satisfied. We note that (3) is a consequence of (4) as $v = 0 \in H(-b/2, \gamma)$. Further as $b_n$ and $v$ vary independently over their respective regions their sum $b+v$ varies over the half-plane $H(b/2, \gamma)$. It is then easily seen that relation (4) is satisfied if $P(b, \gamma)$ is the part common to all the regions $c \cdot H(-b/2, \gamma)$, ($z \in c \cdot H(-b/2, \gamma)$ if $z = c \cdot v$, where $v \in H(-b/2, \gamma)$) where $c$ varies over $H(b/2, \gamma)$. For the proof of this fact we refer the reader to [2, §2], where a very similar fact is proved in detail.

We now proceed to the proof of Theorem B. Under the conditions of that theorem $b_n + z \in H_0(a, \gamma)$ for all $n \geqq 1$. According to the lemma all the approximants of the continued fraction (2) then lie in the half-plane $H(-a/2, \gamma)$. All approximants are finite. This is seen as follows: In order that an approximant be infinite it is necessary that

$$-(b_1 + z) = \frac{a_2}{b_2 + z} + \cdots + \frac{a_n}{b_n + z}.$$

This is impossible as the regions $-H_0(a, \gamma)$ and $H(a/2, \gamma)$ have no point in common.

The approximants of (2) are rational functions of $z$; for $z$ in $H_0(a-b, \gamma)$ they are finite and do not take on certain values (more than two), hence by Montel's Theorem the sequence of approximants forms a normal family of holomorphic functions for $z$ in $H_0(a-b, \gamma)$.

As the $|a_n|$ are bounded there exists an $M$ such that for $|z| > M$, $z \in H_0(a-b, \gamma)$,

$$|b_n + z| > |a_n| + 1$$

for all $n$. For these values of $z$ the continued fraction converges by Pringsheim's Theorem [4, p. 254]. It then follows from the generalized Stieltjes-Vitali Theorem that the continued fraction converges uniformly in every closed region contained in the half-plane $H_0(a-b, \gamma)$. This completes the proof of Theorem B.

Theorem A follows from Theorem B if we set $a = b$ and let $z = \epsilon e^{i\gamma}$.

For the proof of Theorem A′ we consider the continued fraction

$$\frac{a_1}{1 + d_1 e^{i\theta_1 z}} + \frac{a_2}{1 + d_2 e^{i\theta_2 z}} + \cdots, \tag{5}$$

where $a_n \in P(\cos\gamma, \gamma)$, $d_n \geqq 0$ and $|\theta_n - \gamma| < \pi/2 - \epsilon$ for all $n \geqq 1$.

Under these conditions the approximants of (5) form a normal family of holomorphic functions for $z$ in the region $D$ defined by

$$-\delta < \Re(z) < 1 + \delta,$$

where $\delta$ is positive and depends on $\gamma$ and $\epsilon$. The proof of this fact is similar to the proof used in the previous case. For $\Re(z) = 0$ the continued fraction converges. In this case the partial denominators are real and greater than or equal to 1 and hence (5) can be transformed into a continued fraction of the form

$$\frac{g_1}{1} + \frac{g_2}{1} + \cdots,$$

where all $g_n \in P(\cos\gamma, \gamma)$. The convergence of this continued fraction follows from the parabola theorems (we are assuming that all conditions of Theorem A′ are satisfied). The Stieltjes-Vitali Theorem then insures the convergence of (5) for all $z$ in $D$. If we recall that $\sum |c_n| = \infty$ is a necessary condition for convergence we have Theorem A′ by setting $z = 1$ in (5).

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# A THEOREM ON GENERALIZED DERIVATIVES

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1. **General remarks.** Suppose that a function $f(x)$, defined in the neighborhood of a point $x_0$, satisfies a relation

$$f(x_0+t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2/2! + \cdots + \alpha_{k-1}t^{k-1}/(k-1)! + \omega_k(t)t^k/k!, \tag{1.1}$$

where $\alpha_0, \alpha_1, \cdots, \alpha_{k-1}$ are independent of $t$, and the expression $\omega_k(t) = \omega(x_0, t)$ approaches a finite limit $\alpha_k$ as $t$ tends to 0. The function $f$ is then said to possess a $k$th generalized derivative at the point $x_0$, and $\alpha_k$ is the value of that derivative. Instead of $\alpha_k$ we shall write $D_kf(x_0)$. It is clear that the existence of $D_kf(x_0)$ implies that of $D_{k-1}f(x_0)$.

The existence of $D_0f(x_0)$ is simply continuity of the function $f$ at the point $x=x_0$. For $k=1$ the definition of $D_kf(x_0)$ is equivalent to that of the ordinary derivative $f^{(k)}(x_0)$. No such equivalence exists for higher values of $k$, for then the existence of $D_kf(x_0)$ does not even imply continuity of $f$ for $x \neq x_0$. However, if $f^{(k)}(x_0)$ exists and is finite, then $D_kf(x_0)$ also exists and is equal to $f^{(k)}(x_0)$.

It is a classical result of Fatou that, if a function $f(x)$ is everywhere continuous and, say, of period $2\pi$, then the integral

$$\int_0^\pi \frac{f(x+t)-f(x-t)}{t}\,dt = \lim_{\epsilon\to+0}\int_\epsilon^\pi \frac{f(x+t)-f(x-t)}{t}\,dt$$

exists for almost every $x$.[1] This integral may also be written

$$\int_0^\pi \frac{\omega_0(x,t)-\omega_0(x,-t)}{t}\,dt \tag{1.2}$$

or

$$\int_0^\pi \frac{\epsilon_0(x,t)-\epsilon_0(x,-t)}{t}\,dt \tag{1.3}$$

if for any $x$ for which $D_kf(x)$ exists we introduce the notation

$$\omega_k(x,t) = D_kf(x) + \epsilon_k(x,t)$$

(so that $\epsilon(x,t)$ tends to 0 with $t$, for $x$ fixed).

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[1] For this and more general results, see, for example, the author's *Trigonometric Series*, Warsaw, 1935, chap. 7.

The convergence of the integral (1.3) may be due either to the smallness of the numerator $\epsilon_0(x, t) - \epsilon_0(x, -t)$ (which, anyway, tends to 0) or to the interference of positive and negative values of the integrand. In general, it is the second explanation which is right. For there exist continuous functions $f(x)$ such that the integral

$$\int_0^\pi \frac{|\epsilon_0(x, t) - \epsilon_0(x, -t)|}{t} dt \tag{1.4}$$

is infinite for every value of $x$.[2] For what follows it is of interest to observe that there exist continuous functions $f$ such that the integral

$$\int_0^\pi \frac{|\epsilon_0(x, t) - \epsilon_0(x, -t)|^r}{t} dt \tag{1.5}$$

is infinite for every $x$ and for every positive number $r$, however large.[3] Since $\epsilon_0(x, t) - \epsilon_0(x, -t)$ tends to 0 with $t$, the divergence of (1.5) implies the divergence of the integral with any exponent smaller than $r$.

The result of Fatou has been generalized in several ways. In particular, Plessner showed that, if

(i) $f(x)$ is integrable over $(0, 2\pi)$ and is of period $2\pi$;

(ii) $D_k f(x)$ exists for every point $x$ of a set $E$ of positive measure, then the integral

$$\begin{aligned} \int_0^\pi \frac{\omega_k(x, t) - \omega_k(x, -t)}{t} dt &= \int_0^\pi \frac{\epsilon_k(x, t) - \epsilon_k(x, -t)}{t} dt \\ &= \lim_{\delta \to +0} \int_\delta^\pi \end{aligned} \tag{1.6}$$

exists for almost every $x \in E$.[4]

The most interesting is, of course, the behavior of the integrand $[\epsilon_k(x, t) - \epsilon_k(x, -t)]/t$ in the neighborhood of $t = 0$. Since the existence of $D_k f(x_0)$ implies the boundedness of $f$ in the neighborhood of the point $x_0$, the assumption of integrability of $f$ in Plessner's theorem is really no restriction of generality, and is made merely to simplify the statement.

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[2] See, for example, S. Kaczmarz, *Integrale vom Dinischen Typus*, Studia Mathematica vol. 3 (1931) pp. 189–199, or the author's *Trigonometric Series*, p. 77.

Kaczmarz shows that the set of continuous functions $f$ for which the integral (1.4) is finite for *some* $x$ is of the first category in the space of all continuous functions.

[3] The proof does not differ from that of the special case $r = 1$.

[4] Plessner, *Ueber das Verhalten analytischer Funktionen auf dem Rande des Definitionsbereiches*, J. Reine Angew. Math. vol. 158 (1927) pp. 219–227. A different proof will be found in Marcinkiewicz, *Sur les séries de Fourier*, Fund. Math. vol. 27 (1936) pp. 38–69.

Recently, Marcinkiewicz proved the following remarkable result. Suppose that a function $f(x)$, of integrable square and of period $2\pi$, has a finite derivative for every point $x$ of a set $E$ of positive measure. Then the integral

$$\int_0^\pi \frac{[f(x+t)+f(x-t)-2f(x)]^2}{t^3}\,dt \tag{1.7}$$

is finite for almost every $x \in E$.[5]

Consideration of this integral was suggested by its analogy with an important function

$$g(\theta) = \left\{\int_0^1 (1-\rho)\,|F'(\rho e^{i\theta})|^2 d\rho\right\}^{1/2} \qquad (F(z) \text{ regular in } |z| < 1)$$

introduced by Littlewood and Paley.[6] However, the integral (1.7) may also be given a different interpretation. For, if we note that

$$\begin{aligned} f(x+t)+f(x-t)-2f(x) &= t[\omega_1(x,t)-\omega_1(x,-t)] \\ &= t[\epsilon_1(x,t)-\epsilon_1(x,-t)] \end{aligned}$$

at every point where $f'(x)$ exists, we may write the integral (1.7) in the form

$$\int_0^\pi \frac{[\omega_1(x,t)-\omega_1(x,-t)]^2}{t}\,dt = \int_0^\pi \frac{[\epsilon_1(x,t)-\epsilon_1(x,-t)]^2}{t}\,dt. \tag{1.8}$$

The finiteness of this integral at almost every point where $f'$ exists, as compared with the fact that there exist continuous functions $f$ such that

$$\int_0^\pi \frac{[\epsilon_0(x,t)-\epsilon_0(x,-t)]^2}{t}\,dt = +\infty \text{ for every } x$$

(cf. (1.5)), indicates that the behavior of the remainders $\epsilon_0(x,t)$ and $\epsilon_1(x,t)$ is essentially different. It also raises the problem of the extension of Marcinkiewicz's result to functions with a $k$th generalized derivative. The answer to that problem is given by the following theorem, the proof of which is the main object of this note, and which reduces to Marcinkiewicz's theorem for $k=1$.

THEOREM. *Suppose that a function $f(x)$ is of the class $L^2$ and of period $2\pi$, and that the generalized derivative $D_kf(x)$ exists for every point $x$ of*

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[5] Marcinkiewicz, *Sur quelques intégrales du type de Dini*, Annales de la Société Polonaise de Mathématiques, 1938, pp. 42–50.

[6] J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series*, Proc. London Math. Soc. vol. 42 (1937) pp. 52–89.

*a set E of positive measure* $(k=1, 2, \cdots)$. *Let* $\omega_k(x, t)$ *and* $\epsilon_k(x, t)$ *be defined by the equations*

$$\left.\begin{aligned} f(x+t) &= \sum_{\nu=0}^{k-1} D_\nu f(x) t^\nu/\nu! + \omega_k(x, t) t^k/k!, \\ \omega_k(x, t) &= D_k f(x) + \epsilon_k(x, t), \end{aligned}\right\} \qquad x \in E.$$

*Then the integral*

$$(1.9)\quad \int_0^\pi \frac{[\omega_k(x, t) - \omega_k(x, -t)]^2}{t}\,dt = \int_0^\pi \frac{[\epsilon_k(x, t) - \epsilon_k(x, -t)]^2}{t}\,dt$$

*is finite for almost every* $x \in E$.

This result complements the theorem of Plessner stated above, but at individual points is not comparable with the latter. It displays a new property of the remainder $\omega_k(x, t)$, a property which, unlike that in Plessner's theorem, is expressed by the convergence of a *positive* integral.

2. **Auxiliary lemmas.** The proof of the theorem depends on two known lemmas.[7]

LEMMA 1. *Suppose that P is a perfect set of positive measure situated in the interval* $(0, 2\pi)$ *and continued periodically. Let* $\phi(x) = \phi_P(x)$ *denote the function which is equal to* 0 *over P, and is equal to d if x belongs to an interval continguous to P and of length d. Then, whatever* $\lambda > 0$, *the integral*[8] $\int_{-\pi}^{\pi} (\phi^\lambda(x+t)/|t|^{\lambda+1})dt$ *is finite for almost every* $x \in P$.

LEMMA 2. *Suppose that a function* $f(x)$ *of period* $2\pi$ *has a kth generalized derivative for every point x of a set E of positive measure. Then* $f(x) = g(x) + h(x)$, *where*

(i) $g(x)$ *has everywhere a continuous kth derivative* $g^{(k)}(x)$;

(ii) $g(x) = f(x)$ *in a perfect set P contained in E and of positive measure*;

(iii) *except, perhaps, for a few segments* $\Delta$ *contiguous to P*,

$$\operatorname*{Max}_{x \in \Delta} |h(x)| \leq M\Delta^k,$$

*with M independent of* $\Delta$.[9]

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[7] Both lemmas were used by Marcinkiewicz in his proof of the finiteness of the integral (1.7), so that the proof given here of the theorem is partly modelled on Marcinkiewicz's argument.

[8] J. Marcinkiewicz, the paper cited in footnote 4.

[9] J. Marcinkiewicz, loc. cit. Marcinkiewicz's argument shows that the set $P$ may be of measure arbitrarily close to that of $E$.

Passing to the proof of the theorem, let us assume first that the function $f(x)$ has a continuous $k$th derivative $f^{(k)}(x)$. To fix the ideas we assume that $k$ is even. Then

$$
\begin{aligned}
(1/2)[f(x+t) - f(x-t)] = f'(x)t + f'''(x)t^3/3! + \cdots \\
+ f^{(k-1)}(x)t^{k-1}/(k-1)! \\
+ (1/2)[\omega_k(x, t) - \omega_k(x, -t)]t^k/k!.
\end{aligned}
\tag{2.1}
$$

Let us now consider the Fourier series of the function $f(x)$. It may be written in the form

$$
f(x) \sim \sum_{\nu=-\infty}^{+\infty} c_\nu (i\nu)^{-k} e^{i\nu x},
$$

if we assume, for simplicity, that the constant term of the Fourier series is zero. Of course,

$$
\sum_{\nu=-\infty}^{+\infty} |c_\nu|^2 < +\infty,
$$

and the Fourier series of $f^{(l)}(x)$, $0 \leq l \leq k$, is obtained by differentiating $l$ times the Fourier series of $f(x)$. Hence, taking into account that for fixed $t$, $(1/2)[f(x+t) - f(x-t)] \sim \sum_{\nu=-\infty}^{+\infty} ic_\nu (i\nu)^{-k} e^{i\nu x} \sin \nu t$, we get (cf. (2.1))

$$
\begin{aligned}
&\frac{1}{2 \cdot k!}[\omega_k(x, t) - \omega_k(x, -t)] \\
&\sim \sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x} \frac{1}{(i\nu t)^k}\left\{ i \sin \nu t - \left( i\nu t + \frac{(i\nu t)^3}{3!} + \cdots + \frac{(i\nu t)^{k-1}}{(k-1)!} \right) \right\} \\
(2.2) \quad &= i(-1)^{k/2} \sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x} \frac{1}{(\nu t)^k} \\
&\qquad \cdot \left\{ \sin \nu t - \left[ \nu t - \frac{(\nu t)^3}{3!} + \cdots \pm \frac{(\nu t)^{k-1}}{(k-1)!} \right] \right\} \\
&= i(-1)^{k/2} \sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x} \xi_k(\nu t),
\end{aligned}
$$

say, where

$$
\xi_k(u) = \left[ \sin u - \left( u - \frac{u^3}{3!} + \cdots \pm \frac{u^{k-1}}{(k-1)!} \right) \right] \Big/ u^k.
$$

Let us divide (2.2) by $t^{1/2}$ and apply Parseval's formula to the resulting relation. We get

$$\frac{1}{8\pi(k!)^2}\int_{-\pi}^{\pi}\frac{[\omega_k(x,t)-\omega_k(x,-t)]^2}{t}\,dx=\sum_{-\infty}^{+\infty}|c_\nu|^2\frac{\xi_k^2(\nu t)}{t}.$$

Integrating both sides with respect to $t$ over the interval $(0,\pi)$, and inverting the order of integration on the left, we see that

$$\begin{aligned}(2.3)\qquad &\frac{1}{8\pi(k!)^2}\int_{-\pi}^{\pi}dx\left\{\int_0^{\pi}\frac{[\omega_k(x,t)-\omega_k(x,-t)]^2}{t}\,dt\right\}\\ &=\sum_{-\infty}^{+\infty}|c_\nu|^2\int_0^{\pi}\frac{\xi_k^2(\nu t)}{t}\,dt=\sum_{-\infty}^{+\infty}|c_\nu|^2\int_0^{\nu\pi}\frac{\xi_k^2(u)}{u}\,du.\end{aligned}$$

Let us now observe that the integral

$$(2.4)\qquad \int_0^{\infty}\frac{\xi_k^2(u)}{u}\,du$$

is finite. This follows from the fact that in the neighborhood of $u=0$ the integrand is $(u^{-1})\cdot O(u^2)=O(u)=O(1)$, and in the neighborhood of $u=+\infty$ it is $u^{-1}\cdot O(u^{2k-2})u^{-2k}=O(u^{-3})$.

Since $\sum|c_\nu|^2<+\infty$, the right-hand side of (2.3) is finite, so that the integral

$$\int_0^{\pi}\frac{[\omega_k(x,t)-\omega_k(x,-t)]^2}{t}\,dt$$

is finite for almost every $x$.

If $k$ is odd, the argument is similar. Instead of (2.1) we consider the formula

$$\begin{aligned}(1/2)[f(x+t)+f(x-t)]=f(x)&+f''(x)t^2/2!+\cdots\\ &+f^{(k-1)}(x)t^{k-1}/(k-1)!\\ &+(1/2)[\omega_k(x,t)-\omega_k(x,-t)]t^k/k!\end{aligned}$$

and we still get (2.3), where now

$$\xi_k(u)=[\cos u-(1-u^2/2!+\cdots\pm u^{k-1}/(k-1)!)]u^{-k}$$

and the integral (2.4) is again finite. Thus the theorem is proved if $f^{(k)}(x)$ exists everywhere and is continuous.

3. **Proof of the theorem.** In order to complete the proof of the theorem, let us assume that there is a periodic function $f\in L^2$ having a generalized derivative $D_kf(x)$ at every point of a set $E$ of positive measure, and such that the integral (1.9) diverges everywhere in $E$. Let us now consider the decomposition $f=g+h$ of Lemma 2. The function $g$ having everywhere a continuous derivative $g^{(k)}(x)$, the cor-

responding integral (1.9) is convergent almost everywhere, and so almost everywhere in $E$. If we could prove that the integral (1.9) corresponding to the function $h$ is finite in a set $E'$ contained in $E$ and of positive measure, it would follow that the integral (1.9) for the function $f$ is convergent almost everywhere in $E'$, contrary to the assumption that it is divergent at every point of $E$. That contradiction would prove the theorem.

Let us therefore consider the function $h$ of Lemma 2. That function vanishes over a perfect set $P \subset E$. Part (iii) of Lemma 2 asserts that for all, except perhaps a finite number, of the intervals $\Delta$ contiguous to $P$,

$$|h(x)| \leq M\phi^k(x), \qquad x \in \Delta. \tag{3.1}$$

This inequality is also valid for the points of the set $P$, since both $h(x)$ and $\phi(x)$ vanish over $P$.

Let $x_0$ be any point of density of $P$ (hence $x_0$ is not an end point of any interval contiguous to $P$). From the definition of the function $\phi(x)$ it follows that $\phi(x_0+t) = o(|t|)$ as $t \to 0$. Since (3.1) is valid in a neighborhood $(x_0-\delta,\ x_0+\delta)$ of $x_0$, we see that $h(x_0+t) = o(|t|^k)$ as $t \to 0$. Hence the function $\omega_k(x_0,\ t)$ corresponding to $h(x)$ is simply $k!h(x_0+t)t^{-k}$. Thus, in our case,

$$\begin{aligned}
\int_0^\delta \frac{[\omega_k(x_0,\ t) - \omega_k(x_0,\ -t)]^2}{t}\,dt
&\leq 2\int_0^\delta \frac{\omega_k^2(x_0,\ t)}{t}\,dt + 2\int_0^\delta \frac{\omega_k^2(x_0,\ -t)}{t}\,dt \\
&= 2(k!)^2 \int_{-\delta}^{+\delta} \frac{h^2(x_0+t)}{|t|^{2k+1}}\,dt \\
&\leq 2M^2(k!)^2 \int_{-\delta}^{+\delta} \frac{\phi^{2k}(x_0+t)}{|t|^{2k+1}}\,dt.
\end{aligned}$$

If, therefore, $E'$ is the set of points of density of $P$ for which the integral

$$\int_{-\pi}^{\pi} \frac{\phi^{2k}(x+t)}{|t|^{2k+1}}\,dt$$

is finite, the integral (1.9) corresponding to the function $h$ converges at every point of $E'$. This completes the proof of the theorem, since the set $E'$ is obviously of positive measure.

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# CONVERGENCE AND SUMMABILITY PROPERTIES OF SUBSEQUENCES

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In this paper we shall discuss the relation of the convergence or $(C, 1)$ summability of a sequence to that of its subsequences. Some analogous questions for subseries have been considered [6].[2]

Let $\{s_n\}$ be an arbitrary sequence. We can obtain a 1-1 map of its infinite subsequences on the interval $0 < t \leq 1$ as follows. Let $t = .a_1a_2a_3 \cdots$ be the infinite dyadic expansion of a point $t$ of the interval. Corresponding to this point we select the following subsequence: retain $s_m$ if the $m$th place in the expansion, $a_m$, is 1, and drop it otherwise. The inverse correspondence is evident.

In terms of the Lebesgue measure of sets of points on $(0, 1)$ we are now clearly in a position to speak of "almost all" or "almost none" of the subsequences of $\{s_n\}$.

The problem we have set ourselves is to determine under what conditions does the convergence or summability of a sequence carry over to that of its subsequences, and conversely, whether these properties for suitable subsequences imply them for the sequence itself. For simplicity, we restrict ourselves to sequences of real numbers; although it is apparent that the results are more generally true.

In the case of convergence, the answer to our problem is simple: *a sequence is convergent if and only if almost all of its subsequences are convergent.* In the case of $(C, 1)$ summability, the problem is more difficult. It has been established that *all* the subsequences of a sequence cannot be summable by a fixed regular matrix method unless the sequence is in fact convergent [2]. In §§3, 4 we discuss, for $(C, 1)$ summability, the consequences of replacing *all* by *almost all.* We show, for example, that $\{s_n\}$ is $(C, 1)$ summable if almost all of its subsequences are, but not conversely.

The principal tools are the Rademacher functions, $R_n(t)$, and the properties of homogeneous sets.

Our results have certain obvious connections with probability; these are discussed in §5.

1. **Preliminary results.** All sets which occur in the sequel are to be taken as subsets of $(0, 1)$; a.e. will mean "almost everywhere in $(0, 1)$."

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[2] Numbers in brackets refer to the references listed at the end of the paper.

We shall call a measurable set $S$ homogeneous if it has the following property: let $t=.a_1a_2a_3\cdots$ be the infinite dyadic expansion of a point $t$ of $S$; then the point obtained by altering a finite number of the $a_i$ also belongs to $S$. This definition is more restricted than the usual one [7], but is adequate for our purposes.

LEMMA 1. *A homogeneous set has measure* 0 *or* 1.

This is a well known result [7, p. 145]. The sets with which our theorems deal are homogeneous. In view of this lemma, the frequently occurring phrase "almost all the subsequences" can be replaced by the seemingly weaker but actually equivalent phrase "the subsequences corresponding to a set of positive measure."

LEMMA 2. *If $S$ is of measure* 1, *then there exists a subset $E$, also of measure* 1, *with the property that if $t$ belongs to $E$, so does* $1-t$.

This becomes immediately evident if the interval (0, 1/2) is folded symmetrically over onto the interval (1/2, 1). The device of symmetric points was used to advantage in [6].

LEMMA 3. *Let $p_n$ be a sequence of positive numbers, increasing monotonically to infinity. If the series $\sum_{n=1}^{\infty}a_n/p_n$ converges, then* [5, p. 123]

$$\lim_{n\to\infty}\frac{1}{p_n}\sum_1^n a_k = 0.$$

For the properties of the Rademacher functions, we refer the reader to [3]. The principal one is contained in the following lemma.

LEMMA 4. *The series $\sum_1^\infty a_kR_k(t)$ converges on a set of measure* 1 *or measure* 0, *according as $\sum_1^\infty a_k^2$ converges or diverges.*

LEMMA 5. *If $\sum_1^\infty s_k^2/k^2$ converges, then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_1^n s_kR_k(t) = 0 \text{ a.e.} \tag{1.1}$$

For the hypothesis implies, by Lemma 4, that $\sum_1^\infty(s_k^2/k^2)R_k(t)$ converges a.e., so that the result follows from Lemma 3, with $p_n=n$.

LEMMA 6. *If*

$$\lim_{n\to\infty}\frac{1}{n}\sum_1^n s_kR_k(t) = L(t) \tag{1.2}$$

*exists a.e., then $L(t)=0$ a.e.* (*so that* (1.1) *is true*) *and*

$$(1.2) \qquad \lim_{n\to\infty} \frac{1}{n^2} \sum_1^n s_k^2 = 0.$$

(1.1′) must hold uniformly on a set $A$ of positive measure. Hence, for some $M$,

$$|L(t)| \leqq M, \qquad t \in A.$$

But the set of points for which this inequality holds is homogeneous, so that, by Lemma 1, $L(t)$ is bounded a.e. Since it is obviously integrable,

$$(1.3) \qquad \int_0^1 L(t)dt = a$$

exists.

Now, choose an $\epsilon > 0$ and consider the sets $I_1$, $I_2$ and $I_3$ for which

$$L(t) - a > \epsilon,$$
$$L(t) - a < -\epsilon,$$
$$|L(t) - a| \leqq \epsilon,$$

respectively. These sets are homogeneous; since their union is of measure 1, at least one of them must be of measure 1. If either $I_1$ or $I_2$ has measure 1, (1.3) is contradicted. Hence $|L(t)-a| \leqq \epsilon$ a.e., so that $L(t) = a$ on a set $B$ of measure 1.

By Lemma 2, we can pick two points $t_0$ and $1-t_0$ in $B$. By (1.1′)

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n s_k R_k(t_0) = a,$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n s_k R_k(1 - t_0) = a.$$

Adding these, we see that $a = 0$; this establishes (1.1).

To prove (1.2) we employ a device due to Kolmogoroff [4, p. 127]. Let $S_{m,n}(t) = \sum_m^n s_k R_k(t)$. Then

$$(1.4) \qquad \begin{aligned} S^2_{m,n}(t) &= \sum_{j,k=m}^n s_j s_k R_j(t) R_k(t) \\ &= \sum_m^n s_k^2 + 2 \sum_{m \leqq j < k \leqq n} s_j s_k R_j(t) R_k(t). \end{aligned}$$

If (1.1) holds for almost all $t$, it must hold uniformly on a set $E$ of positive measure $|E|$. By (1.4)

$$\int_E S_{m,n}^2(t)\,dt = |E| \sum_m^n s_k^2 + M \tag{1.5}$$

where

$$M = 2 \sum_{m \leq j<k \leq n} s_j s_k \int_E R_j(t) R_k(t)\,dt.$$

From the Schwarz inequality

$$|M| \leq 2\left( \sum_{m \leq j<k \leq n} s_j^2 s_k^2 \right)^{1/2} \left( \sum_{m \leq j<k \leq n} b_{jk}^2 \right)^{1/2} \tag{1.6}$$

where $b_{jk} = \int_E R_j(t) R_k(t)\,dt$.

The functions $R_j(t)R_k(t)$, for $1 \leq j<k<\infty$, are orthonormal on $(0, 1)$. If $X(t)$ is the characteristic function of $E$, it follows from Bessel's inequality that

$$\sum_{1 \leq j<k<\infty} b_{jk}^2 \leq \int_0^1 [X(t)]^2\,dt = |E|.$$

For a large enough value of $m$

$$\left( \sum_{m \leq j<k<\infty} b_{jk}^2 \right)^{1/2} \leq |E|/4.$$

By (1.6)

$$|M| \leq \left( \sum_{j,k=m}^{n} s_j^2 s_k^2 \right)^{1/2} (|E|/2) \leq (|E|/2) \sum_m^n s_j^2.$$

Then, by (1.5) we have

$$\int_E S_{m,n}^2(t)\,dt \geqq (|E|/2) \sum_m^n s_j^2. \tag{1.7}$$

But for this fixed value of $m$,

$$\lim_{n \to \infty} \frac{1}{n^2} S_{m,n}^2(t) = 0$$

boundedly on $E$. (1.7) then implies that

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_m^n s_j^2 = 0$$

and the proof is complete.

2. **Convergence of subsequences.** If a sequence $\{s_n\}$ of real numbers is convergent to $S$, then every subsequence is likewise convergent to $S$. What may be said if $\{s_n\}$ is divergent?

THEOREM 1. *If $\{s_n\}$ is divergent, so are almost all of its subsequences.*

Let $C$ be the set of points corresponding to convergent subsequences. It is homogeneous, and hence of measure 0 or 1. Suppose it is of measure 1. By Lemma 2, we can pick a number $t_0$ not of the form $K/2^n$, such that both $t_0$ and $1-t_0$ belong to $C$. The corresponding subsequences can have no terms in common, for the dyadic expansions of $t_0$ and $1-t_0$ cannot agree anywhere. Furthermore, every term of $\{s_n\}$ is present in one or the other of the subsequences. Thus we have split $\{s_n\}$ into two subsequences, and since $t_0 \in C$, $1-t_0 \in C$, each is convergent. Let their limit points be $S'$ and $S''$.

Every convergent subsequence of $\{s_n\}$ must then have either $S'$ or $S''$ as its limit. Let $C'$ ($C''$) be the set of points corresponding to subsequences convergent to $S'$ ($S''$). Then $C'$ and $C''$ are homogeneous and, since $C = C' \cup C''$, one of them—say $C'$—must be of measure 1. This means that the $t_0$ above could have been chosen from $C'$, so that $\{s_n\}$ can be split into two convergent subsequences, each of which converges to the same limit. Every subsequence is then convergent, and hence $\{s_n\}$ is itself convergent; this contradicts the hypothesis of our theorem, and $C$ therefore has measure 0.

COROLLARY. *If the series $\sum_1^\infty a_k$ converges for almost every bracketing of terms, it is convergent.*

For if $\{s_n\}$ is the sequence of partial sums, and we bracket the series in blocks of length $n_1, n_2, \cdots$, then the partial sums of the resulting series are $s_{n_1}, s_{n_1+n_2}, \cdots$, a subsequence of $\{s_n\}$. By the theorem, if almost all of these converge, the sequence itself does.

3. **(C, 1) summability.** In this and the succeeding section, we consider the corresponding problem for $(C, 1)$ summability. As we have previously noted, we cannot expect all of the subsequences of a sequence to be $(C, 1)$ summable unless the sequence is convergent.

THEOREM 2. *If $\{s_n\}$ is $(C, 1)$ summable to $S$, and*

$$\sum_1^\infty s_k^2/k^2 < \infty, \tag{3.1}$$

*then almost all the subsequences are $(C, 1)$ summable to $S$.*

The problem of the convergence of the Cesàro means of almost all the subsequences of $\{s_n\}$ reduces to the requirement that

$$\lim_{n\to\infty} \frac{\sum_1^n s_k \cdot \frac{1}{2}[1 + R_k(t)]}{\sum_1^n \frac{1}{2}[1 + R_k(t)]} = S \text{ a.e.} \tag{3.2}$$

We can rewrite the expression inside the limit as

$$\frac{\frac{1}{n}\sum_1^n s_k + \frac{1}{n}\sum_1^n s_k R_k(t)}{1 + \frac{1}{n}\sum_1^n R_k(t)}. \tag{3.3}$$

By hypothesis, $\lim (1/n)\sum_1^n s_k = S$; then (3.2) follows immediately by condition (3.1) and Lemma 4.

The question arises whether the condition (3.1) can be removed or at least weakened. We shall show in the next section that it cannot be dropped entirely; on the other hand, this does not preclude the possibility that it might be weakened, perhaps to (1.2) which by Lemma 3 it implies, and which is *necessary* for the validity of the conclusion (Theorem 3, corollary).

4. **Converse.** We now prove the analogue of Theorem 1 for $(C, 1)$ summability. We note again that since $(C, 1)$ summability is a method that preserves translation, our sets are homogeneous.

THEOREM 3. *If almost all the subsequences of* $\{s_n\}$ *are* $(C, 1)$ *summable, then* $\{s_n\}$ *is itself summable to a value* $S$, *and almost all the subsequences are in turn summable to* $S$.

As in the preceding section, the hypothesis implies that (3.3) converges a.e. as $n\to\infty$. By Lemma 5, the denominator of (3.3) approaches the value 1 a.e., so that

$$\sigma_n(t) = \frac{1}{n}\sum_1^n s_k[1 + R_k(t)] \tag{4.1}$$

converges on a set $C$ of measure 1. By Lemma 2, we can pick $t_0$ and $1-t_0$ belonging to $C$; then

$$\sigma_n(t_0) + \sigma_n(1 - t_0) = \frac{2}{n}\sum_1^n s_k$$

so that $(1/n)\sum_1^n s_k$ converges to a value $S$. From (4.1) we see that

$$\frac{1}{n}\sum_1^n s_k R_k(t) \tag{4.2}$$

converges a.e.; by Lemma 6, this last expression then converges a.e. to 0, and, by (4.1),

$$\lim \sigma_n(t) = S \text{ a.e.}$$

Corollary. *Under the hypothesis of the theorem,*

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_1^n s_k^2 = 0. \tag{4.3}$$

For (4.2) converges a.e.; (4.3) follows from Lemma 6.

We are now in a position to answer the question raised in the preceding section by exhibiting a $(C, 1)$ summable sequence with almost none of its subsequences $(C, 1)$ summable. This example is the sequence $s_n = (-1)^n n^{1/2}$. It is readily seen to be summable to zero, but since it violates (4.3), almost none of its subsequences can be $(C, 1)$ summable.

If the sequence $\{s_n\}$ is bounded, condition (3.1) is satisfied. Combining Theorems 2 and 3, we have the following theorem.

Theorem 4. *A bounded sequence is* $(C, 1)$ *summable if and only if almost all of its subsequences are* $(C, 1)$ *summable.*

5. **Probability.** Lemma 5, on which a large part of our paper depends, can also be obtained from the strong law of large numbers [4, p. 59], for we may regard the sequence $s_n R_n(t)$ as a sequence of independent random variables of mean value 0 and standard deviation $s_n$.

An interesting result that may be obtained as a corollary of Theorem 2 is one of Birnbaum and Zuckerman on the v. Mises collective [1].

Theorem A. *If a sequence of* 0*'s and* 1*'s satisfies the first postulate of v. Mises, the second postulate is fulfilled for almost all selections.*

Here the v. Mises postulates are:

P1: *If* $n_0$ *and* $n_1$ *are the number of* 0*'s and* 1*'s respectively among the first* $n$ *terms of the sequence, then the limits* $\lim n_0/n$ *and* $\lim n_1/n$ *exist.*

P2: *For a suitable selection of an infinite subsequence from the given sequence, the same limits exist and their values are unchanged.*

To obtain the theorem we need only observe that $\lim n_1/n$ is the

(*C*, 1) sum of the given sequence, and that (3.1) is automatically satisfied.

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# LAMBERT SUMMABILITY OF ORTHOGONAL SERIES

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If we define Lambert summability of a series, $\sum_1^\infty a_n$, in terms of the existence of the limit

$$(1) \qquad L(a_n) = \lim_{x\to 1-0} (1-x)\sum_1^\infty \frac{n a_n x^n}{1-x^n}$$

we have, by a well known theorem of Hardy-Littlewood [1],[1] that $C(a_n)\to L(a_n)\to A(a_n)$; $C(a_n)$, $A(a_n)$ are respectively the Cesàro and Abel means of the series $\sum_1^\infty a_n$.

The proof of $C(a_n)\to L(a_n)$ is elementary in nature, but the proof of $L(a_n)\to A(a_n)$ requires the prime number theorem, and conversely the theorem $L(a_n)\to A(a_n)$ implies the prime number theorem.

For that reason, it is perhaps interesting to show that for orthogonal series of functions $f(x)$, belonging to $L^2$, the inclusion of $L(a_n)$ between $C(a_n)$ and $A(a_n)$ follows in completely elementary fashion.

That $C(a_n)\sim A(a_n)$ for orthogonal series of $L^2$ is a known result of Kaczmarz [2]. Hence it is sufficient to show that $L(a_n)\to C(a_n)$. In addition, it is further known that $C(a_n)$ is equivalent to the convergence of the partial sums of the orthogonal series $s_{2^n}(\theta)=\sum_1^{2^n} a_k\phi_k(\theta)$ [3]. Therefore, finally, it comes to showing that Lambert summability implies the convergence of the partial sums $s_{2^n}(\theta)$, in order to prove the theorem.

Let $f(\theta)\subset L^2(a, b)$, $a_n=\int_a^b f(\theta)\phi_n(\theta)d\theta$; where $(\phi_n(\theta))$ is an orthonormal sequence in $(a, b)$, $s_n(\theta)=\sum_1^n a_n\phi_n(\theta)$.

Write, where $x$ is $1-1/2^n$,

$$(2) \qquad U_n(\theta) = \sum_1^\infty k a_k\phi_k(\theta)\frac{(1-x)x^k}{1-x^k} - s_{2^n}(\theta) = T_n(\theta) + V_n(\theta)$$

where

$$(3) \qquad T_n(\theta) = \sum_1^{2^n} a_k\phi_k(\theta)\left(\frac{k(1-x)x^k}{1-x^k} - 1\right),$$

$$(4) \qquad V_n(\theta) = \sum_{2^n+1}^\infty k a_k\phi_k(\theta)\frac{(1-x)x^k}{1-x^k}.$$

If $\lim_{n\to\infty} U_n(\theta)=0$, the result is proven. To that end, consider the

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[1] Numbers in brackets refer to the references listed at the end of the paper.

series

$$\sum_1^\infty [U_n(\theta)]^2. \tag{5}$$

To prove convergence almost everywhere in $\theta$, it is sufficient to show

$$\sum_{n=1}^\infty \int_a^b [U_n(\theta)]^2 d\theta < \infty. \tag{6}$$

We have

$$\sum_n \int_a^b [U_n(\theta)]^2 d\theta \leqq 2\sum_n \int_a^b [T_n(\theta)]^2 d\theta + 2\sum_n \int_a^b [V_n(\theta)]^2 d\theta. \tag{7}$$

Let us consider the convergence of each series separately.

$$\begin{aligned}\sum_n \int_a^b [T_n(\theta)]^2 d\theta &= \sum_n \int_a^b \left(\sum_1^{2^n} a_k\phi_k(\theta)\left(\frac{k(1-x)x^k}{1-x^k}-1\right)\right)^2 d\theta \\ &= \sum_n \left(\sum_1^{2^n} a_k^2\left(\frac{k(1-x)x^k}{1-x^k}-1\right)^2\right)\end{aligned} \tag{8}$$

where the $x$ appearing in $\sum_1^{2^n}$ is $1-1/2^n$, $n \geqq 1$.

Now

$$1 - x^k \leqq k(1-x), \qquad 0 \leqq x \leqq 1, \tag{9}$$

$$1 - x^k \geqq 1 - \frac{k(1-x)x^k}{1-x^k} \geqq 0, \tag{10}$$

so that

$$\begin{aligned}\sum_n \int_a^b [T_n(\theta)]^2 d\theta &\leqq \sum_n \sum_1^{2^n} a_k^2(1-x^k)^2 \leqq \sum_n \sum_1^{2^n} k^2 a_k^2 (1-x)^2 \\ &\leqq \sum_n \frac{1}{2^{2n}} \sum_1^{2^n} k^2 a_k^2 \leqq \sum_k a_k^2 k^2 \sum_{n \geqq \log_2 k} 2^{-2n} \leqq A \sum_k a_k^2\end{aligned} \tag{11}$$

and $\sum_k a_k^2 < \infty$ since $f(x) \subset L^2(a, b)$.

Now for the second series $\sum_n \int_a^b [V_n(\theta)]^2 d\theta$:

$$\begin{aligned}\sum_n \int_a^b [V_n(\theta)]^2 d\theta &= \sum_n \int_a^b \left(\sum_{2^n+1}^\infty a_k\phi_k(\theta)\frac{k(1-x)x^k}{1-x^k}\right)^2 d\theta \\ &= \sum_n \left\{\sum_{2^n+1}^\infty k^2 a_k^2 \frac{(1-x)^2 x^{2k}}{(1-x^k)^2}\right\}\end{aligned} \tag{12}$$

where the $x$ appearing in $\sum_{2^n+1}^{\infty}$ is $1-1/2^n$, $n \geqq 1$.

Since $(1-2^{-n})^k$ is a decreasing function of $k$,

$$
\begin{aligned}
&\sum_n \sum_{2^n+1}^{\infty} k^2 a_k^2 \frac{(1-x)^2 x^{2k}}{(1-x^k)^2} \\
(13) \qquad &\leqq \sum_n \frac{1}{1-(1-2^{-n})^{2^n}} \sum_{2^n+1}^{\infty} k^2 a_k^2 (1-x)^2 x^{2k} \\
&\leqq A \sum_n \sum_{2^n+1} k^2 a_k^2 (1-x)^2 x^{2k}.
\end{aligned}
$$

We can majorize $k^2 \sum_1^{\infty} 2^{-2n}(1-2^{-n})^{2k}$ by the integral

$$
\begin{aligned}
&k^2 \int_0^{\infty} 2^{-2x}(1-2^{-x-1})^{2k} dx = 4k^2 \int_1^{\infty} 2^{-2x}(1-2^{-x})^{k} dx \\
(14) \qquad &\qquad < 4k^2 \int_0^{\infty} 2^{-2x}(1-2^{-x})^{2k} dx \\
&\qquad = A k^2/(2k+1)(2k+2)
\end{aligned}
$$

which is obviously bounded.

Therefore we have proven the convergence of the series, which implies that $\lim_{n\to\infty} U_n(\theta) = 0$ almost everywhere in $\theta$, which implies that

$$
(15) \qquad L(a_n) = \lim_{n\to\infty} s_{2^n}(\theta)
$$

almost everywhere in $\theta$.

This is equivalent to what we set out to prove.

## REFERENCES

1. Hardy and Littlewood, *On a Tauberian theorem for Lambert's series*, Proc. Lond. Math. Soc. (2) vol. 19 (1921) pp. 21–29.

2. Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, Warsaw, 1935, Theorem 583.

3. Ibid., Theorem 585.

UNIVERSITY OF WISCONSIN

# A NOTE ON PRIMITIVE SKEW CURVES

DICK WICK HALL

In 1930 Kuratowski[1] established the following result:

THEOREM A. *A locally connected continuum, containing but a finite number of simple closed curves, is homeomorphic with a subset of the plane, provided that it does not contain a primitive skew curve of type* I *or a primitive skew curve of type* II.

By a primitive skew curve of type I we mean any topological image of the complex $C$ which consists of two groups of three vertices each and nine 1-cells, in a fashion that each vertex of one group together with each vertex of the other group bounds a 1-cell. By a primitive skew curve of type II we mean any topological image of the complex $D$ which consists of five vertices and ten 1-cells in a fashion that each pair of vertices bounds a 1-cell.

In 1934 Claytor[2] proved that every cyclic locally connected continuum containing no primitive skew curve of either type must be homeomorphic with a subset of a spherical surface.

In this note we point out that for a large class of locally connected continua the property of being planar may be insured merely by requiring that the given locally connected continuum contain no primitive skew curve of type I. Stated precisely, our principal theorem is the following:

THEOREM 1. *Let $M$ be a locally connected continuum separated by no pair of its points and assume that $M$ contains no primitive skew curve of type* I. *Then $M$ contains no primitive skew curve of type* II.

The proof is immediate. Assuming the existence of a primitive skew curve $D$ of type II in $M$ and using the fact that no two points separate $M$ we may easily find a simple arc $ab$ in $M$ having only the points $a$ and $b$ in common with $D$ and such that $a$ is interior to a free arc of $D$ while $b$ does not lie on the closure of this free arc. Denote the sum of $D$ and the arc $ab$ by $D'$. A careful examination of $D'$ shows that this set must contain a primitive skew curve of type I contrary to our hypothesis on $M$.

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[1] *Sur les problème des courbes gauches en topologie*, Fund. Math. vol. 15 (1930) pp. 271–283.

[2] *Topological immersion of Peanian continua in a spherical surface*, Ann. of Math. vol. 35 (1934) pp. 809–835.

It is natural to inquire whether or not the above result remains true when the types of primitive skew curves there mentioned are interchanged. To show that this is not the case we need only construct a locally connected continuum $K$ which is separated by no pair of its points, but such that $K$ contains a primitive skew curve of type I but no primitive skew curve of type II. To do this let $K$ consist of three 2-cells $E$, $F$, $G$ having their interiors disjoint and exactly three boundary points $a$, $b$, $c$ in common. It is self-evident that no two points separate $K$ while $K$ contains a primitive skew curve of type I. It is easy to show that $K$ contains no primitive skew curve of type II by making use of the fact that no two vertices of such a primitive skew curve can be separared by the removal of any three points of the curve.

As an application of our theorem we prove the following unpublished theorem of F. B. Jones:[3]

THEOREM 2 (JONES). *Let $M$ be a locally connected continuum separated by no pair of its points but by every one of its simple closed curves, and suppose that $M$ contains no primitive skew curve of type* I. *Then $M$ is homeomorphic with a spherical surface.*

PROOF. By Theorem 1 and the theorem of Claytor cited above we know that $M$ is homeomorphic to a subset of a spherical surface $S$. If the theorem is false then $M$ has a complementary domain $D$ in $S$. By a theorem of R. L. Moore[4] the boundary of $D$ is a simple closed curve $J$. Now $J$ must separate $M$. Using this fact together with the knowledge that every component of $M-J$ must lie on the side of $J$ opposite to $D$ we easily see that some pair of points of $M$ must separate $M$. This contradiction completes the proof of the theorem.

UNIVERSITY OF MARYLAND

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[3] See Bull. Amer. Math. Soc. abstract 48-11-340.

[4] *Concerning the common boundary of two domains*, Fund. Math. vol. 6 (1924) pp. 203–213.

# NOTE ON THE NON-EXISTENCE OF ODD PERFECT NUMBERS OF FORM $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$

ALFRED BRAUER

A paper under this title appears in Bull. Amer. Math. Soc. vol. 49 (1943) pp. 712–718. On October 2, 1943, I received a photoprint of a paper of H. J. Kanold, *Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, J. Reine Angew. Math. vol. 184 (1942) pp. 116–124, for reviewing in the Mathematical Reviews. This paper was not available to me previously. In this paper the author, too, proves my theorem. But his proof is so much more difficult than my proof that I believe that the publication of my paper is also now not superfluous. The new method of both papers is the proof of case I. While my proof consists of only 8 lines, Kanold needs almost 5 pages for the proof of this case. In the proof of II, well known methods can be applied, and therefore it is not surprising that both proofs are similar. Difficulties arise here only if we obtain divisors which are greater than the range of D. N. Lehmer's prime factor tables.

Thus in the proof of II c, I had to consider $\sigma(30941^2)$. I checked the small primes and found that every prime factor must be greater than 151. Using this fact, and Lemma 1 and 2, I could prove this case without knowing the factorization of $\sigma(30941^2)$. I learned from Kanold's paper that just the next prime, namely 157, is a divisor,

$$\sigma(30941^2) = 957{,}376{,}423 = 157 \cdot 433 \cdot 14083.$$

Now II c can be proved in a simpler, but perhaps less interesting way. Since $s=13$ and $k=2$ one of the three factors of $\sigma(30941^2)$ equals $p$ and the others must be $q_1$ and $q_2$. But then $(p+1)/2$ and hence $n$ must be divisible by 79 or by 7. This contradicts $k=2$.

Moreover, I learned from Kanold's paper that he proved the special cases $\alpha=1$ and $\alpha=5$ already in an earlier paper *Über eine notwendige Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, Deutsche Mathematik vol. 4 (1939) pp. 53–57, not available here and not mentioned in his paper cited in footnote 5.

UNIVERSITY OF NORTH CAROLINA

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# A CORRECTION TO "ON THE AVERAGE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION"[1]

M. KAC

Dr. R. P. Boas has kindly called my attention to the fact that the proofs of the results of Littlewood and Offord[2] have been published in the Proc. Cambridge Philos. Soc. vol. 35 (1939) pp. 133–148. It should also be mentioned that a similar problem was studied by entirely different methods by S. O. Rice in his paper *The distribution of the maxima of a random curve*, Amer. J. Math. vol. 61 (1939) pp. 409–416. Finally, I take this opportunity to state that under the conditions considered in my note the probability that equation (1) has a multiple root is zero. In fact, this probability is the probability that the discriminant vanish; this probability is clearly equal to zero since the discriminant is an analytic function of the coefficients and the measure considered is nonsingular.

CORNELL UNIVERSITY

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[1] Bull. Amer. Math. Soc. vol. 49 (1943) pp. 314–320.

[2] See the footnote on page 314 of my note.

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# A CORRECTION TO "A LINEAR TRANSFORMATION WHOSE VARIABLES AND COEFFICIENTS ARE SETS OF POINTS"

S. T. SANDERS, JR.

In my paper *A linear transformation whose variables and coefficients are sets of points*, published in Bull. Amer. Math. Soc. vol. 48 (1942), the fourth line, page 442, should read: "there is a subscript $k=k'$ such that $P_{k'}$ involves only factors of $P_1$."

SOUTHWESTERN LOUISIANA INSTITUTE

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# ERRATA TO "TABLE OF THE ZEROS OF THE LEGENDRE POLYNOMIALS OF ORDER 1-16 AND THE WEIGHT COEFFICIENTS FOR GAUSS' MECHANICAL QUADRATURE FORMULA"[1]

ARNOLD N. LOWAN, NORMAN DAVIDS, AND ARTHUR LEVENSON

The following errors should be noted.

Page 740. The statement "where $\xi$ is a point in the interval $(p, q)$ and $k_n$ is the normalizing factor for $P_n(x)$, which is equal to

$$\left(\frac{2n+1}{q-p}\right)^{1/2}."$$

should be replaced by "where $\xi$ is a point in the interval $(p, q)$ and $k_n$ is

$$({}_{2n}C_n)(2n+1)^{1/2}/(q-p)^{n+1/2}."$$

The bottom of the same page should contain the following:

"*Note.* On this and the previous page, the subscripts of $x_n$ and $a_n$ are allowed to range from 1 to $n$. In the table, however, it seemed more convenient to change the notation slightly, and permit both positive and negative subscripts. Thus for $n=3$, the roots are $x_{-1}$, $x_0$, $x_1$. Roots with negative subscripts can be obtained by symmetry from those tabulated. Similarly for the entries $a_n$."

Page 741. The value of $x_2$ should read 0.519096129206812.

Page 742. The value of $x_1$ should read $x_1 = 0.125233408511469$.

The value of $x_2$ should read $x_2 = 0.367831498998180$.

NEW YORK CITY

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Received by the editors March 18, 1943.

[1] Bull. Amer. Math. Soc. vol. 48 (1942) pp. 739–743.

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## ERRATA, VOLUME 49

E. J. Gumbel, *On the plotting of statistical observations*, abstract 49-9-238.

p. 698, line 5 of the abstract. Instead of "The correction Δ for the rank is unlimited and possesses a mode, Δ increases" read "If the variate is unlimited and possesses a single mode the correction Δ for the rank increases."

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